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ON \mathcal{H}^+ Type Multivalued Contractions and Applications in Symmetric and Probabilistic Spaces

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Abstract: The main idea in this article is to establish some fixed and common fixed point results for multivalued \mathcal{H}^+ -type contraction mappings in symmetric spaces. New results are accompanied with illustrative examples. An application of the obtained results to probabilistic spaces is presented.

Keywords: (common) fixed point; Hausdorff distance; α -admissibility; α -complete symmetric spaces; probabilistic spaces

1. Introduction

In a celebrated work, Nadler [1] investigated fixed points of set-valued mappings with the help of the Pompeiu-Hausdorff metric (see definition in next section). In fact, he obtained an extension of the contraction mapping theorem of Banach for set-valued mappings. Later on, the area of fixed points of set-valued functions was developed into a very rich and fruitful theory. Many authors contributed significantly to this (cf. Feng and Liu [2], Kaneko [3], Klim and Wardowski [4], Aydi et al. [5,6], Lim [7], Dozo [8], Mizoguchi and Takahasi [9], etc.). All these legendary works presented fixed point theorems using the Pompeiu-Hausdorff metric. Recently, Pathak and Shahzad [10] obtained fixed point results for \mathcal{H}^+ -contractions (different from set-valued contractions), see also [11,12]. One more category of fixed point results for set-valued mappings was established by Dehaish and Latif [13] without using the Hausdorff metric. For other related results, see [14–24].

The generalization of fixed point theorems not only happened by weakening the contractive conditions but also by relaxing the constraints on (the geometry of) space (see [25,26] and references therein). It is observed that, while proving certain fixed point theorems in metric spaces, the distance function need not satisfy the triangular inequality. This observation inspired Hicks [27] to establish fixed point results for multivalued mappings in a space where distance function does not satisfy the triangular inequality. Such distance functions are called symmetric (or semi-metric). This distance function is comparatively weaker than the metric. Moreover, Hicks and Rhoades in [28] also established the common fixed point results in symmetric spaces (also see Moutawakil [29]). Many authors contributed greatly in the enrichment of fixed point theory in symmetric spaces, see [30–38].

In the present work, we introduce new classes of (set-valued) mappings, namely \mathcal{H}^+ -type contractions, and prove related fixed point results in symmetric spaces. Section 2 presents all the basic notions in the existing literature, which are used while proving our results. Section 3 is divided into three parts: Firstly, we present fixed point results for α - φ - \mathcal{H}^+ -contractive mappings in symmetric spaces. Secondly, we discuss the existence of common fixed points for the α - φ - \mathcal{H}^+ -contractive pair

(T, S) of set-valued mappings T and S on symmetric spaces. Third, we establish a result showing the actuality of fixed points of set-valued mappings without using \mathcal{H} or \mathcal{H}^+ symmetry. Illustrative examples are coined to show the significance of the presented results. The concluding section discusses the application of our results to probabilistic metric spaces.

2. Preliminaries

We start with the definition of symmetric spaces.

Definition 1 ([39]). A function $s : Y \times Y \rightarrow [0, +\infty)$ satisfying

- (W1) $s(a, b) = 0$ if and only if $a = b$ and $s(a, b) \geq 0$, for $a, b \in Y$
- (W2) $s(a, b) = s(b, a)$, for all $a, b \in Y$

is called symmetric (semi-metric) on a nonempty set Y , whereas the pair (Y, s) is called a symmetric space.

Definition 2 ([39]). Let s be symmetric on Y . For $y \in Y$ and $\gamma > 0$, consider $B(y, \gamma) = \{x \in Y : s(y, x) < \gamma\}$.

1. A topology $\tau(s)$ on Y is defined by $\mathcal{U} \in \tau(s)$ if and only if, for each $u \in \mathcal{U}$, there exists $\gamma > 0$ such that $B(u, \gamma) \subset \mathcal{U}$.
2. A subset P of Y is a neighborhood of $y \in Y$ if there exists $\mathcal{U} \in \tau(s)$ such that $y \in \mathcal{U} \subset P$.
3. Such s is a semi-metric if for each $y \in Y$ and each $\gamma > 0$, $B(y, \gamma)$ is a neighborhood of y in the topology $\tau(s)$.

Definition 3 ([28,40]). A sequence $\{a_j\}$ is s -Cauchy if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $s(a_j, a_k) < \epsilon$ for every $j, k \geq N$ with $k \neq j$.

Definition 4 ([28,40]). Let (Y, s) be a symmetric space.

- (a) (Y, s) is S -complete if, for every s -Cauchy sequence $\{a_j\}$, there exists $a \in Y$ with $\lim_{j \rightarrow \infty} s(a_j, a) = 0$.
- (b) (Y, s) is s -Cauchy complete if, for every s -Cauchy sequence $\{a_j\}$, there exists $a \in Y$ with $\lim_{j \rightarrow \infty} a_j = a$ with respect to $\tau(s)$.
- (c) $T : Y \rightarrow Y$ is s -continuous if, whenever $\lim_{j \rightarrow \infty} s(a_j, a) = 0$, we have $\lim_{j \rightarrow \infty} s(Ta_j, Ta) = 0$.
- (d) $T : Y \rightarrow Y$ is $\tau(s)$ -continuous if, whenever $\lim_{j \rightarrow \infty} a_j = a$ with respect to $\tau(s)$, we have $\lim_{j \rightarrow \infty} Ta_j = Ta$ with respect to $\tau(s)$.

Due to relaxing triangular inequality in case of semi-metrics, some alternate concepts need to be satisfied and are listed below.

(W3) [39] Assume $\{a_j\}$, a , and b in Y such that

$$\left. \begin{aligned} \lim_{j \rightarrow \infty} s(a_j, a) = 0 \text{ and} \\ \lim_{j \rightarrow \infty} s(a_j, b) = 0 \end{aligned} \right\} . \text{ Then } a = b.$$

(W4) [39] Assume $\{a_j\}$, $\{b_j\}$, and a in Y such that

$$\left. \begin{aligned} \lim_{j \rightarrow \infty} s(a_j, a) = 0 \text{ and} \\ \lim_{j \rightarrow \infty} s(a_j, b_j) = 0 \end{aligned} \right\} . \text{ Then } \lim_{j \rightarrow \infty} s(b_j, a) = 0.$$

(CC) [41] Assume $\{a_j\}$ and a in Y such that $\lim_{j \rightarrow \infty} s(a_j, a) = 0$. Then $\lim_{j \rightarrow \infty} s(a_j, b) = s(a, b)$ for some $b \in Y$.

Definition 5 ([27]). A symmetric space (Y, s) is complete if, whenever $\sum_{j=1}^{\infty} s(a_j, a_{j+1}) < \infty$, there exists $a \in Y$ such that $\lim_{j \rightarrow \infty} s(a_j, a) = 0$.

Definition 6 ([29]). Let $P \neq \emptyset$ be a subset in a symmetric space (Y, s) . We say that P is

- (i) s -closed if and only if $\bar{P}^s = P$, where $\bar{P}^s = \{y \in Y : s(y, P) = 0\}$ and $s(y, P) = \inf\{s(y, p) : p \in P\}$;
- (ii) bounded if and only if $\sup\{d(p, q) : p, q \in P\} < \infty$.

