mathematics

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Brauer-Type Inclusion Sets of Zeros for Chebyshev Polynomial

Xiao Feng and Yaotang Li *

School of Mathematics and Statistics, Yunnan University, Kunming 650091, China; fengxiao@mail.ynu.edu.cn
* Correspondence: liyaotang@ynu.edu.cn

Received: 5 January 2019; Accepted: 3 February 2019; Published: 7 February 2019

Abstract: The generalized polynomials such as Chebyshev polynomial and Hermite polynomial are widely used in interpolations and numerical fittings and so on. Therefore, it is significant to study inclusion regions of the zeros for generalized polynomials. In this paper, several new inclusion sets of zeros for Chebyshev polynomials are presented by applying Brauer theorem about the eigenvalues of the comrade matrix of Chebyshev polynomial and applying the properties of ovals of Cassini. Some examples are given to show that the new inclusion sets are tighter than those provided by Melman (2014) in some cases.

Keywords: the inclusion region of polynomial zeros; Chebyshev polynomials; Brauer theorem; oval of Cassini; comrade matrix

1. Introduction

The inclusion region of polynomial zeros is widely used in the theory of differential equations, the complex functions and the numerical analysis. There are some inclusion regions for polynomial zeros in power basis [1–3]. However, the structures of comrade matrix for generalized polynomials are different from this of polynomial in power basis [4], so it is difficult to use them to generalized polynomials such as Chebyshev polynomial and Hermite polynomial, which are widely used in interpolations and numerical fittings. Therefore, it is necessary to use some new methods to study inclusion regions of the zeros for polynomials in generalized basis. In [5], Melman used linear algebra techniques to derive two Gershgorin-type inclusion disks of the zeros for polynomials in generalized basis, specially the Chebyshev basis.

Definition 1 ([6])**.** *The Chebyshev polynomials $\{T_i(z)\}$ and $\{U_i(z)\}$ of the first and second kind, respectively, are defined by the relation*

$$T_0(z) = 1, \quad T_1(z) = z,$$
$$T_i(z) = 2zT_{i-1}(z) - T_{i-2}(z) \quad (i = 2, 3, \cdots),$$

and

$$U_0(z) = 1, \quad U_1(z) = 2z,$$
$$U_i(z) = 2zU_{i-1}(z) - U_{i-2}(z) \quad (i = 2, 3, \cdots).$$

In addition, there is a relationship between the Chebyshev polynomials of the first and second kinds:

$$T_i(z) = \frac{1}{2}\left(U_i(z) - U_{i-2}(z)\right), \quad (i = 2, 3, \cdots). \tag{1}$$

As for practical applications, Chebyshev polynomials can be used to differential equations, approximation theory, combinatorics, Fourier series, numerical analysis, geometry, graph theory, number theory, and statistics.

Chebyshev differential equations were put forward by mathematicians when studing of differential equations, which were

$$\left(1-x^2\right)y''-xy'+n^2y=0,$$

and

$$\left(1-x^2\right)y''-3xy'+n\left(n+2\right)y=0.$$

Correspondingly, the first and second kind of chebyshev polynomials are the solutions of these two equations respectively. Next, we give the main results about Chebyshev polynomials obtained by Malman in [5].

Theorem 1 ([5]). *Let*

$$P_U(z)=U_n(z)+b_{n-1}U_{n-1}(z)+\cdots+b_1U_1(z)+b_0U_0(z),$$

with $b_j\in\mathbb{C}$, and $U_j\left(z\right)$ is the Chebyshev polynomial of the second kind, and let μ_U be the largest positive solution of the equation

$$x^n-\left|b_{n-1}\right|x^{n-1}+\left(1-\left|1-b_{n-2}\right|\right)x^{n-2}-\sum_{j=0}^{n-3}\left|b_j\right|x^j=0.$$

Then all the zeros of P_U are contained in $\bar{O}\left(0;\frac{1}{2}\left(\mu_U+\mu_U{}^{-1}\right)\right)$, where we denote by $\bar{O}(a;r)$ the closed disk with center a and radius r.

Theorem 2 ([5]). *Let*

$$P_U(z)=U_n(z)+b_{n-1}U_{n-1}(z)+\cdots+b_1U_1(z)+b_0U_0(z),$$

with $b_j\in\mathbb{C}$, and $U_j\left(z\right)$ is the Chebyshev polynomial of the second kind, and let V_U be the largest positive solution of the equation

$$x^n+\left|b_{n-1}\right|x^{n-1}+\left(1-\left|1-b_{n-2}\right|\right)x^{n-2}-\sum_{j=0}^{n-3}\left|b_j\right|x^j=0.$$

Then all the zeros of P_U are contained in $\bar{O}\left(-\frac{b_{n-1}}{2};\left|\frac{b_{n-1}}{2}\right|+\frac{1}{2}\left(V_U+V_U{}^{-1}\right)\right)$.

