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Article

Sehgal Type Contractions on Dislocated Spaces

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Abstract: In this paper, we investigate the contractive type inequalities for the iteration of the mapping at a given point in the setting of dislocated metric space. We consider an example to illustrate the validity of the given result. Further, as an application, we propose a solution for a boundary value problem of the second order differential equation.

Keywords: quasi-metric spaces; fixed point; self-mappings

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1. Introduction and Preliminaries

In 1968, Bryant relaxed the assumption of Banach contraction mapping principle by using an iteration of the mapping.

Theorem 1 ([1])**.** *Let T be a self mapping on the complete metric space $(\mathcal{M}, d)$, and m a positive integer. Suppose that there exists $q \in [0,1)$ such that T satisfies the inequality*

$$d(T^m v, T^m w) \leq q d(v,w), \tag{1}$$

for all $v, w \in \mathcal{M}$, where T^m denotes the mth iterate of T. Then, there exists exactly one fixed point of T.

After then a number of authors deepen the research by considering an iteration of the mapping, see e.g., [2–7]. We recollect some significant results in this direction. One of the pioneer report in this way was given Seghal [4].

Theorem 2 ([4])**.** *Let $(\mathcal{M}, d)$ be a complete metric space, T a continuous self-mapping of $\mathcal{M}$ which satisfies the condition that there exists a real number q, $0 < q < 1$ such that, for each $v \in \mathcal{M}$ there exists a positive integer $m(v)$ such that, for each $w \in \mathcal{M}$,*

$$d(T^{m(v)} v, T^{m(v)} w) \leq q d(v,w). \tag{2}$$

Then T has a unique fixed point in $\mathcal{M}$.

Guseman [2] extended this result by removing the condition of continuity of T and later, other extensions for a single mapping were discussed in several papers, see e.g., Iseki [8], Matkowski [3], Singh [5] and the reference therein. One of the most interesting results for mappings which satisfy a general contractive conditions were announced by Singh.

Theorem 3 ([5]). *Let $(\mathcal{M}, d)$ be a complete metric space and $T : \mathcal{M} \to \mathcal{M}$ be a mapping such that for all $v, w \in \mathcal{M}$ we can find a positive integer $m(v)$ such that*

$$\begin{aligned} &d(T^{m(v)}v, T^{m(v)}w) \leq q(v,w)d(v,w) + r(v,w)d(v, T^{m(v)}v) + s(v,w)(w, T^{m(v)}w) \\ &+t(v,w)d(w, T^{m(v)}v) + p(v,w)d(v, T^{m(v)}w), \end{aligned} \tag{3}$$

where $q(v,w), r(v,w), s(v,w), t(v,w), p(v,w)$ are nonnegative functions such that

$$\sup\{2t(v,w) + q(v,w) + r(v,w) + s(v,w) + p(v,w)\} = \lambda < 1.$$

Then T has a unique fixed point v^*.

In this paper, we consider more general contractive condition in the setting of dislocated metric space. For sake of completeness, we shall recollect some basic notions and fundamental results.

Definition 1. *For a nonempty set $\mathcal{M}$ a dislocated metric is a function $\mathcal{D} : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ such that for all $v, w, u \in \mathcal{M}$:*

($\mathcal{D}1$) $\mathcal{D}(v,w) = 0 \Rightarrow v = w$,
($\mathcal{D}2$) $\mathcal{D}(v,w) = \mathcal{D}(w,v)$,
($\mathcal{D}3$) $\mathcal{D}(v,w) \leq \mathcal{D}(v,u) + \mathcal{D}(u,w)$.

The space $(\mathcal{M}, \mathcal{D})$ is said to be a dislocated metric space (DMS).

Example 1. *Let $\mathcal{M} = \mathbb{R}_0^+$ and $\mathcal{D} : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ defined by $\mathcal{D}(v,w) = \max\{v, w\}$. The pair $(\mathcal{M}, \mathcal{D})$ forms a dislocated metric space.*

It is obvious that any metric space is a dislocated metric space, but conversely this is not true.

Definition 2. *Let $(\mathcal{M}, \mathcal{D})$ be a DMS. A sequence $\{v_n\}$ in $\mathcal{M}$ is called :*

(a) *convergent to a point $v \in \mathcal{M}$ if the following limit exists and is finite*

$$\lim_{n\to\infty} \mathcal{D}(v_n, v) = \mathcal{D}(v, v); \tag{4}$$

(b) *Cauchy if the following limit*

$$\lim_{n\to\infty} \mathcal{D}(v_n, v_m)$$

exists and is finite.

Moreover, if $\lim_{n\to\infty} \mathcal{D}(v_n, v_m) = 0$, then is said that $\{v_n\}$ is a 0-Cauchy sequence.

Definition 3. *The DMS $(\mathcal{M}, \mathcal{D})$ is complete if for each Cauchy sequence $\{v_n\}$ in $\mathcal{M}$, there is some $v \in \mathcal{M}$ such that*

$$l = \lim_{n\to\infty} \mathcal{D}(v_n, v) = \mathcal{D}(v, v) = \lim_{n,m\to\infty} \mathcal{D}(v_n, v_m). \tag{5}$$

Particularly, if each 0-Cauchy sequence $\{v_n\}$ converges to a point $v \in \mathcal{M}$ the pair $(\mathcal{M}, \mathcal{D})$ is said to be 0- complete-DMS.

Definition 4. *Let $(\mathcal{M}, \mathcal{D})$ be a DMS. A mapping $T : \mathcal{M} \to \mathcal{M}$ is continuous if for any sequence $\{v_n\}$ in $\mathcal{M}$ converging to $v \in \mathcal{M}$, we have $\{Tv_n\}$ converges to Tv.*

Proposition 1 ([9]). *Let $(\mathcal{M}, \mathcal{D})$ be a DMS. For any $v, w \in \mathcal{M}$ we have the following*

(i) If $\mathcal{D}(v, w) = 0$ then $\mathcal{D}(v, v) = \mathcal{D}(w, w) = 0$.
(ii) If $v \neq w$ then $\mathcal{D}(v, w) > 0$.
(iii) If $\{v_n\}$ is a sequence in $\mathcal{M}$ such that $\lim_{n\to\infty} \mathcal{D}(v_n, v_{n+1}) = 0$ then

$$\lim_{n\to\infty} \mathcal{D}(v_n, v_n) = \lim_{n\to\infty} \mathcal{D}(v_{n+1}, v_{n+1}) = 0.$$

Definition 5. *By a comparison function we mean a function $\varphi : [0, \infty) \to [0, \infty)$ with the following properties:*

(cf_1) φ *is increasing;*
(cf_2) $\lim_{n\to\infty} \varphi^n(x) = 0$, *for* $x \in [0, \infty)$.

We denote by Φ the class of the comparison function $\varphi : [0, \infty) \to [0, \infty)$.

Next we list some basic properties of the comparison functions.

Proposition 2 ([10,11]). *If φ is a comparison function then:*

(cf_i) *each φ^k is a comparison function, for all $k \in \mathbb{N}$;*
(cf_{ii}) *φ is continuous at 0;*
(cf_{iii}) *$\varphi(x) < x$ for all $x > 0$.*

Definition 6 ([10]). *A function $\varphi_c : [0, \infty) \to [0, \infty)$ is called a $c-$comparison function if:*

(ccf_1) *φ_c is monotone increasing;*
(ccf_2) $\sum_{n=0}^{\infty} \varphi_c^n(x) < \infty$, *for all $x \in (0, \infty)$.*

We denote by Φ_c the family of $c-$comparison functions.

It can be shown that every $c-$comparison function is a comparison function.

Throughout this paper we denote by Ψ the collection of all $c-$comparison functions $\psi : [0, \infty) \to [0, \infty)$ that satisfy the following condition

(ccf_3) $\lim_{x\to\infty} (x - \psi(x)) = \infty$.