The following families of subsets of a nonempty set Y are considered for the rest of the paper:

$$\begin{aligned} \mathcal{N}(Y) &= \{P : P \subseteq Y \text{ and } P \neq \emptyset\}; \\ \mathcal{CL}_s(Y) &= \{P : P \in \mathcal{N}(Y) \text{ and } \bar{P}^s = P\}; \\ \mathcal{B}_s(Y) &= \{P : P \in \mathcal{N}(Y) \text{ and } P \text{ is bounded}\}; \\ \mathcal{CB}_s(Y) &= \{P : P \in \mathcal{CL}_s(Y) \cap \mathcal{B}_s(Y)\}; \text{ and} \\ \mathcal{C}(Y) &= \{P : P \in \mathcal{N}(Y) \text{ and } P \text{ is compact}\}. \end{aligned}$$

For $P, Q \in \mathcal{CB}_s(Y)$, the \mathcal{H} and \mathcal{H}^+ distance functions are defined as

$$\mathcal{H}(P, Q) = \max \left\{ \sup_{a \in P} s(a, Q), \sup_{b \in Q} s(b, P) \right\} \tag{1}$$

and

$$\mathcal{H}^+(P, Q) = \frac{1}{2} \left[\sup_{a \in P} s(a, Q) + \sup_{b \in Q} s(b, P) \right] \tag{2}$$

where $s(a, Q) = \inf_{b \in Q} s(a, b)$. \mathcal{H} is called a Pompeiu-Hausdorff distance.

Proposition 1 ([29]). $(\mathcal{CB}_s(Y), \mathcal{H})$ is a symmetric space if (Y, s) is a symmetric space.

Proposition 2. $(\mathcal{CB}_s(Y), \mathcal{H}^+)$ is a symmetric space if (Y, s) is a symmetric space.

Proof. Clearly, \mathcal{H}^+ satisfies (W2) because s satisfies (W2).

Next, we show that $\mathcal{H}^+(P, Q) = 0$ if and only if $P = Q$. We only need to show that $\mathcal{H}^+(P, Q) = 0 \implies P = Q$. The converse will be true due to property (W1) on s . The fact that $\mathcal{H}^+(P, Q) = 0$ for any $P, Q \in \mathcal{CB}_s(X)$ implies that $\sup\{s(q, P) | q \in Q\} = 0$ and $\sup\{s(p, Q) | p \in P\} = 0$. Thus, $s(q, P) = 0$ for $q \in Q$ and $s(p, Q) = 0$ for $p \in P$. This yields that $q \in \bar{P}^s$ and $p \in \bar{Q}^s$. Hence, $Q \subset \bar{P}^s = P$ and $P \subset \bar{Q}^s = Q$. Therefore, $P = Q$, so \mathcal{H}^+ satisfies (W1). \square

Remark 1 ([10]). \mathcal{H}^+ and \mathcal{H} are topologically equivalent, i.e.,

$$k_1 \mathcal{H}(P, Q) \leq \mathcal{H}^+(P, Q) \leq k_2 \mathcal{H}(P, Q),$$

where $k_1 = \frac{1}{2}$ and $k_2 = 1$.

It is worth mentioning here that the equivalence of two symmetric spaces does not mean that the results proved with one are equivalent to others. This is shown by means of some examples in [10,12] in case of metric spaces.

Lemma 1 ([27]). Let (Y, s) be a symmetric space and $T : Y \rightarrow \mathcal{B}_s(Y)$. Then $\lim_{j \rightarrow \infty} s(a_j, Ta) = 0$ if and only if there exists $b_j \in Ta$, satisfying $\lim_{j \rightarrow \infty} s(a_j, b_j) = 0$.

In order to relax the requirement of satisfying the contractive condition at every pair of points in a space without altering the outcome, Samet et al. [42] coined the notion of α -admissibility. The idea of α -admissible mappings is interesting, as it includes the case of discontinuous mappings, unlike the contraction mapping. Nowadays, the literature dealing with fixed point problems via α -admissible mappings has been developed extensively in various directions. For the rest of the paper, the used mapping α (unless mentioned) is considered as $\alpha : Y \times Y \rightarrow [0, \infty)$, where Y is nonempty.

Definition 7 ([42]). A self-mapping $T : Y \rightarrow Y$ is called α -admissible if for $a, b \in Y$, the condition $\alpha(a, b) \geq 1$ implies that $\alpha(Ta, Tb) \geq 1$.

In order to extend the notion of α admissibility to set-valued mappings, Asl et al. [43] came up with the following definition in metric spaces.

Definition 8 ([43]). A set-valued mapping $T : Y \rightarrow \mathcal{N}(Y)$ is called α_* -admissible if, for all $a, b \in Y$, $\alpha(a, b) \geq 1$ implies $\alpha_*(Ta, Tb) \geq 1$, where $\alpha_*(Ta, Tb) = \inf\{\alpha(x, y) : x \in Ta, y \in Tb\}$.

Afterwards, a new definition of multivalued α -admissible mappings is proposed by Mohammadi et al. [44] as follows:

Definition 9 ([44]). A set-valued mapping $T : Y \rightarrow \mathcal{N}(Y)$ is called an α -admissible mapping if, for all $a \in Y$ and $b \in Ta$, $\alpha(a, b) \geq 1$ implies $\alpha(b, c) \geq 1$ for each $c \in Tb$.

Remark 2 ([44]). A mapping with α_* -admissibility also has α -admissibility. The converse may not hold.

Definition 10 ([44]). Let $T, S : Y \rightarrow \mathcal{N}(Y)$ be two mappings. The ordered pair (T, S) is said to be α -admissible if, for all $a, b \in Y$, $\alpha(a, b) \geq 1$ implies $\alpha(p, q) \geq 1$, for all $p \in Ta$ and $q \in Sb$.

The notion of α -completeness of a metric space defined by Hussain et al. [45] (see also [46]) weakens the metric completeness.

Definition 11 ([45]). A metric space (Y, s) is called α -complete if and only if every Cauchy sequence $\{a_j\}$ in Y with $\alpha(a_j, a_{j+1}) \geq 1$ for all j , converges in Y .

Remark 3 ([45]). If (Y, s) is complete, then it is also α -complete. The converse may not hold.

In 2015, Kutbi and Sintunavarat [47] weakened the notion of continuity by introducing α -continuity as follows:

Definition 12 ([47]). A set-valued mapping T is said to be α -continuous on $(\mathcal{CL}(Y), \mathcal{H})$, if for each sequence $\{a_j\}$ with $a_j \rightarrow a \in Y$ as $j \rightarrow \infty$ and $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N}$, we have $Ta_j \xrightarrow{\mathcal{H}} Ta$ as $j \rightarrow \infty$.

Let Φ denote the set of all monotone nondecreasing functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{j=1}^{\infty} \varphi^j(t) < \infty$ for each $t > 0$, where φ^j is the j th iterate of φ .

Lemma 2. Assume the following statements:

- (i) $\varphi \in \Phi$;
- (ii) $\lim_{j \rightarrow \infty} \varphi^j(t) = 0$;
- (iii) $\varphi(t) < t$ for all $t > 0$.