In this paper, we continue to research the inclusion regions of generalized polynomial zeros. We will give a tighter inclusion sets for generalized polynomial zeros. Since Chebyshev polynomials are reprensentative of all polynomials which satisfy three-term recurrence relation, we only discuss Chebyshev polynomials. We firstly give some previous results.

In mathematics, the recurrence relation are equations defined by successive terms of a sequence or multidimensional array of values, therefore, once one or more initial terms of the sequence are given, we can calculate the value of the sequence. The property of recurrence relation makes it useful in many fields. And three-term recurrence relation is a special kind which is defined by successive three terms. Its definition is as following:

Definition 2 ([4]). *We define the families of the polynomial $\{\phi_i\}$ $(i=0,1,2,3,\cdots)$ satisfying three-term recurrence relation as following*

$$\begin{aligned} \phi_0(z) = 1, \quad \phi_1(z) = \alpha_1 z + \beta_1, \\ \phi_i(z) = (\alpha_i z + \beta_i)\phi_{i-1}(z) - \gamma_i \phi_{i-2}(z). \end{aligned} \tag{2}$$

where α_i, β_i, $\gamma_i \in \mathbb{C}$, and $\alpha_i \neq 0$.

Among all the three-term recurrence relation, the Fibonacci sequence is a typical one [7]. Besides, Mathieu functions, is an example of three-term recurrence relation appears in physical problems involving elliptical shapes or periodic potentials [8].

Theorem 3 ([4]). *All the zeros of the polynomial*

$$p(z) = \phi_n(z) + a_{n-1}\phi_{n-1}(z) + \cdots + a_1\phi_1(z) + a_0\phi_0(z),$$

are the eigenvalues of the comrade matrix

$$\begin{pmatrix} \frac{-\beta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & & & & \frac{-a_0}{\alpha_n} \\ \frac{1}{\alpha_1} & \frac{-\beta_2}{\alpha_2} & \frac{\gamma_3}{\alpha_3} & & & \frac{-a_1}{\alpha_n} \\ & \frac{1}{\alpha_2} & \frac{-\beta_3}{\alpha_3} & \frac{\gamma_4}{\alpha_4} & & \frac{-a_2}{\alpha_n} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \frac{1}{\alpha_{n-2}} & \frac{-\beta_{n-1}}{\alpha_{n-1}} & \frac{-a_{n-2}+\gamma_n}{\alpha_n} \\ & & & & \frac{1}{\alpha_{n-1}} & \frac{-a_{n-1}-\beta_n}{\alpha_n} \end{pmatrix}, \tag{3}$$

where blank spaces indicate zero entries, $\phi_i(z)$ is defined in (2), and $a_i \in \mathbb{C}$.

Because Chebyshev polynomials satisfy three-term recurrence relation, we can easily obtain the following corollaries from Theorem 3.

Corollary 1. *Let polynomial*

$$P_T(z) = T_n(z) + b_{n-1}T_{n-1}(z) + \cdots + b_1T_1(z) + b_0T_0(z),$$

where $T_i(z)$ is the first Chebyshev polynomial. Then the comrade matrix of $P_T(z)$ is

$$C^{(1)}(P_T) = \begin{pmatrix} 0 & \frac{1}{2} & & & & \frac{-b_0}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & & & \frac{-b_1}{2} \\ & \frac{1}{2} & 0 & \frac{1}{2} & & \frac{-b_2}{2} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \frac{1}{2} & 0 & \frac{-b_{n-2}+1}{2} \\ & & & & \frac{1}{2} & \frac{-b_{n-1}}{2} \end{pmatrix}. \tag{4}$$

Corollary 2. *Let polynomial*

$$P_U(z) = U_n(z) + b_{n-1}U_{n-1}(z) + \cdots + b_1U_1(z) + b_0U_0(z),$$

where $U_i(z)$ is the second Chebyshev polynomial. Then the comrade matrix of $P_U(z)$ is

$$C^{(2)}(P_U) = \begin{pmatrix} 0 & \frac{1}{2} & & & & \frac{-b_0}{2} \\ 1 & 0 & \frac{1}{2} & & & \frac{-b_1}{2} \\ & \frac{1}{2} & 0 & \frac{1}{2} & & \frac{-b_2}{2} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \frac{1}{2} & 0 & \frac{-b_{n-2}+1}{2} \\ & & & & \frac{1}{2} & \frac{-b_{n-1}}{2} \end{pmatrix}. \tag{5}$$

Now, we give the Brauer theorem for the eigenvalues of a matrix and Descartes' rule of signs of polynomial zeros for using in the later.