In the following we recall the concept of α-admissible mappings. A function $T : \mathcal{M} \to \mathcal{M}$ is said to be α-admissible if

(A) $\alpha(v, w) \geq 1 \Rightarrow \alpha(Tv, Tw) \geq 1$,

for all $v, w \in \mathcal{M}$ where $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is a given function. An α-admissible map $T : \mathcal{M} \to \mathcal{M}$ which satisfies the condition

(TA) $\alpha(v, w) \geq 1$ and $\alpha(w, u) \geq 1$ implies that $\alpha(v, u) \geq 1$, $v, w, u \in \mathcal{M}$

is said to be triangular α-admissible.

Later, the notion of α-admissible mapping and triangular α-admissible mappings are refined by Popescu [12], as follows:

Definition 7 ([12]). *Let $T : \mathcal{M} \to \mathcal{M}$ and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$. We say that T is an $\alpha-$orbital admissible mapping if for all $v \in \mathcal{M}$ we have*

(O) $\alpha(v, Tv) \geq 1 \Rightarrow \alpha(Tv, T^2v) \geq 1$.

Every $\alpha-$admissible mapping is an α-orbital admissible mapping, for more details on admissible mapping, see e.g., [13–24].

Definition 8 ([12]). *Let $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$. An α-orbital admissible function $T : \mathcal{M} \to \mathcal{M}$ is said to be triangular α-orbital admissible if it satisfies*

(TO) $\alpha(v, w) \geq 1$ and $\alpha(w, Tw) \geq 1$ implies that $\alpha(v, Tw) \geq 1$, for all $v, w \in \mathcal{M}$.

At the end of this section, we present two further concepts that will be essential in our next considerations.

A set $\mathcal{M}$ is *regular* with respect to mapping $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ if the following condition is satisfied:

(R) for any sequence $\{v_n\}$ in $\mathcal{M}$ such that $\alpha(v_n, v_{n+1}) \geq 1$ for all n and $v_n \to v \in \mathcal{M}$ as $n \to \infty$ we have $\alpha(v, v_n) \geq 1$, for all n.

A map $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is said to *satisfy the condition* (U) if

(U) for any fixed point v of $T^{m(v)}$ we have $\alpha(v, w) \geq 1$ for any $w \in \mathcal{M}$, where $m(v)$ is a positive integer.

2. Main Results

We are now prepared to establish the main result of this paper.

Theorem 4. *Let $(\mathcal{M}, \mathcal{D})$ be a complete DMS, a function $T : \mathcal{M} \to \mathcal{M}$, $\psi \in \Psi$ and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$. Suppose that for all $v \in \mathcal{M}$ we can find a positive integer $m(v)$ such that for any $w \in \mathcal{M}$*

$$\alpha(v,w)\mathcal{D}(T^{m(v)}v, T^{m(v)}w) \leq \psi\left(\max\left\{\mathcal{D}(v,w), \frac{\mathcal{D}(v,T^{m(v)}v)}{2}, \frac{\mathcal{D}(w,T^{m(v)}w)}{2}, \frac{\mathcal{D}(w,T^{m(v)}v)+\mathcal{D}(v,T^{m(v)}w)}{3}\right\}\right). \tag{6}$$

Suppose also that:

(i) T is triangular α-orbital admissible;
(ii) there exists v_0 in $\mathcal{M}$ such that $\alpha(v_0, Tv_0) \geq 1$;
(iii) either T is continuous, or
(iv) the $\mathcal{M}$ space is regular and α satisfies the condition (U).

Then the function T has exactly one fixed point.

Proof. Consider the initial value $v_0 \in \mathcal{M}$ and define a sequence $\{v_n\}$ as follows:

$$v_1 = T^{m(v_0)}v_0,\ v_2 = T^{m(v_1)}v_1, \ldots\ v_{k+1} = T^{m(v_k)}v_k, \ldots \tag{7}$$

If we denote $m_k = m(v_k)$ for any $k \in \mathbb{N}$, then we can write $v_{k+1} = T^{m_k}v_k$.

Now, T is α-orbital admissible and $\alpha(v_0, Tv_0) \geq 1$. Thus, from condition (O), we have $\alpha(Tv_0, T^2v_0) \geq 1$ and so forth

$$\alpha(T^{n-1}v_0, T^n v_0) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{8}$$

Taking into account (TO) and (8) we easily infer that

$$\alpha(v_k, Tv_k) \geq 1 \text{ and } \alpha(Tv_k, T^2v_k) \geq 1 \text{ imply that } \alpha(v_k, T^2v_k) \geq 1.$$

Recursively, we can conclude that

$$\alpha(v_k, T^m v_k) \geq 1, \tag{9}$$

for all $m \in \{1, 2, \ldots\}$.

In the initial inequality (6) letting $v = v_{k-1}$, $w = T^{m_k}v_{k-1}$ and using (9) we can find a positive integer, m_{k-1} such that

$$\begin{aligned}
\mathcal{D}(v_k, v_{k+1}) &= \mathcal{D}(v_k, T^{m_k}v_k)) = \mathcal{D}(T^{m_{k-1}}v_{k-1}, T^{m_k}(T^{m_{k-1}}v_{k-1})) \\
&= \mathcal{D}(T^{m_{k-1}}v_{k-1}, T^{m_{k-1}}(T^{m_k}v_{k-1})) \\
&\leq \alpha(v_{k-1}, T^{m_k}v_{k-1})\mathcal{D}(T^{m_{k-1}}v_{k-1}, T^{m_{k-1}}(T^{m_k}v_{k-1})) \\
&\leq \psi\left(\max\left\{\begin{array}{l} \mathcal{D}(v_{k-1}, T^{m_k}v_{k-1}), \frac{\mathcal{D}(v_{k-1},T^{m_{k-1}}v_{k-1})}{2}, \frac{\mathcal{D}(T^{m_k}v_{k-1},T^{m_{k-1}}(T^{m_k}v_{k-1}))}{2}, \\ \frac{\mathcal{D}(T^{m_k}v_{k-1},T^{m_{k-1}}v_{k-1})+\mathcal{D}(v_{k-1},T^{m_{k-1}}(T^{m_k}v_{k-1}))}{3} \end{array}\right\}\right).
\end{aligned} \tag{10}$$

Since $\psi \in \Psi$, the condition (cf_1) is satisfied and applying $(\mathcal{D}3)$ we obtain

$$\begin{aligned}
\mathcal{D}(v_k, v_{k+1}) &= \mathcal{D}(v_k, T^{m_k}v_k)) \\
&< \max\left\{\begin{array}{l} \mathcal{D}(v_{k-1}, T^{m_k}v_{k-1}), \frac{\mathcal{D}(v_{k-1},T^{m_{k-1}}v_{k-1})}{2}, \frac{\mathcal{D}(T^{m_k}v_{k-1},v_{k-1})+\mathcal{D}(v_{k-1},T^{m_k+m_{k-1}}v_{k-1})}{2}, \\ \frac{\mathcal{D}(T^{m_k}v_{k-1},v_{k-1})+\mathcal{D}(v_{k-1},T^{m_{k-1}}v_{k-1})+\mathcal{D}(v_{k-1},T^{m_k+m_{k-1}}v_{k-1})}{3} \end{array}\right\}.
\end{aligned} \tag{11}$$

Let $p_1 \in \{m_{k-1}, m_k, m_k + m_{k-1}\}$ such that

$$\max\left\{\mathcal{D}(v_{k-1}, T^{m_{k-1}}v_{k-1}), \mathcal{D}(v_{k-1}, T^{m_k}v_{k-1}), \mathcal{D}(v_{k-1}, T^{m_{k-1}+m_k}v_{k-1})\right\} = \mathcal{D}(v_{k-1}, T^{p_1}v_{k-1}).$$

Then from (11) together with (cf_{iii}) we get that

$$\mathcal{D}(v_k, T^{m_k}v_k)) \leq \psi(\mathcal{D}(v_{k-1}, T^{p_1}v_{k-1})) < \mathcal{D}(v_{k-1}, T^{p_1}v_{k-1}). \tag{12}$$