Then (i) implies (ii), which implies (iii).

Very recently in [48], fixed point results for single valued α - φ -contractive mappings in symmetric spaces are obtained.

Definition 13. A self-mapping T on a symmetric space (Y, s) is called α - φ -contractive if there exist φ in Φ and $\alpha : Y \times Y \rightarrow [0, \infty)$ such that

$$\alpha(a, b)s(Ta, Tb) \leq \varphi(s(a, b)) \text{ for all } a, b \in Y.$$

3. Main Results

First, we extend the idea of α -completeness to the symmetric space (Y, s) .

Definition 14. A symmetric space (Y, s) is said to be α -complete if, for every sequence $\{a_j\}$ in Y , satisfying $\sum_{j=1}^{\infty} s(a_j, a_{j+1}) < \infty$ with $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N}$, there exists $a \in Y$ such that $\lim_{j \rightarrow \infty} s(a_j, a) = 0$.

Remark 4. If (Y, s) is complete, then it is also α -complete. The converse need not be true (see Example 1).

Example 1. Let $Y = \{\frac{1}{j} : j \in \mathbb{N}\} \cup \{1 + \frac{1}{j} : j \in \mathbb{N}\}$. Define $s : Y \times Y \rightarrow [0, \infty)$ by $s(a, b) = |b - a|$ for all $a, b \in Y$. Then (Y, s) is a complete symmetric space. Consider $\alpha : Y \times Y \rightarrow [0, \infty)$ as

$$\alpha(a, b) = \begin{cases} 1 & \text{if } a, b \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here, (Y, s) is also α -complete. In fact, for every sequence $\{a_j\}$ in Y , satisfying $\sum_{j=1}^{\infty} s(a_j, a_{j+1}) < \infty$ with $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N}$, we have $a_j \in \{1 + \frac{1}{j} : j \in \mathbb{N}\}$. There exists $a = 1 \in Y$ such that $\lim_{j \rightarrow \infty} s(1 + \frac{1}{j}, 1) = 0$.

Definition 15. A set-valued mapping $T : Y \rightarrow \mathcal{CL}(Y)$ is called α - \mathcal{H}^+ -continuous on $\mathcal{CL}(Y)$ if $\lim_{j \rightarrow \infty} s(a_j, a) = 0$ and $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N}$ implies $\lim_{j \rightarrow \infty} \mathcal{H}^+(Ta_j, Ta) = 0$.

3.1. Fixed Point Theorems Using \mathcal{H}^+ Distance Functions

Definition 16. Let (Y, s) be a symmetric space. A set-valued mapping $T : Y \rightarrow \mathcal{N}(Y)$ is called α - φ - \mathcal{H}^+ -contractive

(1) if there exist two functions $\varphi \in \Phi$ and α such that

$$\alpha(a, b)\mathcal{H}^+(Ta, Tb) \leq \varphi(s(a, b)) \text{ for all } a, b \in Y, \tag{3}$$

(2) for every $a \in Y, b \in Ta, q \geq 1$, there exists $c \in Tb$ such that

$$s(b, c) \leq q\mathcal{H}^+(Ta, Tb).$$

Definition 17. In the above definition, if we put $m(a, b) = \max \{s(a, b), s(a, Ta), s(b, Tb)\}$ instead of $s(a, b)$ in Equation (3), then the mapping T is called generalized α - φ - \mathcal{H}^+ -contractive.

Remark 5. The condition (2) in the above definition holds automatically if we replace \mathcal{H}^+ by \mathcal{H} . In case of the multivalued contractions of Nadler, there is no need to assume it.

Theorem 1. Let (Y, s) be an α -complete symmetric space with (W4). Let $T : Y \rightarrow \mathcal{CB}_s(Y)$ be a generalized α - φ - \mathcal{H}^+ -contraction. Assume that

- (i) T is α -admissible;
- (ii) there exist a_0 in Y and a_1 in Ta_0 such that $\alpha(a_0, a_1) \geq 1$;
- (iii) T is α - \mathcal{H}^+ -continuous.

Then T admits a fixed point.

Proof. From (ii), we have $a_0 \in Y$ and $a_1 \in Ta_0$ such that $\alpha(a_0, a_1) \geq 1$. Assume $a_0 \neq a_1$. Otherwise, a_0 is a fixed point of T . Assume also $a_1 \notin Ta_1$. Otherwise, a_1 will be a fixed point of T .

Define a sequence $\{a_j\}$ in Y by $a_1 \in Ta_0, a_2 \in Ta_1, \dots, a_{j+1} \in Ta_j$, for all $j \in \mathbb{N}$ such that $a_j \notin Ta_j$. Further, using (i), we obtain $\alpha(a_j, a_{j+1}) \geq 1$. Because of condition (2) in Definition 16, we now have

$$\begin{aligned} s(a_j, a_{j+1}) &\leq \alpha(a_{j-1}, a_j) \mathcal{H}^+(Ta_{j-1}, Ta_j) \\ &\leq \varphi\left(\max\{s(a_{j-1}, a_j), s(a_{j-1}, Ta_{j-1}), s(a_j, Ta_j)\}\right) \\ &\leq \varphi\left(\max\{s(a_{j-1}, a_j), s(a_j, a_{j+1})\}\right). \end{aligned} \tag{4}$$

If $\max\{s(a_{j-1}, a_j), s(a_j, a_{j+1})\} = s(a_j, a_{j+1})$, then from Equation (4) we have $s(a_j, a_{j+1}) \leq \varphi(s(a_j, a_{j+1}))$ and then by Lemma 2, we have $s(a_j, a_{j+1}) < s(a_j, a_{j+1})$, a contradiction. Therefore,

$$s(a_j, a_{j+1}) \leq \varphi(s(a_{j-1}, a_j)).$$

Applying the above process, we have

$$s(a_j, a_{j+1}) \leq \varphi^j(s(a_0, a_1)).$$

Since $\sum_{j=1}^\infty \varphi^j(t) < \infty$ for all $t > 0$, so we have

$$\sum_{j=1}^\infty s(a_j, a_{j+1}) < \infty.$$

As Y is an α -complete symmetric space, there exists $a \in Y$ such that $\lim_{j \rightarrow \infty} s(a_j, a) = 0$. The α - \mathcal{H}^+ -continuity of T gives us

$$\lim_{j \rightarrow \infty} \mathcal{H}^+(Ta_j, Ta) = 0.$$

Since $a_{j+1} \in Ta_j$, by using condition (2) in Definition 16 for $q \geq 1$, we have

$$s(a_{j+1}, Ta) \leq q\mathcal{H}^+(Ta_j, Ta) \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Thus, $\lim_{j \rightarrow \infty} s(a_{j+1}, Ta) = 0$. This is equivalent to $\lim_{j \rightarrow \infty} s(a_j, Ta) = 0$. Therefore, by Lemma 1, there exists $b_j \in Ta$ such that $\lim_{j \rightarrow \infty} s(a_j, b_j) = 0$. Since $\lim_{j \rightarrow \infty} s(a_j, a) = 0$, (W4) implies $\lim_{j \rightarrow \infty} s(b_j, a) = 0$ which in turn implies $s(a, Ta) = 0$. Since Ta is closed, $a \in Ta$. \square

The following result can be proved in similar lines of proof of Theorem 1.