Theorem 4 ((Brauer theorem) [9])**.** *All the eigenvalues of a matrix* $A=(a_{ij})\in\mathbb{C}^{n\times n}$*,* $n\geq 2$*, are contained in the set of*

$$\Delta(A) = \bigcup_{\substack{i,j\in N \\ i\neq j}} \Delta_{ij}(A) = \bigcup_{\substack{i,j\in N \\ i\neq j}} \left\{z\in\mathbb{C} : |z-a_{ii}|\left|z-a_{jj}\right| \leq r_i(A)r_j(A)\right\},$$

where $r_i(A) = \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|$ *is the i-th deleted absolute row sum of A, and* $N = \{1,2,3,\cdots,n\}$*.* $\Delta(A)$ *is called Brauer set of A.*

Remark 1. *Because A and* A^T *have the same eigenvalues, so we have that all the eigenvalues of A are contained in the following set*

$$\mathcal{K}(A) = \bigcup_{\substack{i,j\in N \\ i\neq j}} \mathcal{K}_{ij}(A) = \bigcup_{\substack{i,j\in N \\ i\neq j}} \left\{z\in\mathbb{C} : |z-a_{ii}|\left|z-a_{jj}\right| \leq c_i(A)c_j(A)\right\},$$

where $c_i(A) = \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ji}\right|$ *is the i-th deleted absolute column sum of A, and* $N = \{1,2,3,\cdots,n\}$*.* $\mathcal{K}(A)$ *is called as Brauer column set of A. It is well to be reminded that Theorem 3 and Theorem 4 are very important and can be applicable to estimate the Estrada index of weighted graphs [10,11].*

Theorem 5. $(Descartes'\ rule\ of\ signs\ of\ polynomial\ zeros)$ [12] *Let* $P(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ *be a polynomial with nonzero real coefficients* a_i *, where the* b_i *are integers satisfying* $0 \neq b_0 < b_1 < b_2 < \cdots < b_n$*. Then the number of positive real zeros of* $P(x)$ $(counted\ with\ multiplicities)$ *is either equal to the number of variations in sign in the sequence* $a_0,\cdots,a_n$ *of the coefficients or less than that by an even whole number.*

2. Brauer-Type Inclusion Sets for Chebyshev Polynomials Zeros

In this section, we use Brauer theorem and the properties of ovals of Cassini to derive a tighter inclusion sets for the zeros of Chebyshev polynomials.

Theorem 6. *Let*

$$P_U(z) = U_n(z) + b_{n-1}U_{n-1}(z) + \cdots + b_1U_1(z) + b_0U_0(z),$$

with $b_j \in \mathbb{C}$*, and* $U_j(z)$ *be the Chebyshev polynomial of the second kind, and let* τ_U *be the largest positive solution of the the following real equation*

$$x^{n+2} - |b_{n-1}|\, x^{n+1} + (2 - |1 - b_{n-2}|)\, x^n - (|b_{n-1}| + |b_{n-3}|)\, x^{n-1}$$
$$+ (1 - |b_{n-4}| - |1 - b_{n-2}|)\, x^{n-2} - \sum_{j=2}^{n-3} \left(|b_j| + |b_{j-2}|\right) x^j - |b_1| x - |b_0| = 0.$$

Then all the zeros of $P_U(z)$ are contained in $\left\{z \in \mathbb{C} : |z| \leq \left(\tau_U + \tau_U^{-1}\right)/2\right\}$.