Using the same arguments, we can find a positive integer m_{k-2} such that

$$\begin{aligned}
\mathcal{D}(v_{k-1}, T^{p_1}v_{k-1})) &= \mathcal{D}(T^{m_{k-2}}v_{k-2}, T^{p_1}(T^{m_{k-2}}v_{k-2})) \\
&\leq \alpha(v_{k-2}, T^{p_1}v_{k-2})\mathcal{D}(T^{m_{k-2}}v_{k-2}, T^{m_{k-2}}(T^{p_1}v_{k-2})) \\
&\leq \psi\left(\max\left\{\begin{array}{l} \mathcal{D}(v_{k-2}, T^{p_1}v_{k-2}), \frac{\mathcal{D}(v_{k-2},T^{m_{k-2}}v_{k-2})}{2}, \frac{\mathcal{D}(T^{p_1}v_{k-2},T^{m_{k-2}}(T^{p_1}v_{k-2}))}{2}, \\ \frac{\mathcal{D}(T^{p_1}v_{k-2},T^{m_{k-2}}v_{k-2})+\mathcal{D}(v_{k-2},T^{m_{k-2}}(T^{p_1}v_{k-2})}{3} \end{array}\right\}\right) \\
&< \max\left\{\begin{array}{l} \mathcal{D}(v_{k-2}, T^{p_1}v_{k-2}), \frac{\mathcal{D}(v_{k-2},T^{m_{k-2}}v_{k-2})}{2}, \frac{\mathcal{D}(T^{p_1}v_{k-2},v_{k-2})+\mathcal{D}(v_{k-2},T^{p_1+m_{k-2}}v_{k-2})}{2}, \\ \frac{\mathcal{D}(T^{p_1}v_{k-2},v_{k-2})+\mathcal{D}(v_{k-2},T^{p_1}v_{k-2})+\mathcal{D}(v_{k-2},T^{p_1+m_{k-2}}v_{k-2})}{3} \end{array}\right\} \\
&= \mathcal{D}(v_{k-2}, T^{p_1}v_{k-2}),
\end{aligned} \tag{13}$$

where $p_2 \in \{m_{k-2}, p_1, m_{k-2} + p_1\}$ is chosen such that

$$\mathcal{D}(v_{k-2}, T^{p_2}v_{k-2}) = \max\left\{\mathcal{D}(v_{k-2}, T^{p_1}v_{k-2}), \mathcal{D}(v_{k-2}, T^{m_{k-2}}v_{k-2}), \mathcal{D}(v_{k-2}, T^{m_{k-2}+p_1}v_{k-2})\right\}.$$

Very easily we can see from (12), (13) and taking into account $(ccf1)$, that

$$\mathcal{D}(v_k, T^{m_k}v_k)) \leq \psi(\mathcal{D}(v_{k-1}, T^{p_1}v_{k-1})) < \psi^2(\mathcal{D}(v_{k-2}, T^{p_2}v_{k-2})). \tag{14}$$

Since ψ is monotone increasing, by continuing this process, we find that

$$\mathcal{D}(v_k, v_{k+1}) = \mathcal{D}(v_k, T^{m_k}v_k)) \leq \psi^k(\mathcal{D}(v_0, T^{p_k}v_0)), \tag{15}$$

for $p_1, p_2, ..., p_k \in \mathbb{N}$. On one hand the inequality (15) shows us, taking into account $(cf2)$ from Definition 5 that

$$\mathcal{D}(v_k, v_{k+1}) = \mathcal{D}(v_k, T^{m_k} v_k)) \to 0. \tag{16}$$

On the other hand using triangle inequality, for $l \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{D}(v_k, v_{k+l}) &\leq \mathcal{D}(v_k, v_{k+1}) + \mathcal{D}(v_{k+1}, v_{k+2}) + ... + \mathcal{D}(v_{k+l-1}, v_{k+l}) \\ &\leq \psi^k(\mathcal{D}(v_0, T^{p_k} v_0)) + \psi^{k+1}\mathcal{D}(v_0, T^{p_{k+1}} v_0)) + ... + \psi^{k+l-1}(\mathcal{D}(v_0, T^{p_{k+l-1}} v_0)). \end{aligned} \tag{17}$$

We should focus our attention on the set $\{\mathcal{D}(v_0, T^i v_0), i \in \mathbb{N}\}$. More precisely, we will show that this set is bounded. In order to prove that, we mention first that by hypothesis there exists a positive integer $m_0 = m(v_0)$ such that (6) holds. Let $x_i = \mathcal{D}(v_0, T^{im_0+s} v_0)$, where $s \in \{1, 2, ..., m_0\}$ and $l = \max\{x_0, \mathcal{D}(v_0, T^{m_0} v_0)\}$. According to (ccf_3), we can find $a > l > 0$ such that

$$t - \psi(t) > l \text{ for any } t \in [a, \infty). \tag{18}$$

It is clear then that $x_0 \leq l < a$ and we will show that $x_i < a$ for all $i \in \mathbb{N}$. We suppose the contrary, that there exists $k \in \mathbb{N}$ such that $x_k < a \leq x_{k+1}$. Note that (according to (6), (9) and triangle inequality)

$$\begin{aligned} x_k = \mathcal{D}(v_0, T^{km_0+s} v_0) &\leq \mathcal{D}(v_0, T^{m_0} v_0) + \mathcal{D}(T^{m_0} v_0, T^{km_0+s} v_0) \\ &= \mathcal{D}(v_0, T^{m_0} v_0) + \mathcal{D}(T^{m_0} v_0, T^{m_0}(T^{(k-1)m_0+s} v_0) \\ &\leq l + \alpha(v_0, (T^{(k-1)m_0+s} v_0))\mathcal{D}(T^{m_0} v_0, T^{m_0}(T^{(k-1)m_0+s} v_0) \\ &\leq l + \psi\left(\max\left\{ \begin{array}{c} \mathcal{D}(v_0, T^{(k-1)m_0+s} v_0), \frac{\mathcal{D}(v_0, T^{m_0} v_0)}{2}, \frac{\mathcal{D}(T^{(k-1)m_0+s} v_0, T^{km_0+s} v_0)}{2}, \\ \frac{\mathcal{D}(v_0, T^{km_0+s} v_0) + \mathcal{D}(T^{(k-1)m_0+s} v_0, T^{m_0} v_0)}{3} \end{array} \right\}\right). \end{aligned} \tag{19}$$

But,

$$\begin{aligned} &\max\left\{ \begin{array}{c} \mathcal{D}(v_0, T^{(k-1)m_0+s} v_0), \frac{\mathcal{D}(v_0, T^{m_0} v_0)}{2}, \frac{\mathcal{D}(T^{(k-1)m_0+s} v_0, T^{km_0+s} v_0)}{2}, \\ \frac{\mathcal{D}(v_0, T^{km_0} v_0 + s) + \mathcal{D}(T^{(k-1)m_0+s} v_0, T^{m_0} v_0)}{3} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} \mathcal{D}(v_0, T^{(k-1)m_0+s} v_0), \frac{\mathcal{D}(v_0, T^{m_0} v_0)}{2}, \frac{\mathcal{D}(T^{(k-1)m_0+s} v_0, v_0) + \mathcal{D}(v_0, T^{km_0+s} v_0)}{2}, \\ \frac{\mathcal{D}(v_0, T^{km_0+s} v_0) + \mathcal{D}(T^{(k-1)m_0+s} v_0, v_0) + \mathcal{D}(v_0, T^{m_0} v_0)}{3} \end{array} \right\} \\ &= \max\left\{ x_{k-1}, \frac{x_0}{2}, \frac{x_{k-1} + x_k}{2}, \frac{x_k + x_{k-1} + \mathcal{D}(v_0, T^{m_0} v_0)}{3} \right\} \\ &\leq \max\left\{ x_{k-1}, x_0, \frac{x_{k-1} + x_k}{2}, \frac{x_k + x_{k-1} + x_0}{3} \right\} \\ &< \max\left\{ l, \frac{l}{2}, x_k, l \right\} = x_k. \end{aligned}$$