Theorem 2. Let (Y, s) be an α -complete symmetric space with (W4) and let $T : Y \rightarrow \mathcal{CB}_s(Y)$ be an α - φ - \mathcal{H}^+ -contractive mapping. Assume that (i)–(iii) of Theorem 1 are true. Then T admits a fixed point.

Corollary 1. Let (Y, s) be an α -complete symmetric space with (W4) and let $T : Y \rightarrow \mathcal{CB}_s(Y)$ be a generalized α - φ - \mathcal{H}^+ -contractive (or α - φ - \mathcal{H}^+ -contractive) mapping. Assume that

- (i) T is α_* -admissible;
- (ii) there exist a_0 in Y and a_1 in Ta_0 such that $\alpha(a_0, a_1) \geq 1$;
- (iii) T is $\alpha\mathcal{H}^+$ -continuous.

Then T admits a fixed point.

Corollary 2. Let (Y, s) be a complete symmetric space with (W4) and let $T : Y \rightarrow \mathcal{CB}_s(Y)$ be a generalized $\alpha\text{-}\varphi\text{-}\mathcal{H}^+$ -contractive (or $\alpha\text{-}\varphi\text{-}\mathcal{H}^+$ -contractive) mapping. We assume that (i) and (ii) in Corollary 1 hold. If T is a continuous multivalued mapping, then T has a fixed point.

We now prove our second main result.

Theorem 3. Let (Y, s) be an α -complete symmetric space with (CC) and let $T : Y \rightarrow \mathcal{CB}_s(Y)$ be a generalized $\alpha\text{-}\varphi\text{-}\mathcal{H}^+$ -contractive mapping. Assume that

- (i) T is α -admissible;
- (ii) there exist a_0 in Y and a_1 in Ta_0 such that $\alpha(a_0, a_1) \geq 1$;
- (iii) if $\{a_j\}$ is a sequence in Y with $\lim_{j \rightarrow \infty} s(a_j, a) = 0$ and $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N}$ then $\alpha(a_j, a) \geq 1$.

Then T admits a fixed point.

Proof. Following the proof of Theorem 1, we have that $\sum_{j=1}^{\infty} s(a_j, a_{j+1}) < \infty$ and $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N}$. Then by α -completeness of (Y, s) , there exists $a \in Y$ such that $\lim_{j \rightarrow \infty} s(a_j, a) = 0$. Using (iii), we have $\alpha(a_j, a) \geq 1$ for all $j \in \mathbb{N}$. We now claim that $a \in Ta$. Assume that $a \notin Ta$. Then $s(a, Ta) > 0$. By using (3), we have

$$\begin{aligned} s(a_{j+1}, Ta) &\leq \alpha(a_j, a)\mathcal{H}^+(Ta_j, Ta) \\ &\leq \varphi(\max\{s(a_j, a), s(a_j, Ta_j), s(a, Ta)\}). \end{aligned} \tag{5}$$

Let $\epsilon = \frac{s(a, Ta)}{2}$. Since $\lim_{j \rightarrow \infty} s(a_j, a) = 0$, we can find $n_1 \in \mathbb{N}$ such that $s(a_j, a) < \frac{s(a, Ta)}{2}$ for all $j > n_1$.

Moreover, as $\lim_{j \rightarrow \infty} s(a_j, a_{j+1}) = 0$, we can find $n_2 \in \mathbb{N}$ such that $s(a_j, Ta_j) \leq s(a_j, a_{j+1}) < \frac{s(a, Ta)}{2}$ for all $j > n_2$. Thus, we have

$$\max\{s(a_j, a), s(a_j, Ta_j), s(a, Ta)\} = s(a, Ta)$$

for all $j \geq n_0 = \max\{n_1, n_2\}$. Therefore, Equation (5) yields

$$s(a_{j+1}, Ta) \leq \varphi(s(a, Ta)) \tag{6}$$

for $j \geq n_0$. Taking limit as $j \rightarrow \infty$ in (6) and in view of condition (CC), we get $s(a, Ta) \leq \varphi(s(a, Ta))$, which is a contradiction to the consequence of Lemma 2. Thus, our assumption is wrong. Hence, $a \in Ta$. \square

Following the proof of the above theorem, the next result can be proved easily.

Theorem 4. Let (Y, s) be an α -complete symmetric space with (CC). Let $T : Y \rightarrow \mathcal{CB}_s(Y)$ be an $\alpha\text{-}\varphi\text{-}\mathcal{H}^+$ -contraction. Then if Conditions (i), (ii), and (iii)' of Theorem 3 hold, then T admits a fixed point.

Corollary 3. Let (Y, s) be an α -complete symmetric space with (CC). Let $T : Y \rightarrow \mathcal{CB}_s(Y)$ be a generalized $\alpha\text{-}\varphi\text{-}\mathcal{H}^+$ -contractive (or, $\alpha\text{-}\varphi\text{-}\mathcal{H}^+$ -contractive) mapping. Suppose that Conditions (ii) and (iii) in Theorem 3 hold. If in addition T is α_* -admissible, then T admits a fixed point.

Corollary 4. Let (Y, s) be a complete symmetric space with (CC). Let $T : Y \rightarrow \mathcal{CB}_s(Y)$ be a generalized α - φ - \mathcal{H}^+ -contraction (or α - φ - \mathcal{H}^+ -contraction). Suppose that Conditions (ii) and (iii) in Theorem 3 hold. If in addition T is α_* -admissible, then T admits a fixed point.

Remark 6. If we replace multivalued mappings by single valued mappings in all of the above results, then we get corresponding results for the single valued mappings as corollaries. The main results in [48] are corollaries to Theorems 1 and 3 for single valued mappings.

Example 2. Let $Y = (-5, 5)$ and $s : Y \times Y \rightarrow [0, \infty)$ be defined by $s(a, b) = (a - b)^2$. Then (Y, s) is a symmetric space, but not complete. Define $T : Y \rightarrow \mathcal{CB}_s(Y)$ by

$$T(a) = \begin{cases} [-4, |a + 1|] & \text{if } a \in (-5, 0), \\ [0, \frac{a}{2}] & \text{if } a \in [0, 2], \\ [\frac{a+5}{5}, 4] & \text{if } a \in (2, 5). \end{cases}$$

Let us define $\alpha : Y \times Y \rightarrow [0, \infty)$ by

$$\alpha(a, b) = \begin{cases} 2 & \text{if } a, b \in [0, 2] \\ 0 & \text{otherwise.} \end{cases}$$

Then one can easily observe that the symmetric space (Y, s) is α -complete and the mapping T is not continuous, but α - \mathcal{H}^+ -continuous. Moreover, Y satisfies (W4) (e.g., the sequences $\{a_n = \frac{1}{n}\}$, $\{b_n = \frac{1}{2n}\}$, and $a = 0$). Moreover, the mapping T is α -admissible and there exist $a_0 = 1$ and $a_1 = \frac{1}{2} \in Ta_0 = [0, \frac{1}{2}]$ such that $\alpha(a_0, a_1) \geq 2$. Now, for $a, b \in [0, 2]$,

$$\begin{aligned} \alpha(a, b)\mathcal{H}^+(Ta, Tb) &= 2\mathcal{H}^+(Ta, Tb) \\ &= \frac{1}{2}(a - b)^2 \\ &\leq \frac{1}{2}s(a, b) = \varphi(s(a, b)). \end{aligned}$$

Since $\alpha(a, b) = 0$ in other cases, the condition (3) holds vacuously. It is also easy to verify that for every $a \in Y$, $b \in Ta$, and $q \geq 1$, there exists $c \in Tb$ such that $s(b, c) \leq q\mathcal{H}^+(Ta, Tb)$.