Proof. According to Corollary 2, the comrade matrix of the polynomial

$$P_U(z) = U_n(z) + b_{n-1}U_{n-1}(z) + \cdots + b_1U_1(z) + b_0U_0(z)$$

is the matrix (5). For a real number $x > 0$, denote $C_x^{(2)}(P_U) = D_x^{-1}C^{(2)}(P_U)D_x$, where D_x is the diagonal matrix with diagonal $\left(x^n, x^{n-1}, \cdots, x\right)$. By simple calculations, we have

$$C_x^{(2)}(P_U) = \begin{pmatrix} 0 & \frac{1}{2x} & & & & \frac{-b_0}{2x^{n-1}} \\ \frac{x}{2} & 0 & \frac{1}{2x} & & & \frac{-b_1}{2x^{n-2}} \\ & \frac{x}{2} & 0 & \frac{1}{2x} & & \frac{-b_2}{2x^{n-3}} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \frac{x}{2} & 0 & \frac{-b_{n-2}+1}{2x} \\ & & & & \frac{x}{2} & \frac{-b_{n-1}}{2} \end{pmatrix}. \tag{6}$$

Here, $C_x^{(2)}(P_U)$ and $C^{(2)}(P_U)$ have the same eigenvalues. The Brauer column set of $C_x^{(2)}(P_U)$ is the union of 2 parts:

$$\left\{z \in \mathbb{C} : |z| \leq \frac{x + x^{-1}}{2}\right\} \bigcup \left\{z \in \mathbb{C} : |z| \left|z + \frac{b_{n-1}}{2}\right| \leq g(x)\right\}$$

where

$$g(x) = \frac{1}{4}\left(|1 - b_{n-2}| + \frac{|b_{n-3}|}{x} + \frac{|b_{n-4}| + |1 - b_{n-2}|}{x^2} + \sum_{j=3}^{n-2} \frac{|b_{n-j-2}| + |b_{n-j}|}{x^j} + \frac{|b_1|}{x^{n-1}} + \frac{|b_0|}{x^n}\right). \tag{7}$$

It is the n-th deleted absolute column sum of $C_x^{(2)}(P_U)$. From [13], We know the fact that the entire oval of Cassini $|z|\left|z + \frac{b_{n-1}}{2}\right| \leq g(x)$ is contained in a circle whose center is 0, radius is

$$r = \frac{1}{2}\left(\left|\frac{b_{n-1}}{2}\right| + \sqrt{\left|\frac{b_{n-1}}{2}\right|^2 + 4g(x)}\right).$$

So the oval of Cassini is encompassed in the disk and will be tangent to it when the value of x satisfies

$$\frac{1}{2}\left(x + x^{-1}\right) = r = \frac{1}{2}\left(\left|\frac{b_{n-1}}{2}\right| + \sqrt{\left|\frac{b_{n-1}}{2}\right|^2 + 4g(x)}\right).$$

Taking the x and multiplying this equation by x^n yields

$$\begin{aligned} &x^{n+2} - |b_{n-1}|\, x^{n+1} + (2 - |1 - b_{n-2}|)\, x^n - (|b_{n-1}| + |b_{n-3}|)\, x^{n-1} \\ &+ (1 - |b_{n-4}| - |1 - b_{n-2}|)\, x^{n-2} - \sum_{j=2}^{n-3} \left(|b_j| + |b_{j-2}|\right) x^j - |b_1| x - |b_0| = 0. \end{aligned} \tag{8}$$

By Theorem 5, this equation have one positive solution. Let τ_U be the largest positive solution of the Equation (8), then all the zeros of $P_U(z)$ are contained in $\left\{z \in \mathbb{C} : |z| \leq \left(\tau_U + \tau_U^{-1}\right)/2\right\}$. $\square$

Remark 2. *Any positive solution of the Equation (8) can be used to get the inclusion sets, but it is the largest one that guarantee the smallest inclusion set because $g(x)$ is a decreasing function of x, for $x > 0$.*

With Theorem 6, we naturally think of making a similar transformation on $C^{(1)}(P_T)$, denoting $C_x^{(1)}(P_T) = D_x^{-1}C^{(1)}(P_T)D_x$, through calculation, We have that

$$C_x^{(1)}(P_T) = \begin{pmatrix} 0 & \frac{1}{2x} & & & & \frac{-b_0}{2x^{n-1}} \\ x & 0 & \frac{1}{2x} & & & \frac{-b_1}{2x^{n-2}} \\ & \frac{x}{2} & 0 & \frac{1}{2x} & & \frac{-b_2}{2x^{n-3}} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \frac{x}{2} & 0 & \frac{-b_{n-2}+1}{2x} \\ & & & & \frac{x}{2} & \frac{-b_{n-1}}{2} \end{pmatrix}.$$

And the Brauer column set of $C_x^{(1)}(P_T)$ is

$$\left\{z \in \mathbb{C} : |z| \leq max\left\{x, \frac{x+x^{-1}}{2}\right\}\right\} \bigcup \left\{z \in \mathbb{C} : |z|\left|z + \frac{b_{n-1}}{2}\right| \leq g(x)\right\},$$

where the radius of the former part $\left\{z \in \mathbb{C} : |z| \leq max\left\{x, \frac{x+x^{-1}}{2}\right\}\right\}$ is a non-smooth function, which makes the subsequent proof relatively complicated. In order to avoid this situation, we use the relation (1) to obtain the following theorem.