Since ψ is increasing, from (19) we get

$$x_k \leq l + \psi(x_k) \tag{20}$$

which contradicts (18). This contradiction shows that our assumption was false. Thus, for all $i \in \mathbb{N}$

$$x_i = \mathcal{D}(v_0, T^{im_0+s} v_0) < a.$$

We have thus demonstrated that the set $\{\mathcal{D}(v_0, T^{im_0+s} v_0) : i \in \mathbb{N}\}$ is bounded, and also, varying $s \in \{0, 1, 2, ..., m_0\}$, the set $\{\mathcal{D}(v_0, T^i v_0), i \in \mathbb{N}\}$ is bounded. Hence,

$$r(v_0) = \sup_i \left\{\mathcal{D}(v_0, T^i v_0)\right\} < \infty. \tag{21}$$

With this observation, we return to (17) and we get

$$\begin{aligned} \mathcal{D}(v_k, v_{k+l}) &\leq \mathcal{D}(v_k, v_{k+1}) + \mathcal{D}(v_{k+1}, v_{k+2}) + \ldots + \mathcal{D}(v_{k+l-1}, v_{k+l}) \\ &\leq \psi^k(r(v_0)) + \psi^{k+1}(r(v_0)) + \ldots + \psi^{k+l-1}(r(v_0)) \\ &= \sum_{j=k}^{k+l-1} \psi^j(r(v_0)). \end{aligned} \tag{22}$$

The series $\sum_{j=0}^{\infty} \psi^j(r(v_0))$ is convergent due to (ccf_2) and its sequence of partial sums, denoted by $\{Sn\}$, is convergent at S. Then

$$\mathcal{D}(v_k, v_{k+l}) \leq S_{k+l-1} - S_k \to 0 \tag{23}$$

as $k \to \infty$, and, therefore $\{v_k\}$ is a 0-Cauchy sequence. By completeness of $(\mathcal{M}, \mathcal{D})$, there is some point $v^* \in \mathcal{M}$ such that

$$\lim_{n\to\infty} \mathcal{D}(v_n, v^*) = 0 = \lim_{n,m\to\infty} \mathcal{D}(v_n, v_m). \tag{24}$$

From the continuity of T it easily follows that

$$\lim_{n\to\infty} \mathcal{D}(v_{n+1}, Tv^*) = \lim_{n\to\infty} \mathcal{D}(Tv_n, Tv^*) = \lim_{n,m\to\infty} \mathcal{D}(Tv_n, Tv_m) = \lim_{n,m\to\infty} \mathcal{D}(v_{n+1}, v_{m+1}) = 0$$

and by the uniqueness of the limit, we get $Tv^* = v^*$.

We claim now that v^* is a fixed point of T under the hypothesis (iv). The first step in our proof is to show that $T^{m(v^*)}v^* = v^*$ which means that v^* is a fixed point of $T^{m(v^*)}$. Firstly we claim that $\lim\limits_{k\to\infty} \mathcal{D}(T^{m(v^*)}v_k, v_k) = 0$. Taking $v = T^{m(v^*)}v_{k-1}$ and $w = v_{k-1}$ in (6), there exists m_{k-1} such that for any $k \geq 1$ we have

$$\begin{aligned} \mathcal{D}(T^{m(v^*)}v_k, v_k) &= \mathcal{D}(T^{m(v^*)}(T^{m_{k-1}}v_{k-1}), T^{m_{k-1}}v_{k-1}) \\ &\leq \alpha(T^{m(v^*)}v_{k-1}, v_{k-1})\mathcal{D}(T^{m_{k-1}}(T^{m(v^*)}v_{k-1}), T^{m_{k-1}}v_{k-1}) \\ &\leq \psi\left(\max\left\{\begin{array}{c} \mathcal{D}(T^{m(v^*)}v_{k-1}, v_{k-1}), \frac{\mathcal{D}(T^{m(v^*)}v_{k-1}, T^{m_{k-1}+m(v^*)}v_{k-1})}{2}, \frac{\mathcal{D}(v_{k-1}, T^{m(v_{k-1})}v_{k-1})}{2}, \\ \frac{\mathcal{D}(T^{m(v^*)}v_{k-1}, T^{m_{k-1}}v_{k-1}) + \mathcal{D}(v_{k-1}, T^{m_{k-1}+m(v^*)}v_{k-1})}{3} \end{array}\right\}\right) \end{aligned} \tag{25}$$

Let $q_1 \in \{m(v^*), m_{k-1}, m(v^*) + m_{k-1}\}$ such that

$$\max\left\{\mathcal{D}(T^{m(v^*)}v_{k-1}, v_{k-1}), \mathcal{D}(T^{m_{k-1}}v_{k-1}, v_{k-1}), \mathcal{D}(T^{m(v^*)+m_{k-1}}v_{k-1}, v_{k-1})\right\} = \mathcal{D}(T^{q_1}v_{k-1}, v_{k-1}).$$

Using triangle inequality, we have

$$\begin{aligned} &\frac{\mathcal{D}(T^{m(v^*)}v_{k-1}, T^{m_{k-1}+m(v^*)}v_{k-1})}{2} \leq \\ &\leq \frac{\mathcal{D}(T^{m(v^*)}v_{k-1}, v_{k-1}) + \mathcal{D}(v_{k-1}, T^{m_{k-1}+m(v^*)}v_{k-1})}{2} \\ &\leq \mathcal{D}(T^{q_1}v_{k-1}, v_{k-1}) \end{aligned}$$

and

$$\begin{aligned} &\frac{\mathcal{D}(T^{m(v^*)}v_{k-1}, T^{m_{k-1}}v_{k-1}) + \mathcal{D}(v_{k-1}, T^{m_{k-1}+m(v^*)}v_{k-1})}{3} \leq \\ &\leq \frac{\mathcal{D}(T^{m(v^*)}v_{k-1}, v_{k-1}) + \mathcal{D}(v_{k-1}, T^{m_{k-1}}v_{k-1}) + \mathcal{D}(v_{k-1}, T^{m_{k-1}+m(v^*)}v_{k-1})}{3} \\ &\leq \mathcal{D}(T^{q_1}v_{k-1}, v_{k-1}). \end{aligned}$$

Then, from (25) it follows that

$$\mathcal{D}(T^{m(v^*)}v_k, v_k) \leq \psi(\mathcal{D}(T^{q_1}v_{k-1}, v_{k-1})). \tag{26}$$

Repeating this process and keeping in mind the properties (cf_1), (cf_2) we find that

$$\mathcal{D}(T^{m(v^*)}v_k, v_k) \leq \psi^k(\mathcal{D}(T^{q_k}v_0, v_0)) \to 0. \tag{27}$$

Suppose now that $T^{m(v^*)}v^* \neq v^*$. Then $\mathcal{D}(T^{m(v^*)}v^*, v^*) > 0$. Since $v_k \to v^*$ as $k \to \infty$ and the space $\mathcal{M}$ is regular, by triangle inequality we have

$$\begin{aligned}
0 < \mathcal{D}(T^{m(v^*)}v^*, v^*) &\leq \mathcal{D}(T^{m(v^*)}v^*, T^{m(v^*)}v_k) + \mathcal{D}(T^{m(v^*)}v_k, v_k) + \mathcal{D}(v_k, v^*) \\
&\leq \alpha(v^*, v_k)\mathcal{D}(T^{m(v^*)}v^*, T^{m(v^*)}v_k) + \mathcal{D}(T^{m(v^*)}v_k, v_k) + \mathcal{D}(v_k, v^*) \\
&\leq \psi\left(\max\left\{\mathcal{D}(v^*, v_k), \tfrac{\mathcal{D}(v^*, T^{m(v^*)}v^*)}{2}, \tfrac{\mathcal{D}(v_k, T^{m(v^*)}v_k)}{2}, \tfrac{\mathcal{D}(v_k, T^{m(v^*)}v^*) + \mathcal{D}(v^*, T^{m(v^*)}v_k)}{3}\right\}\right) \\
&\quad + \mathcal{D}(T^{m(v^*)}v_k, v_k) + \mathcal{D}(v_k, v^*) \\
&< \max\left\{\begin{array}{c} \mathcal{D}(v^*, v_k), \mathcal{D}(v^*, T^{m(v^*)}v^*), \frac{\mathcal{D}(v_k, T^{m(v^*)}v_k)}{2}, \\ \frac{\mathcal{D}(v_k, v^*) + \mathcal{D}(v^*, T^{m(v^*)}v^*) + \mathcal{D}(v^*, v_k) + \mathcal{D}(v_k, T^{m(v^*)}v_k)}{3} \end{array}\right\} \\
&\quad + \mathcal{D}(T^{m(v^*)}v_k, v_k) + \mathcal{D}(v_k, v^*).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, and taking (24) respectively (27) into account, we find that

$$0 < \mathcal{D}(T^{m(v^*)}v^*, v^*) < \mathcal{D}(v^*, T^{m(v^*)}v^*) \tag{28}$$

which implies that $\mathcal{D}(v^*, T^{m(v^*)}v^*) = 0$. Hence, $T^{m(v^*)}v^* = v^*$.