Therefore, the mapping T is α - φ - \mathcal{H}^+ -contractive for $\varphi(t) = \frac{1}{2}t$. Thus, all conditions of Theorem 2 hold, so T admits a fixed point, $a = 0$.

3.2. Common Fixed Point Theorems Using \mathcal{H}^+ Distance

Definition 18. Let (Y, s) be a symmetric space. Assume $T, S : Y \rightarrow \mathcal{N}(Y)$. (T, S) is called an α - φ - \mathcal{H}^+ -contractive pair if

(1) there exist $\varphi \in \Phi$ and a symmetric function $\alpha : Y \times Y \rightarrow [0, \infty)$ such that

$$\alpha(a, b)\mathcal{H}^+(Ta, Sb) \leq \varphi(m_{T,S}(a, b)) \tag{7}$$

for all $a, b \in Y$, where $m_{T,S}(a, b) = \max \{s(a, b), s(a, Ta), s(b, Sb)\}$;

(2) for every $a \in Y$,

(a) $b \in Ta$, $q \geq 1$, and there exists $c \in Sb$ such that

$$s(b, c) \leq q\mathcal{H}^+(Ta, Sb);$$

(b) $b \in Sa, q \geq 1$, and there exists $c \in Tb$ such that

$$s(b, c) \leq q\mathcal{H}^+(Sa, Tb).$$

Theorem 5. Let (Y, s) be an α -complete symmetric space with (CC) and let $(T, S) [T, S : Y \rightarrow \mathcal{CB}_s(Y)]$ be an α - φ - \mathcal{H}^+ -contractive pair. Assume that

- (i) (T, S) is α -admissible;
- (ii) there exist a_0 in Y and $a_1 \in Ta_0$ such that $\alpha(a_0, a_1) \geq 1$;
- (iii) if $\{a_j\}$ is any sequence in Y with $\lim_{j \rightarrow \infty} s(a_j, a) = 0$ and $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N}$, then we have $\alpha(a_j, a) \geq 1$.

Then T and S admit a fixed point.

Proof. Let $a_0 \in Y$ be arbitrary and $a_1 \in Ta_0$. We assume $a_0 \neq a_1$. Otherwise, there is nothing to prove. It means $s(a_0, a_1) > 0$. From (ii), we have $\alpha(a_0, a_1) \geq 1$. Thus, by virtue of 2(a) of Definition 18, we choose $a_2 \in Sa_1$ such that

$$\begin{aligned} s(a_1, Sa_1) &\leq s(a_1, a_2) \\ &\leq \alpha(a_0, a_1)\mathcal{H}^+(Ta_0, Sa_1) \\ &\leq \varphi(m_{S,T}(a_0, a_1)) \\ &\leq \varphi(\max\{s(a_0, a_1), s(a_0, Ta_0), s(a_1, Sa_1)\}) \\ &= \varphi(\max\{s(a_0, a_1), s(a_1, Sa_1)\}). \end{aligned} \tag{8}$$

Clearly, from the above inequality, we can conclude that $\max\{s(a_0, a_1), s(a_1, Sa_1)\} = s(a_0, a_1)$. Otherwise, the second case would lead us to a contradiction. Thus, Equation (8) yields us

$$s(a_1, a_2) \leq \varphi(s(a_0, a_1)). \tag{9}$$

As $a_1 \in Ta_0$ and $a_2 \in Sa_1$ and due to α -admissibility of (T, S) , we have $\alpha(a_1, a_2) \geq 1$. Thus, by virtue of 2(b) of Definition 18, we choose $a_3 \in Ta_2$ such that

$$\begin{aligned} s(a_2, Ta_2) &\leq s(a_2, a_3) \\ &\leq \alpha(a_1, a_2)\mathcal{H}^+(Sa_1, Ta_2) \\ &\leq \varphi(m_{S,T}(a_1, a_2)) \\ &\leq \varphi(\max\{s(a_1, a_2), s(a_1, Sa_1), s(a_2, Ta_2)\}) \\ &= \varphi(\max\{s(a_1, a_2), s(a_2, Ta_2)\}). \end{aligned} \tag{10}$$

Again, we have $\max\{s(a_1, a_2), s(a_2, Ta_2)\} = s(a_1, a_2)$. Otherwise, the second case would lead to a contradiction. Thus, from (10), we have

$$s(a_2, a_3) \leq \varphi(s(a_1, a_2)) = \varphi^2(s(a_0, a_1)). \tag{11}$$

Persisting this way, a sequence $\{a_j\}$ in Y is generated such that $a_{2j+1} \in Ta_{2j}, a_{2j+2} \in Sa_{2j+1}$ satisfying $\alpha(a_j, a_{j+1}) \geq 1$, and

$$s(a_j, a_{j+1}) = \varphi^j(s(a_0, a_1)) \text{ for all } j \in \mathbb{N}. \tag{12}$$

Since $\sum_{j=1}^{\infty} \varphi^j(t) < \infty$, we have $\sum_{j=1}^{\infty} s(a_j, a_{j+1}) < \infty$. As the symmetric space (Y, s) is α -complete, there exists $a \in Y$ such that

$$\lim_{j \rightarrow \infty} s(a_j, a) = 0.$$

From (iii), we have $\alpha(a_{j+1}, a) \geq 1$ for all $j \in \mathbb{N}$. We now claim that $a \in Ta \cap Sa$. Firstly, let us assume $a \notin Ta$, then $s(a, Ta) > 0$. By 2(a), we have

$$s(a_{2j+2}, Ta) \leq \alpha(a_{2j+1}, a)\mathcal{H}^+(Sa_{2j+1}, Ta) \leq \varphi(\max\{s(a_{2j+1}, a), s(a, Ta), s(a_{2j+1}, Sa_{2j+1})\}). \tag{13}$$

Since $\lim_{j \rightarrow \infty} s(a_j, a) = 0$, we can find integer $N_1 \in \mathbb{N}$ such that $s(a_{2j+1}, a) < \epsilon = \frac{s(a, Ta)}{2}$ for all $j > N_1$. Furthermore, as $\{a_j\}$ is a sequence such that $\lim_{j \rightarrow \infty} s(a_j, a_{j+1}) = 0$, we can find integer $N_2 \in \mathbb{N}$ such that $d(a_{2j+1}, Sa_{2j+1}) \leq s(a_{2j+1}, a_{2j+2}) < \epsilon = \frac{s(a, Ta)}{2}$ for all $j > N_2$. Thus, we get

$$\max\{s(a, a_{2j+1}), s(a, Ta), s(a_{2j+1}, Sa_{2j+1})\} = s(a, Ta),$$

for all $j \geq N_0 = \max\{N_1, N_2\}$. Therefore, we have

$$s(Ta, a_{2j+2}) \leq \varphi(s(a, Ta)) \text{ for all } j \geq N_0.$$

Taking $j \rightarrow \infty$ and in view of (CC), we get $s(Ta, a) < s(a, Ta)$, which gives us $s(a, Ta) = 0$. As Ta is closed, we have $a \in Ta$. Arguing in a similar way, we can get $a \in Sa$ and hence $a \in Ta \cap Sa$. \square