Theorem 7. *Let*

$$P_T(z) = T_n(z) + a_{n-1}T_{n-1}(z) + \cdots + a_1T_1(z) + a_0T_0(z),$$

with $a_j \in \mathbb{C}$, and $T_j(z)$ be the Chebyshev polynomial of the first kind, and let τ_T be the largest positive solution of the following real equation

$$\begin{aligned} &x^{n+2} - |a_{n-1}|\, x^{n+1} + (2 - |2 - a_{n-2}|)\, x^n - (|a_{n-1}| + |a_{n-3} - a_{n-1}|)\, x^{n-1} \\ &+ (1 - |a_{n-4} - a_{n-2}| - |2 - a_{n-2}|)\, x^{n-2} - \sum_{j=2}^{n-3} \left(|a_j - a_{j+2}| + |a_{j-2} - a_j|\right) x^j - |a_1 - a_3|x - |2a_0 - a_2| = 0. \end{aligned}$$

Then all the zeros of $P_T(z)$ are contained in $\left\{z \in \mathbb{C} : |z| \leq \left(\tau_T + \tau_T^{-1}\right)/2\right\}$.

Proof. According to the relations of the two kinds of chebyshev polynomials, the polynomial

$$P_T(z) = T_n(z) + a_{n-1}T_{n-1}(z) + \cdots + a_1T_1(z) + a_0T_0(z)$$

can be expressed as

$$P_T(z) = \frac{1}{2}\left(U_n(x) + a_{n-1}U_{n-1}(z) + \sum_{j=1}^{n-2}\left(a_j - a_{j+2}\right)U_j(z) + (2a_0 - a_2)\,U_0(z)\right).$$

By Theorem 6, though changing the corresponding coefficients, we have all the zeros of $P_T(z)$ are contained in $\left\{z \in \mathbb{C} : |z| \leq \left(\tau_T + \tau_T^{-1}\right)/2\right\}$. $\square$

Theorem 8. *Let*

$$P_U(z) = U_n(z) + b_{n-1}U_{n-1}(z) + \cdots + b_1U_1(z) + b_0U_0(z),$$

with $b_j \in \mathbb{C}$, and $U_j(z)$ be the Chebyshev polynomial of the second kind, and let η_U be the largest positive solution of the following real equation

$$x^{n+2} + |b_{n-1}|\, x^{n+1} + (2 - |1 - b_{n-2}|)\, x^n + (|b_{n-1}| - |b_{n-3}|)\, x^{n-1} \\ + (1 - |b_{n-4}| - |1 - b_{n-2}|)\, x^{n-2} - \sum_{j=2}^{n-3} \left(|b_j| + |b_{j-2}|\right) x^j - |b_1|x - |b_0| = 0.$$

Then all the zeros of $P_U(z)$ are contained in

$$\left\{ z \in \mathbb{C} : |z| \left| z + \frac{b_{n-1}}{2} \right| \le \left(|b_{n-1}| \left(\eta_U + {\eta_U}^{-1} \right) + {\eta_U}^2 + {\eta_U}^{-2} + 2 \right) /4 \right\}$$

Proof. In [13], it is given that the point closest to 0 in the oval of Cassini lies at a distance given by $s = \frac{1}{2}\left(-\left|\frac{b_{n-1}}{2}\right| + \sqrt{\left|\frac{b_{n-1}}{2}\right|^2 + 4g(x)} \right)$. Here, we take x to make the oval of Cassini encompass the disk and be tangent to it, thus

$$\frac{1}{2}\left(x + x^{-1}\right) = s = \frac{1}{2}\left(-\left|\frac{b_{n-1}}{2}\right| + \sqrt{\left|\frac{b_{n-1}}{2}\right|^2 + 4g(x)} \right),$$

where $g(x)$ is defined as in (7). Multiplying this equation by x^n yields

$$x^{n+2} + |b_{n-1}|\, x^{n+1} + (2 - |1 - b_{n-2}|)\, x^n + (|b_{n-1}| - |b_{n-3}|)\, x^{n-1} \\ + (1 - |b_{n-4}| - |1 - b_{n-2}|)\, x^{n-2} - \sum_{j=2}^{n-3} \left(|b_j| + |b_{j-2}|\right) x^j - |b_1|x - |b_0| = 0. \tag{9}$$