Let $w^* \in \mathcal{M}$ another point such that $T^{m(v^*)}w^* = w^*$ and $v^* \neq w^*$. Since T satisfies (6) and the function α satisfies the condition (U) we get

$$\begin{aligned}
0 < \mathcal{D}(v^*, w^*) &= \mathcal{D}(T^{m(v^*)}v^*, T^{m(v^*)}w^*)) \leq \alpha\,(v^*, w^*)\,\mathcal{D}(T^{m(v^*)}v^*, T^{m(v^*)}w^*)) \\
&\leq \psi\left(\max\left\{\begin{array}{c} \mathcal{D}(v^*, w^*), \frac{\mathcal{D}(v^*, T^{m(v^*)}v^*)}{2}, \frac{\mathcal{D}(w^*, T^{m(v^*)}w^*)}{2}, \\ \frac{\mathcal{D}(w^*, T^{m(v^*)}v^*) + \mathcal{D}(v^*, T^{m(v^*)}w^*)}{3} \end{array}\right\}\right).
\end{aligned}$$

Since

$$\frac{\mathcal{D}(w^*, T^{m(v^*)}w^*)}{2} \leq \frac{\mathcal{D}(w^*, v^*) + \mathcal{D}(v^*, T^{m(v^*)}w^*)}{2} = \mathcal{D}(v^*, w^*)$$

we obtain

$$0 < \psi(\mathcal{D}(v^*, w^*)) < \mathcal{D}(v^*, w^*).$$

But the above inequality is possible only if $\mathcal{D}(v^*, w^*) = 0$, that is $v^* = w^*$. This is a contradiction.

From the uniqueness of the fixed point we can conclude that v^* is a fixed point for T. Indeed,

$$Tv^* = T(T^{m(v^*)}v^*) = T^{m(v^*)}(Tv^*) \tag{29}$$

shows that Tv^* is also fixed point of $T^{m(v^*)}$. But, $T^{m(v^*)}$ has a unique fixed point v^*. Hence, $Tv^* = v^*$. □

Remark 1. *Denoting by $S(x) = \sum_{n=0}^{\infty} \psi^n(x)$, we have from (22)*

$$\mathcal{D}(v_k, v_{k+l}) \leq \sum_{j=k}^{k+l-1} \psi^j(r(v_0)) = \sum_{j=0}^{l-1} \psi^j(\psi^k(r(v_0))) < \sum_{j=0}^{l-1} \psi^j(\psi^k(r(v_0))). \tag{30}$$

Letting $l \to \infty$ in the above inequality we obtain

$$\mathcal{D}(v_k, v^*) < S(\psi^k(r(v_0))).$$

Example 2. *Let $\mathcal{M} = \{a, b, c, d\}$ and $\mathcal{D} : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ with $\mathcal{D}(v, w) = \mathcal{D}(w, v)$ for any $v, w \in \mathcal{M}$, defined as follows*

$$\begin{aligned} &\mathcal{D}(a, a) = \mathcal{D}(b, b) = \mathcal{D}(c, c) = 0,\ \mathcal{D}(d, d) = 1, \\ &\mathcal{D}(a, b) = \mathcal{D}(a, c) = \mathcal{D}(b, c) = 1, \\ &\mathcal{D}(a, d) = \mathcal{D}(b, d) = \mathcal{D}(c, d) = 2. \end{aligned}$$

(It is easy to see that the pair $(\mathcal{M}, \mathcal{D})$ is a DMS but not a metric space.) Let $T : \mathcal{M} \to \mathcal{M}$ be defined by

$$Ta = Tc = a,\ Tb = c,\ Td = b.$$

For $x = b$ and $y = c$ we have

$$\mathcal{D}(Tb, Tc) = \mathcal{D}(c, a) = 1 \geq q\mathcal{D}(b, c) = 1$$

for any $q \in (0, 1)$. Thus, T does not satisfy the Banach contraction condition. We show that the function T satisfies all the assumptions of Theorem 4 with $\psi(x) = \frac{2x}{3}$ for any $x \geq 0$ and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$, $\alpha(v, w) = 1$. Since $\alpha(v, w) = 1$ for all $v, w \in \mathcal{M}$ the assumptions (i), (ii) and (iv) are fulfilled. We discuss the following possible cases:

1. *If $v, w \in \{a, b, c\}$ and $v = w$ then for $m(v) = 1$ we have $\mathcal{D}(Tv, Tv) = 0$ and inequality (6) holds.*
2. *If $v = w = d$ then $\mathcal{D}(T^2d, T^2d) = \mathcal{D}(c, c) = 0$ and also (6) holds.*
3. *If $v = b$, $w = c$ then $T^2b = T^2c = a$. Then, for $m(b) = 2$ we have $\mathcal{D}(T^2b, T^2c)\mathcal{D}(a, a) = 0$. Thus, the condition (6) is satisfied.*
4. *If $v = b$, $w = d$ then $T^2b = a$ and $T^2d = c$. Then, for $m(b) = 2$ we have $\mathcal{D}(T^2b, T^2d) = \mathcal{D}(a, c) = 1$ and $\mathcal{D}(b, d) = 2$. Thus,*

$$\alpha(b, d)\mathcal{D}(T^2b, T^2d) = \mathcal{D}(a, c) = 1 \leq \frac{2 \cdot 2}{3} = \psi(\mathcal{D}(b, d)).$$

5. *If $v = c, w = d$ then $T^2c = a$ and $T^2d = c$. Then, for $m(c) = 2$ we have $\mathcal{D}(T^2c, T^2d) = \mathcal{D}(a, c) = 1$ and $\mathcal{D}(c, d) = 2$. Thus,*

$$\alpha(c, d)\mathcal{D}(T^2b, T^2d) = \mathcal{D}(a, c) = 1 \leq \frac{2 \cdot 2}{3} = \psi(\mathcal{D}(c, d)).$$

On the other hand, we can note that:

$$T^3a = T^3b = T^3c = T^3d = a.$$

For this reason, there exists $m(a) = 3$ such that for any $w \in \mathcal{M}$ the condition (6) is satisfied (since $\mathcal{D}(a, a) = 0$).

The conclusion is that T satisfies all the assumptions of Theorem 4. Therefore T has exactly one fixed point, $v = a$.

Taking, in Theorem 4, $\alpha(v, w) = 1$ we get the following result:

Corollary 1. *Let $(\mathcal{M}, \mathcal{D})$ be a complete DMS, a function $T : \mathcal{M} \to \mathcal{M}$ and $\psi \in \Psi$. Suppose that for all $v, w \in \mathcal{M}$ we can find a positive integer $m(v)$ such that*

$$\begin{aligned} &\mathcal{D}(T^{m(v)}v, T^{m(v)}w) \leq \\ &\quad \leq \psi\left(\max\left\{\mathcal{D}(v, w), \tfrac{\mathcal{D}(v, T^{m(v)}v)}{2}, \tfrac{\mathcal{D}(w, T^{m(v)}w)}{2}, \tfrac{\mathcal{D}(w, T^{m(v)}v) + \mathcal{D}(v, T^{m(v)}w)}{3}\right\}\right). \end{aligned} \tag{31}$$

Then the function T has exactly one fixed point.