Example 3. Let $Y = [0, 5]$ and $s : Y \times Y \rightarrow [0, \infty)$ defined by $s(a, b) = (a - b)^2$. Then (Y, s) is a symmetric space and not complete. Consider $T, S : Y \rightarrow \mathcal{CB}_s(Y)$ given by

$$T(a) = \begin{cases} [0, \frac{a^2}{8}] & \text{if } a \in [0, 2], \\ \{1\} & \text{if } a > 1, \end{cases} \quad \text{and} \quad S(a) = \begin{cases} [0, \frac{a}{5}] & \text{if } a \in [0, 2], \\ \{\frac{9}{2}\} & \text{if } a > 1. \end{cases}$$

We now define $\alpha : Y \times Y \rightarrow [0, \infty)$ by

$$\alpha(a, b) = \begin{cases} 1 & \text{if } a, b \in [0, 2], \\ 0 & \text{otherwise.} \end{cases}$$

- Here symmetric space (Y, s) is α -complete with (CC). In fact, for a given sequence $\{a_j = \frac{1}{j} + 1\}$, $a = 1$ there is $b = \frac{1}{2}$ such that $\lim_{j \rightarrow \infty} s(a_j, b) = s(a, b)$.
- The pair (T, S) is α -admissible because for $a, b \in Y$ such that $\alpha(a, b) \geq 1$, a, b should lie in $[0, 2]$. Then $Ta = [0, \frac{a^2}{8}]$, $Sb = [0, \frac{b}{5}]$ which are again subsets of $[0, 2]$. Thus, for any $u \in Ta$ and $v \in Sb$, $\alpha(u, v) \geq 1$.
- There exist $a_0 = 1$ and $a_1 = \frac{1}{8} \in Ta_0 = [0, \frac{1}{8}]$ such that $\alpha(a_0, a_1) \geq 1$.
- If every sequence $\{a_j\}$ in Y such that $\lim_{j \rightarrow \infty} s(a_j, a) = 0$ satisfies $\alpha(a_j, a_{j+1}) \geq 1$, then $a_j \in [0, 2]$, which in turn gives $a \in [0, 2]$. Thus, we have $\alpha(a_j, a) \geq 1$.
- Now for $a, b \in [0, 2]$, one can easily verify that $\alpha(a, b)\mathcal{H}^+(Ta, Tb) \leq \varphi(m_{T,S}(a, b))$. In other cases, since $\alpha(a, b) = 0$, the condition (7) always holds.
- It is also easy to verify that, for all $a \in Y$, $b \in Ta$, and $q \geq 1$, there exists $c \in Sb$ such that $s(b, c) \leq q\mathcal{H}^+(Ta, Sb)$. For every $a \in Y$, $b \in Sa$, $q > 1$, there exists $c \in Tb$ such that $s(b, c) \leq q\mathcal{H}^+(Sa, Tb)$.

This means the pair (T, S) is α - φ - \mathcal{H}^+ -contractive for $\varphi(t) = \frac{1}{2}t$. Thus, all the requirements of Theorem 5 are satisfied and hence $a = 0 \in Ta \cap Sa$.

3.3. Fixed Point Results Without Using \mathcal{H} or \mathcal{H}^+ Distance Functions

Definition 19. Let (Y, s) be a symmetric space. A set-valued mapping $T : Y \rightarrow \mathcal{CB}_s(Y)$ is called generalized pointwise α - φ -contractive if there exist functions $\varphi \in \Phi$ and $\alpha : Y \times Y \rightarrow [0, \infty)$ such that for $a_1, a_2 \in Y$, $b_1 \in Ta_1, b_2 \in Ta_2$,

$$\alpha(a_1, a_2)s(b_1, b_2) \leq \varphi(M_3(a_1, a_2)) \tag{14}$$

where $M_3(a_1, a_2) = \max \{s(a_1, a_2), s(a_1, b_1), s(a_2, b_2)\}$.

Definition 20. A mapping T is called pointwise α - φ -contractive if we replace $M_3(a_1, a_2)$ by $s(a_1, a_2)$ in Definition 19.

Theorem 6. Let (Y, s) be an α -complete symmetric space with (W4), and let mapping $T : Y \rightarrow \mathcal{CB}_s(Y)$ be generalized pointwise α - φ -contractive. Then T admits a fixed point if the following hold:

- (i) T is α -admissible;
- (ii) there exist a_0 in Y and $a_1 \in Ta_0$ such that $\alpha(a_0, a_1) \geq 1$;
- (iii) for every sequence $\{a_j\}$ in Y such that $\lim_{n \rightarrow \infty} s(a_j, a) = 0$ with $\alpha(a_j, a_{j+1}) \geq 1$, there exist a sequence $\{b_j\}$ in Ta_j such that $\lim_{j \rightarrow \infty} s(b_j, b) = 0$ for some $b \in Ta$.

Proof. Initiating with arbitrary $a_0 \in Y$ and $a_1 \in Ta_0$ such that $\alpha(a_0, a_1) \geq 1$, then following the proof of Theorem 1 we get a sequence $\{a_j\}$ defined by $a_1 \in Ta_0, a_2 \in Ta_1, \dots, a_{j+1} \in Ta_j$ for all $j \in \mathbb{N}$ such that $a_j \notin Ta_j$. Since T is α -admissible, we have $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N} \cup \{0\}$. By (14), we have

$$\begin{aligned} s(a_j, a_{j+1}) &\leq \alpha(a_{j-1}, a_j)s(a_j, a_{j+1}) \\ &\leq \varphi(M_3(a_{j-1}, a_j)) \\ &\leq \varphi\left(\max \{s(a_{j-1}, a_j), s(a_{j-1}, a_j), s(a_j, a_{j+1})\}\right) \\ &\leq \varphi\left(\max \{s(a_{j-1}, a_j), s(a_j, a_{j+1})\}\right). \end{aligned} \tag{15}$$

If $\max \{s(a_{j-1}, a_j), s(a_j, a_{j+1})\} = s(a_j, a_{j+1})$, then from Equation (15) we have $s(a_j, a_{j+1}) \leq \varphi(s(a_j, a_{j+1}))$. Using Lemma 2, we get $s(a_j, a_{j+1}) < s(a_j, a_{j+1})$, that is a contradiction. Thus, Equation (15) gives

$$s(a_j, a_{j+1}) \leq \varphi(s(a_{j-1}, a_j)).$$

Repeating this process, we get

$$s(a_j, a_{j+1}) \leq \varphi^j(s(a_0, a_1)).$$

As $\sum_{j=1}^{\infty} \varphi^j(t) < \infty$ for all $t > 0$, so we obtain

$$\sum_{j=1}^{\infty} s(a_j, a_{j+1}) < \infty.$$

Due to α -completeness of the symmetric space Y , there exists $a \in Y$ such that $\lim_{j \rightarrow \infty} s(a_j, a) = 0$ and by (iii), we obtain a sequence $\{b_j\} \in Ta_j$ such that $\lim_{j \rightarrow \infty} s(b_j, b) = 0$ for some $b \in Ta$.

Since $a_{j+1} \in Ta_j$, we obtain

$$\begin{aligned} s(a_{j+1}, Ta) &= \inf\{s(a_j, Ta) : b_j \in Ta_j\} \\ &\leq s(b_j, b) \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Thus, we find that $\lim_{j \rightarrow \infty} s(a_{j+1}, Ta) = 0$. This is equivalent to $\lim_{j \rightarrow \infty} s(a_j, Ta) = 0$. Therefore, by Lemma 1, there exists $c_j \in Ta$ such that $\lim_{j \rightarrow \infty} s(a_j, c_j) = 0$. Since $\lim_{j \rightarrow \infty} s(a_j, a) = 0$, (W4) implies $\lim_{j \rightarrow \infty} s(c_n, a) = 0$, which in turn implies $s(a, Ta) = 0$ and, since Ta is closed, $a \in Ta$. \square

The following results follow in a similar way as the above proof.

Theorem 7. Let (Y, s) be an α -complete symmetric space with (W4), and let mapping $T : Y \rightarrow \mathcal{CB}_s(Y)$ be point-wise α - φ -contractive. If conditions (i)-(iii) in Theorem 6 hold, then T admits a fixed point.

Example 4. Let $Y = (-1, 1]$ and $s : Y \times Y \rightarrow [0, \infty)$ defined by $s(a, b) = e^{|a-b|} - 1$. Then (Y, s) is a symmetric space but not complete. Consider $T : Y \rightarrow \mathcal{CB}_s(Y)$ defined as

$$T(a) = \begin{cases} \{\frac{1}{4}\} & \text{if } a \in (-1, 0), \\ [0, \frac{a}{3}] & \text{if } a \in [0, \frac{1}{2}], \\ \{\frac{1}{5}\} & \text{if } a \in (\frac{1}{2}, 1] \end{cases}$$

and define $\alpha : Y \times Y \rightarrow [0, \infty)$ by

$$\alpha(a, b) = \begin{cases} e^{ab} & \text{if } a, b \in [0, \frac{1}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the symmetric space (Y, s) is α -complete with (W4). If every sequence $\{a_j\}$ in Y such that $\lim_{j \rightarrow \infty} s(a_j, a) = 0$ satisfies $\alpha(a_j, a_{j+1}) \geq 1$, then $a_j \in [0, \frac{1}{2}]$, which in turn gives $a \in [0, \frac{1}{2}]$. So $Ta_j, Ta \subseteq [0, \frac{1}{6}]$. Therefore, there exist a sequence $\{b_j\}$ in Ta_j and $b \in Ta$ such that $\lim_{j \rightarrow \infty} s(b_j, b) = 0$. Thus, condition (iii) is satisfied.

Further, α -admissibility of T can be verified easily. In addition, for $a_0 = \frac{1}{3}$ and $a_1 = \frac{1}{15} \in [0, \frac{1}{15}] = Ta_0$, we have $\alpha(a_0, a_1) = e^{\frac{1}{45}} = 1.022... \geq 1$, and mapping T is generalized point-wise α - φ -contractive with $\varphi(t) = \frac{5}{6}t$. Thus, all the requirements of Theorem 6 are fulfilled and $a = 0 \in Ta$.

Remark 7. Theorems 6 and 7 also hold if condition (iii) is replaced by the α -continuity of T .

4. An Application to Probabilistic Spaces

Definition 21. Let \mathcal{L} be a collection of nondecreasing and left-continuous functions $g : (-\infty, +\infty) \rightarrow [0, 1]$ such that $\sup g(t) = 1$ and $\inf g(t) = 0$. Such mappings are generally called distribution functions.

Definition 22 ([28]). Let Y be a set. Assume a mapping $\mathcal{G} : Y \times Y \rightarrow \mathcal{L}$ such that $\mathcal{G}(a, b) = G_{a,b}$. Consider the following conditions:

- (1) for all $a, b \in Y$, $G_{a,b}(0) = 0$, where $G_{a,b}$ is value of \mathcal{G} at $(a, b) \in Y \times Y$;
- (2) $G_{a,b} = K$ if and only if $a = b$, where K is the distribution function given as $K(a) = 1$ if $a > 0$ and $K(a) = 0$ if $a \leq 0$;
- (3) $G_{a,b} = G_{b,a}$;
- (4) If $G_{a,b}(\epsilon) = 1$ and $G_{b,c}(\delta) = 1$, then $G_{a,c}(\epsilon + \delta) = 1$.

If \mathcal{G} satisfies (1) and (2), then it is said to be a PPM-structure on Y , and the pair (Y, \mathcal{G}) is said to be a PPM-space. The mapping \mathcal{G} with (3) is called symmetric. A PPM-space (Y, \mathcal{G}) is called a probabilistic metric space (in short, PM-space) if \mathcal{G} satisfies (4).

The set $\mathcal{B}_a(\epsilon, \lambda) = \{b \in Y : G_{a,b}(\epsilon) > 1 - \lambda\}$ for all $\epsilon, \lambda > 0$, is called (ϵ, λ) -neighborhood of $a \in Y$. $t_{\mathcal{G}}$ is the topology on (Y, \mathcal{G}) generated by the collection

$$\mathbb{U} = \{\mathcal{B}_a(\epsilon, \lambda) : a \in Y, \epsilon, \lambda > 0\}.$$

A T_1 topology $t_{\mathcal{G}}$ on Y can be defined as $U \in t_{\mathcal{G}}$ if for any $a \in U$, there exists $\epsilon > 0$ such that $\mathcal{B}_a(\epsilon, \epsilon) \subset U$. If $\mathcal{B}_a(\epsilon, \epsilon) \in \tau(\mathcal{G})$, then $t_{\mathcal{G}}$ is said to be topological.

Definition 23 ([28]). A sequence $\{a_j\}$ in a symmetric PPM-space (Y, \mathcal{G}) is called fundamental if $\lim_{i,j \rightarrow \infty} G_{a_i, a_j}(t) = 1$ for all $t > 0$.

Definition 24 ([28]). A symmetric PPM-space (Y, \mathcal{G}) is called complete if for every fundamental sequence $\{a_j\}$ there exists $a \in Y$ such that $\lim_{j \rightarrow \infty} G_{a_j, a}(t) = 1$ for all $t > 0$.

Definition 25. A symmetric PPM-space (Y, \mathcal{G}) is called α -complete if for every fundamental sequence $\{a_j\}$ in Y with $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N}$, there exists $a \in Y$ such that $\lim_{j \rightarrow \infty} G_{a_j, a}(t) = 1$ for all $t > 0$, where $\alpha : Y \times Y \rightarrow [0, \infty)$.