Corollary 2. *Let $(\mathcal{M},\mathcal{D})$ be a DMS, a function $T:\mathcal{M}\to\mathcal{M}$ and $\psi\in\Psi$. Suppose that for all $v,w\in\mathcal{M}$ we can find a positive integer $m(v)$ such that*

$$\mathcal{D}(T^{m(v)}v,T^{m(v)}w)\leq\psi\left(\mathcal{D}(v,w)\right). \tag{32}$$

Then the function T has exactly one fixed point.

If we take $\psi(x)=qx$, $q\in[0,1)$ in Corollary 1 respectively in Corollary 2 we find the following consequences:

Corollary 3. *Let $(\mathcal{M},\mathcal{D})$ be a complete DMS, a function $T:\mathcal{M}\to\mathcal{M}$. Suppose that for all $v,w\in\mathcal{M}$ we can find a positive integer $m(v)$ such that*

$$\begin{aligned}&\mathcal{D}(T^{m(v)}v,T^{m(v)}w)\leq\\&\quad\leq q\left[\max\left\{\mathcal{D}(v,w),\frac{\mathcal{D}(v,T^{m(v)}v)}{2},\frac{\mathcal{D}(w,T^{m(v)}w)}{2},\frac{\mathcal{D}(w,T^{m(v)}v)+\mathcal{D}(v,T^{m(v)}w)}{3}\right\}\right].\end{aligned} \tag{33}$$

Then the function T has exactly one fixed point.

Corollary 4. *Let $(\mathcal{M},\mathcal{D})$ be a complete DMS, a function $T:\mathcal{M}\to\mathcal{M}$. Suppose that for all $v,w\in\mathcal{M}$ we can find a positive integer $m(v)$ such that*

$$\mathcal{D}(T^{m(v)}v,T^{m(v)}w)\leq q\mathcal{D}(v,w). \tag{34}$$

Then the function T has exactly one fixed point.

Corollary 5. *Let $(\mathcal{M},\mathcal{D})$ be a complete DMS, a function $T:\mathcal{M}\to\mathcal{M}$. Suppose that for all $v,w\in\mathcal{M}$ we can find a positive integer $m(v)$ such that*

$$\mathcal{D}(T^{m(v)}v,T^{m(v)}w)\leq a_1\mathcal{D}(v,w)+a_2\left[\mathcal{D}(v,T^{m(v)}v)+\mathcal{D}(w,T^{m(v)}w)\right]+a_3\left[\mathcal{D}(w,T^{m(v)}v)+\mathcal{D}(v,T^{m(v)}w)\right], \tag{35}$$

where $a_1,a_2,a_3\in\mathbb{R}_0$ and $a_1+4a_2+3a_3<1$. Then the function T has exactly one fixed point.

Proof. Since

$$\begin{aligned}&\mathcal{D}(T^{m(v)}v,T^{m(v)}w)\leq a_1\mathcal{D}(v,w)+2a_2\left[\frac{\mathcal{D}(v,T^{m(v)}v)}{2}+\frac{\mathcal{D}(w,T^{m(v)}w)}{2}\right]+3a_3\left[\frac{\mathcal{D}(w,T^{m(v)}v)+\mathcal{D}(v,T^{m(v)}w)}{3}\right]\\&\leq a_1\mathcal{D}(v,w)+4a_2\max\left\{\frac{\mathcal{D}(v,T^{m(v)}v)}{2},\frac{\mathcal{D}(w,T^{m(v)}w)}{2}\right\}+3a_3\frac{\mathcal{D}(w,T^{m(v)}v)+\mathcal{D}(v,T^{m(v)}w)}{3}\\&\leq(a_1+4a_2+3a_3)\cdot\max\left\{\mathcal{D}(v,w),\frac{\mathcal{D}(v,T^{m(v)}v)}{2},\frac{\mathcal{D}(w,T^{m(v)}w)}{2},\frac{\mathcal{D}(w,T^{m(v)}v)+\mathcal{D}(v,T^{m(v)}w)}{3}\right\}.\end{aligned}$$

Letting $\psi(x)=(a_1+4a_2+3a_3)x$ the result follows from Theorem 4. □

If we take $m(v)=1$ in Theorem 4 we get:

Corollary 6. *Let $(\mathcal{M},\mathcal{D})$ be a complete DMS, a function $T:\mathcal{M}\to\mathcal{M}$ and $\psi\in\Psi$. Suppose that for all $v,w\in\mathcal{M}$*

$$\alpha(v,w)\mathcal{D}(Tv,Tw)\leq\psi\left(\max\left\{\mathcal{D}(v,w),\frac{\mathcal{D}(v,Tv)}{2},\frac{\mathcal{D}(w,Tw)}{2},\frac{\mathcal{D}(w,Tv)+\mathcal{D}(v,Tw)}{3}\right\}\right). \tag{36}$$

Suppose also that:

(i) *T is triangular α-orbital admissible;*
(ii) *there exists v_0 in $\mathcal{M}$ such that $\alpha(v_0, Tv_0) \geq 1$;*
(iii) *either T is continuous, or*
(iv) *the space $\mathcal{M}$ is regular and the condition (U) is satisfied.*

Then the function T has exactly one fixed point.

Example 3. *Let a dislocated metric space $(\mathcal{M}, \mathcal{D})$, where $X = [0,1]$ and $\mathcal{D}(v,w) = \max\{v,w\}$ for any $v, w \in \mathcal{M}$. Let a self mapping T on $\mathcal{M}$ be defined as follows:*

$$T(v) = \begin{cases} \frac{v}{3} & \text{for } v \in [0, \frac{1}{2}) \cup \{1\} \\ 1 & \text{for } v = 1/2 \\ \frac{v^2}{4} + \frac{1}{2} & \text{for } v \in (\frac{1}{2}, 1) \end{cases}$$

Let the functions $\psi(x) = \frac{2x}{3}$, $x \geq 0$ and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$,

$$\alpha(v,w) = \begin{cases} 2 & \text{for } (v,w) \in A \times A \\ 1 & \text{for } (v,w) = \left(\frac{1}{2}, \frac{1}{2}\right) \\ v+1 & \text{for } (v,w) \in \left\{(v, \frac{1}{2}), (\frac{1}{2}, v) : v \in A\right\} \\ 0 & \text{otherwise} \end{cases}$$

Let us first notice that $v_n = T^n v = \frac{v}{3^n} \to 0$ for any $v \in A$ and $T^n 1 = \frac{1}{3^{n-1}} \to 0$. Since $\alpha(v_n, 0) = 2$ we get that assumption (iv) of Theorem 4 is satisfied. Also, since $\alpha(0,0) = 2 \geq 1$ by simple calculation we can conclude that the assumptions (ii) and (iv) are satisfied. We remark that if $v = \frac{1}{3}$ and $w = \frac{1}{2}$ then $T\frac{1}{3} = \frac{1}{9}$, $T\frac{1}{2} = 1$. Hence,

$$\mathcal{D}\left(T\frac{1}{3}, T\frac{1}{2}\right) = \max\left\{\frac{1}{9}, 1\right\} = 1 \geq q \cdot \frac{1}{2} = \max\left\{\frac{1}{3}, \frac{1}{2}\right\} = \mathcal{D}\left(\frac{1}{3}, \frac{1}{2}\right)$$

and

$$\begin{aligned} \alpha\left(\tfrac{1}{3}, \tfrac{1}{2}\right) \mathcal{D}\left(T\tfrac{1}{3}, T\tfrac{1}{2}\right) &= \left(\tfrac{1}{3} + 1\right) \max\left\{\tfrac{1}{9}, 1\right\} = \tfrac{4}{3} \geq \tfrac{1}{3} = \tfrac{2}{3} \cdot \tfrac{1}{2} \\ &= \psi\left(\max\left\{\mathcal{D}\left(\tfrac{1}{3}, \tfrac{1}{2}\right), \tfrac{\mathcal{D}(\frac{1}{3}, \frac{1}{9})}{2}, \tfrac{\mathcal{D}(\frac{1}{2}, 1)}{2}, \tfrac{\mathcal{D}(\frac{1}{3}, 1) + \mathcal{D}(\frac{1}{2}, \frac{1}{9})}{3}\right\}\right), \end{aligned}$$

which shows us that T does not satisfy the contraction condition of Banach, neither condition (36) of Corollary 6. We must discuss the next cases:

1. *If $v, w \in A$ then for $m(v) = 3$ we have $T^{m(v)}v = \frac{v}{27}$ and $T^{m(v)}w = \frac{w}{27}$. Thus,*

$$\alpha(v,w)\mathcal{D}\left(T^{m(v)}v, T^{m(v)}w\right) = 2 \cdot \frac{\max\{v,w\}}{27} \leq \frac{2}{3} \cdot \max\{v,w\} = \psi(\mathcal{D}(v,w)) \leq \psi(M(v,w)),$$

where

$$M(v,w) = \max\left\{\mathcal{D}(v,w), \frac{\mathcal{D}(v, T^{m(v)}v)}{2}, \frac{\mathcal{D}(w, T^{m(v)}w)}{2}, \frac{\mathcal{D}(w, T^{m(v)}v) + \mathcal{D}(v, T^{m(v)}w)}{3}\right\}$$

2. *If $v = w = \frac{1}{2}$ we can choose $m(v) = 4$. Then, $T^4\frac{1}{2} = \frac{1}{27}$ and*

$$\alpha(\frac{1}{2}, \frac{1}{2})\mathcal{D}\left(T^4\frac{1}{2}, T^4\frac{1}{2}\right) = 1 \cdot \frac{1}{27} \leq \frac{1}{3} = \frac{2}{3} \cdot \frac{1}{2} = \psi(\mathcal{D}(\frac{1}{2}, \frac{1}{2})) \leq \psi(M(\frac{1}{2}, \frac{1}{2})).$$

3. *If $v = \frac{1}{2}$ and $w \in [0, \frac{1}{2})$ then for $m(v) = 4$ we have $T^4\frac{1}{2} = \frac{1}{27}$ and $T^4 w = \frac{w}{81}$*

$$\begin{aligned}\alpha(\tfrac{1}{2}, w)\mathcal{D}\left(T^4\tfrac{1}{2}, T^4 w\right) &= (w+1)\cdot \max\left\{\tfrac{1}{27}, \tfrac{w}{81}\right\} = \tfrac{w+1}{27}\\ &\leq \tfrac{1}{3} = \tfrac{2}{3}\cdot \max\left\{\tfrac{1}{2}, w)\right\} = \psi(\mathcal{D}(\tfrac{1}{2}, w)) \leq \psi(M(\tfrac{1}{2}, w)).\end{aligned}$$

4. *If $v = \frac{1}{2}$, $w = 1$ and $m(v) = 4$ then $T^4\frac{1}{2} = \frac{1}{27}$ and $T^4 1 = \frac{1}{81}$*

$$\alpha(\frac{1}{2}, 1)\mathcal{D}\left(T^4\frac{1}{2}, T^4 1\right) = (1+1)\cdot\frac{1}{27} \leq \frac{2}{3}\cdot 1 = \psi(\mathcal{D}(\frac{1}{2}, 1)) \leq \psi(M(\frac{1}{2}, 1)).$$

The other cases are not interesting since $\alpha(v, w) = 0$. Therefore $v = 0$ is the unique fixed point for T.

Inspired by Proposition 3 from [7] we will establish a new fixed point result for a T function on a DMS, not necessarily complete.

Corollary 7. *Let $(\mathcal{M}, \mathcal{D})$ be a DMS and a function $T : \mathcal{M} \to \mathcal{M}$. Suppose that for a given $v \in \mathcal{M}$ such that $\mathcal{D}(v, Tv) > 0$ we can find a positive integer $m(v)$ such that the following two conditions hold:*

$$\mathcal{D}(v, T^{m(v)}v) < \mathcal{D}(v, Tv), \tag{37}$$

and

$$\begin{aligned}\mathcal{D}(T^{m(v)}v, T^{m(v)+1}v) \quad &\leq a_1\mathcal{D}(v, Tv) + a_2\left[\mathcal{D}(v, T^{m(v)}v) + \mathcal{D}(Tv, T^{m(v)+1}v)\right]\\ &+a_3\left[\mathcal{D}(Tv, T^{m(v)}v) + \mathcal{D}(v, T^{m(v)+1}v)\right]\end{aligned} \tag{38}$$

for any $a_1, a_2, a_3 \in \mathbb{R}_0$ and $a_1 + 4a_2 + 4a_3 < 1$.

Suppose also that there exists a point $v^* \in \mathcal{M}$ such that $\sigma(v^*) = \inf\{\sigma(v) : v \in \mathcal{M}\}$, where $\sigma(v) = \mathcal{D}(v, Tv)$. Then v^* is fixed point of T.

Proof. Suppose that $\mathcal{D}(v^*, T^{m(v^*)} > 0$. Then we can find a positive integer $m(v^*)$ such that

$$\mathcal{D}(v^*, T^{m(v^*)}v^*) < \mathcal{D}(v^*, Tv^*). \tag{39}$$

Replacing $v = v^*$ in (38), using triangle inequality and keeping in mind (39) we have

$$\begin{aligned}&\mathcal{D}(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*) \leq a_1\mathcal{D}(v^*, Tv^*) +\\ &\quad +a_2\left[\mathcal{D}(v^*, T^{m(v^*)}v^*) + \mathcal{D}(Tv^*, v^*) + \mathcal{D}(v^*, T^{m(v^*)}v^*) + \mathcal{D}(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*)\right]\\ &\quad +a_3\left[\mathcal{D}(Tv^*, v^*) + \mathcal{D}(v^*, T^{m(v^*)}v^*) + \mathcal{D}(v^*, T^{m(v^*)}v^*) + \mathcal{D}(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*)\right]\\ &\quad < (a_1 + 3a_2 + 3a_3)\,\mathcal{D}(v^*, Tv^*) + (a_2 + a_3)\,\mathcal{D}(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*)\end{aligned} \tag{40}$$

or, since $a_1 + 4a_2 + 4a_3 < 1$ we get

$$\sigma(T^{m(v^*)}v^*) = \mathcal{D}(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*) < \frac{a_1 + 3a_2 + 3a_3}{1 - a_2 - a_3}\mathcal{D}(v^*, Tv^*) < \mathcal{D}(v^*, Tv^*) = \sigma(v^*).$$

This is a contradiction. Hence $\mathcal{D}(v^*, Tv^*) = 0$ and v^* a is fixed point of T. □

3. Ulam-Stability

Definition 9. *Let $(\mathcal{M}, \mathcal{D})$ be a DMS and a function $T : \mathcal{M} \to \mathcal{M}$. We say that the fixed point equation*

$$v = Tv,\ v \in \mathcal{M} \tag{41}$$

is generalized Ulam-stable if for each $\varepsilon > 0$ and $w \in \mathcal{M}$ there exists $m(w) \in \{1, 2, \ldots\}$ such that for any $w^* \in \mathcal{M}$ satisfying the inequality

$$q(T^{m(w^*)}w^*, w^*) \leq \varepsilon \tag{42}$$

there exists an increasing function $\eta : [0, \infty) \to [0, \infty)$ continuous at 0, with $\eta(0) = 0$ and $v^* \in \mathcal{M}$ a solution of Equation (41) such that

$$\mathcal{D}(v^*, w^*) \leq \eta(\varepsilon). \tag{43}$$

Remark 2. *If $\eta(x) = ax$ for all $x \geq 0$, where $a > 0$, the fixed point Equation (41) is said to be Ulam-stable.*

Theorem 5. *Let the function $\eta : [0, \infty) \to [0, \infty)$, defined by $\eta(x) := x - \psi(x)$, with $\psi \in \Psi$. Under the hypothesis of Corollary 2 the fixed point Equation (41) is generalized Ulam-stable.*

Proof. There exists exactly one point $v^* \in \mathcal{M}$ such that $Tv^* = v^*$, which means that v^* is a unique solution of fixed point Equation (41). Let $w^* \in \mathcal{M}$. There exists $m(w^*) \in \{1, 2, \ldots\}$ such that (42) holds. Keeping in mind the properties of function ψ, the condition imposed on the alpha function and using the triangle inequality we obtain

$$\begin{aligned} \mathcal{D}(v^*, w^*) &\leq \mathcal{D}(v^*, T^{m(w^*)}w^*) + \mathcal{D}(T^{m(w^*)}w^*, w^*) \\ &= \mathcal{D}(T^{m(w^*)}v^*, T^{m(w^*)}w^*) + \mathcal{D}(T^{m(w^*)}w^*, w^*) \\ &\leq \alpha(v^*, w^*)\mathcal{D}(T^{m(w^*)}v^*, T^{m(w^*)}w^*) + \mathcal{D}(T^{m(w^*)}w^*, w^*) \\ &\leq \psi\left(\mathcal{D}(v^*, w^*)\right) + \varepsilon. \end{aligned} \tag{44}$$

Taking into account the definition of the function β we have

$$\mathcal{D}(v^*, w^*) - \psi\left(\mathcal{D}(v^*, w^*)\right) = \eta(\mathcal{D}(v^*, w^*)) \leq \varepsilon$$

which is equivalent with

$$\mathcal{D}(v^*, w^*) \leq \eta^{-1}(\mathcal{D}(v^*, w^*))$$

From the assumption, η is continuous and strictly increasing. Thus, η^{-1} is also continuous and increasing, with $\eta^{-1}(0) = 0$ Therefore, the Equation (41) is generalized Ulam-stable.
□

4. Application to Boundary Value Problem

Here we consider the following two point boundary value problems for the second order differential equation.

$$\begin{cases} -\frac{d^2\vartheta}{dt^2} = \ell(t, \vartheta(t)); \quad t \in [0, 1] \\ \vartheta(0) = \vartheta(1) = 0 \end{cases} \tag{45}$$

where $\ell : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Recall that the Green's function associated to (45) is

$$\Lambda(t, s) = \begin{cases} t(1 - s), & \text{if } 0 \leq t \leq s \leq 1 \\ s(1 - t), & \text{if } 0 \leq s \leq t \leq 1 \end{cases} \tag{46}$$

Let $\mathcal{M} = \mathbb{C}([0, 1])$ be the space of all continuous functions defined on $I = [0, 1]$. We consider on $\mathcal{M}$, the dislocated metric $\mathcal{D}$ given by $\mathcal{D}(\vartheta, \omega) = ||\vartheta - \omega|| + ||\vartheta|| + ||\omega||$ for all $\vartheta, \omega \in \mathcal{M}$, where $||\vartheta|| = \max_{t \in I} |\vartheta(t)|$ for each $\vartheta \in \mathcal{M}$.

Clearly, $(\mathcal{M}, \mathcal{D})$ is a complete DMS.

It is well known that $\vartheta \in \mathbb{C}^2(I)$ is a solution of (45) is equivalent to that $\vartheta \in \mathcal{M} = \mathbb{C}(I)$ is a solution of the integral equation.

$$\vartheta(t) = \int_0^1 \Lambda(t,s)\ell(s,\vartheta(s))ds, \ \ \forall t \in I \tag{47}$$

Theorem 6. *Let $(\mathcal{M}, \mathcal{D})$ be a complete DMS as defined above. Further, we will assume the following conditions hold:*

1. *there exists a continuous function $\varrho : I \to \mathbb{R}^+$ such that*

$$|\ell(s,x_1) - \ell(s,x_2)| \leq 8\varrho(s)|x_1 - x_2|$$

   for each $s \in I$ and $x_1, x_2 \in \mathbb{R}$;
2. *there exists a continuous function $\Upsilon : I \to \mathbb{R}_0^+$ such that*

$$|\ell(s,x_1)| \leq 8\Upsilon(s)|x_1|$$

   for each $s \in I$ and $|x_1 \in \mathbb{R}$;
3. $\sup_{s\in I} \varrho(s) = z_1 < \frac{1}{3}$;
4. $\sup_{s\in I} \Upsilon(s) = z_2 < \frac{1}{3}$.

   The problem (45) has a solution $\vartheta \in \mathcal{M}$.

Proof. Define the mapping $T : \mathbb{C}(I) \to \mathbb{C}(I)$ by

$$T^{m(\vartheta)}\vartheta(t) = \int_0^1 \Lambda(t,s)\ell(s,\vartheta(s))ds$$

for all $\vartheta \in \mathcal{M}, s, t \in I$ and $m(\vartheta)$ be a positive integer. Then the Equation (47) is equivalent to finding $\vartheta \in \mathcal{M}$ that is a fixed point of T.

Now let $\vartheta, \omega \in \mathcal{M} = (\mathbb{C}[0,1])$. We have,

$$\begin{aligned} |T^{m(\vartheta)}\vartheta(t) - T^{m(\vartheta)}\omega(t)| &= |\int_0^1 \Lambda(t,s)\ell(s,\vartheta(s))ds - \int_0^1 \Lambda(t,s)\ell(s,\omega(s))ds| \\ &\leq \int_0^1 \Lambda(t,s)|\ell(s,\vartheta(s)) - \ell(s,\omega(s))|ds \\ &\leq 8\int_0^1 \Lambda(t,s)\varrho(s)|\vartheta(s)) - \omega(s)|ds \\ &\leq 8z_1||\vartheta - \omega|| \sup_{t\in I} \int_0^1 \Lambda(t,s)ds \end{aligned} \tag{48}$$

for each $t \in I$. On the other hand,

$$\int_0^1 \Lambda(t,s)ds = \frac{t}{2} - \frac{t^2}{2} \quad \text{and so} \quad \sup_{t\in I} \int_0^1 \Lambda(t,s)ds = \frac{1}{8}.$$

From (48), we get,

$$|T^{m(\vartheta)}\vartheta(t) - T^{m(\vartheta)}\omega(t)| \leq z_1||\vartheta - \omega|| \tag{49}$$

$$
\begin{aligned}
|T^{m(\vartheta)}\vartheta(t)| &= |\int_0^1 \Lambda(t,s)\ell(s,\vartheta(s))ds| \\
&\leq \int_0^1 \Lambda(t,s)|\ell(s,\vartheta(s))|ds \\
&\leq 8\int_0^1 \Lambda(t,s)\Upsilon(s)|\vartheta(s)|ds \\
&\leq 8z_2||\vartheta||\sup_{t\in I}\int_0^1 \Lambda(t,s)ds \\
&= z_2||\vartheta||.
\end{aligned} \tag{50}
$$

Thus $||T^{m(\vartheta)}\vartheta(t)|| \leq z_2||\vartheta||$. Similarly, we derive that

$$||T^{m(\vartheta)}\omega(t)|| \leq z_2||\omega||. \tag{51}$$

Take $z = z_1 + 2z_2$. Thus $z < 1$. By using (49)–(51), we get,

$$
\begin{aligned}
\mathcal{D}(T^{m(\vartheta)}\vartheta(t) - T^{m(\vartheta)}\omega(t)) &= ||T^{m(\vartheta)}\vartheta(t) - T^{m(\vartheta)}\omega(t)|| + ||T^{m(\vartheta)}\vartheta(t)|| + ||T^{m(\vartheta)}\omega(t)|| \\
&\leq z_1||\vartheta-\omega|| + z_2||\vartheta|| + z_2||\omega|| \\
&\leq (z_1+2z_2)(||\vartheta-\omega|| + ||\vartheta|| + ||\omega||) \\
&= z\mathcal{D}(\vartheta,\omega).
\end{aligned} \tag{52}
$$

Hence $\mathcal{D}(T^{m(\vartheta)}\vartheta(t), T^{m(\vartheta)}\omega(t)) \leq z\mathcal{D}(\vartheta(t),\omega(t))$. Thus all the conditions of Corollary 4 are satisfied. Hence T has exactly one fixed point $\vartheta \in \mathcal{M}$, i.e., the problem (45) has a solution $\vartheta \in \mathbb{C}^2(I)$. □

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