Remark 8. (W4) is equivalent to the

$$(P4) \quad \lim_{j \rightarrow \infty} G_{a_j, a}(t) = 1 \text{ and } \lim_{j \rightarrow \infty} G_{a_j, b_j}(t) = 1 \text{ imply } \lim_{j \rightarrow \infty} G_{b_j, a}(t) = 1 \text{ for all } t > 0.$$

Each symmetric PPM-space has a compatible symmetric mapping [28] as follows:

Theorem 8 ([28]). Let (Y, \mathcal{G}) be a symmetric PPM-space. Let $h : Y \times Y \rightarrow [0, \infty)$ be a function defined as

$$h(a, b) = \begin{cases} 0 & \text{if } b \in \mathcal{B}_a(\delta, \delta) \text{ for all } \delta > 0 \\ \sup\{\delta : y \notin \mathcal{B}_a(\delta, \delta), 0 < \delta < 1\} & \text{otherwise.} \end{cases}$$

Then

- (1) $h(a, b) < t$ if and only if $G_{a,b}(t) > 1 - t$.
- (2) h is a compatible symmetric for $t_{\mathcal{G}}$.
- (3) (Y, \mathcal{G}) is complete if and only if (Y, s) is S-complete.

We now present the following proposition which is required for establishing our results.

Proposition 3. Let (Y, \mathcal{G}) be a symmetric PPM-space and h be a compatible symmetric function for $t_{\mathcal{G}}$. For $\mathcal{P}, \mathcal{Q} \in \mathcal{CL}(Y)$, set

$$\mathcal{E}_{\mathcal{P}, \mathcal{Q}}(\epsilon) = \frac{1}{2} \left\{ \inf_{a \in \mathcal{P}} \sup_{b \in \mathcal{Q}} G_{a,b}(\epsilon) + \inf_{a \in \mathcal{Q}} \sup_{b \in \mathcal{P}} G_{a,b}(\epsilon) \right\}$$

for $\epsilon > 0$ and

$$\mathcal{H}^+(\mathcal{P}, \mathcal{Q}) = \frac{1}{2} \left\{ \sup_{a \in \mathcal{P}} \inf_{b \in \mathcal{Q}} h(a, b) + \sup_{b \in \mathcal{Q}} \inf_{a \in \mathcal{P}} h(a, b) \right\}.$$

If $T : Y \rightarrow \mathcal{C}(Y)$ is a set-valued mapping, then $G_{a,b}(t) > 1 - t$ implies $\mathcal{E}_{Ta, Tb}(\alpha_0 \varphi(t)) > 1 - \alpha_0 \varphi(t)$ for every $t > 0$ and $a, b \in Y$, where $\alpha_0 > 0$ such that $\sup_{a, b \in Y} \alpha(a, b) \leq \frac{1}{\alpha_0}$ and the mapping φ satisfying $\lim_{\epsilon \rightarrow 0} \varphi(t + \epsilon) = \varphi(t)$ implies that $\alpha(a, b) \mathcal{H}^+(Ta, Tb) \leq \varphi(h(a, b))$.

Proof. Let $t > 0$ be given and set $v = h(a, b) + t$. Then $h(a, b) = v - t < v$ gives us $G_{a,b}(v) > 1 - v$ and hence $\mathcal{E}_{Ta, Tb}(\alpha_0\varphi(v)) > 1 - \alpha_0\varphi(v)$. This gives us

$$\frac{1}{2} \left\{ \inf_{p \in Ta} \sup_{q \in Tb} G_{p,q}(\alpha_0\varphi(v)) + \inf_{q \in Tb} \sup_{p \in Ta} G_{p,q}(\alpha_0\varphi(v)) \right\} > 1 - \alpha_0\varphi(v).$$

This implies for every $p \in Ta$ there exists $q \in Tb$ such that

$$\frac{1}{2} \{ G_{p,q}(\alpha_0\varphi(v)) + G_{p,q}(\alpha_0\varphi(v)) \} > 1 - \alpha_0\varphi(v),$$

or for every $q \in Tb$ there exists $p \in Ta$ such that

$$\frac{1}{2} \{ G_{p,q}(\alpha_0\varphi(v)) + G_{p,q}(\alpha_0\varphi(v)) \} > 1 - \alpha_0\varphi(v).$$

Therefore, for every $p \in Ta$ there exists $q \in Tb$ (or for every $q \in Tb$ there exists $p \in Ta$) such that

$$h(p, q) < \alpha_0\varphi(v).$$

Then

$$\frac{1}{2} \left\{ \sup_{p \in Ta} \inf_{q \in Tb} h(p, q) + \sup_{q \in Tb} \inf_{p \in Ta} h(p, q) \right\} < \alpha_0\varphi(v).$$

Therefore,

$$\mathcal{H}^+(Ta, Tb) < \alpha_0\varphi(v) = \alpha_0\varphi(h(a, b) + t).$$

This gives

$$\alpha(a, b)\mathcal{H}^+(Ta, Tb) \leq \frac{1}{\alpha_0}\mathcal{H}^+(Ta, Tb) < \varphi(h(a, b) + t).$$

Taking $t \rightarrow 0$, we have $\alpha(a, b)\mathcal{H}^+(Ta, Tb) \leq \varphi(h(a, b))$. \square

Theorem 9. Let (Y, \mathcal{G}) be an α -complete symmetric PPM-space with (P4) and admit a compatible symmetric function h for $t_{\mathcal{G}}$. Consider the set-valued mapping $T : Y \rightarrow C(Y)$ to be α -admissible with $\lim_{\epsilon \rightarrow 0} \varphi(t + \epsilon) = \varphi(t)$.

Assume that

- (i) there exist $a_0 \in Y$ and $a_1 \in Ta_0$ such that $\alpha(a_0, a_1) \geq 1$;
- (ii) there exists $\alpha_0 > 0$ such that $\sup_{a,b \in Y} \alpha(a, b) \leq \frac{1}{\alpha_0}$ and $G_{a,b}(t) > 1 - t$ implies $\mathcal{E}_{Ta, Tb}(\alpha_0\varphi(t)) > 1 - \alpha_0\varphi(t)$ for all $t > 0$;
- (iii) if there is a sequence $\{a_j\}$ such that $\alpha(a_j, a_{j+1}) \geq 1$ for all $j \in \mathbb{N}$ and $\lim_{j \rightarrow \infty} G_{a_j, a}(\epsilon) = 1$ for all $\epsilon > 0$, then we have $\alpha(a_j, a) \geq 1$.

Then T admits a fixed point.

Proof. It is clear that (Y, h) is complete and bounded. Now, let $\epsilon > 0$ and $t = h(a, b) + \epsilon$. Then $h(a, b) < t$ gives $G_{a,b}(t) < 1 - t$, which implies that $\mathcal{E}_{Ta, Tb}(\alpha_0\varphi(t)) > 1 - \alpha_0\varphi(t)$; therefore, by Proposition 3, we have $\alpha(a, b)\mathcal{H}^+(Ta, Tb) \leq \varphi(h(a, b))$. Applying Theorem 2, we can guarantee the existence of a fixed point. \square

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