By Theorem 5, this equation has positive roots. Let η_U be the largest positive solution of Equation (9). All the zeros of $P_U(z)$ must therefore be contained in the following set

$$\left\{ z \in \mathbb{C} : |z| \left| z + \frac{b_{n-1}}{2} \right| \le \left(|b_{n-1}| \left(\eta_U + {\eta_U}^{-1} \right) + {\eta_U}^2 + {\eta_U}^{-2} + 2 \right) /4 \right\}.$$

□

Theorem 9. *Let*

$$P_T(z) = T_n(z) + a_{n-1}T_{n-1}(z) + \cdots + a_1T_1(z) + a_0T_0(z),$$

with $a_j \in \mathbb{C}$, and $T_j(z)$ is the Chebyshev polynomial of the first kind, and let η_T be the largest positive solution of the the following real equation

$$x^{n+2} + |a_{n-1}|\, x^{n+1} + (2 - |2 - a_{n-2}|)\, x^n + (|a_{n-1}| + |a_{n-3} - a_{n-1}|)\, x^{n-1} \\ + (1 - |a_{n-4} - a_{n-2}| - |2 - a_{n-2}|)\, x^{n-2} - \sum_{j=2}^{n-3} \left(|a_j - a_{j+2}| + |a_{j-2} - a_j|\right) x^j - |a_1 - a_3|x - |2a_0 - a_2| = 0.$$

Then all the zeros of $P_T(z)$ are contained in

$$\left\{ z \in \mathbb{C} : |z| \left| z + \frac{a_{n-1}}{2} \right| \le \left(|a_{n-1}| \left(\eta_T + {\eta_T}^{-1} \right) + {\eta_T}^2 + {\eta_T}^{-2} + 2 \right) /4 \right\}.$$

Proof. Similar to Theorem 7, using the relation of the two kinds of chebyshev polynomials, the polynomial

$$P_T(z) = T_n(z) + a_{n-1}T_{n-1}(z) + \cdots + a_1T_1(z) + a_0T_0(z)$$

can be expressed as

$$P_T(z) = \frac{1}{2}\left(U_n(x) + a_{n-1}U_{n-1}(z) + \sum_{j=1}^{n-2}(a_j - a_{j+2})\,U_j(z) + (2a_0 - a_2)\,U_0(z)\right).$$

According to Theorem 8, by changing the corresponding coefficients, we have the fact that all the zeros of $P_T(z)$ are contained in

$$\left\{z \in \mathbb{C} : |z|\left|z + \frac{a_{n-1}}{2}\right| \leq \left(|a_{n-1}|\left(\eta_T + \eta_T{}^{-1}\right) + \eta_T{}^2 + \eta_T{}^{-2} + 2\right)/4\right\}.$$

□

3. Examples

In this section, we give two examples to compare our results with Theorems 1 and 2 given by Melman in [5]. Theorems 7 and 9 are similar to Theorems 6 and 8, therefore, we don't give example on Theorems 7 and 9, here.

Example 1. *Consider the polynomial*

$$P_1(z) = U_5(z) + (-1+2i)U_4(z) + 1.5U_3(z) + 1.1U_2(z) + (2+1i)U_1(z),$$

In Figure 1, the black area is the disk obtained from Theorem 1, the blue area is the Cassini oval obtained from the Theorem 6. The red dots are the zeros of $P_1(z)$. It is easy to see that our result is tighter than Melman's.

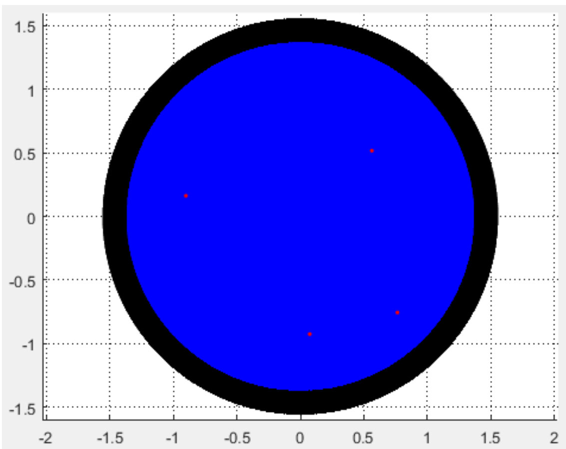

Figure 1. $P_1(z) = U_5(z) + (-1+2i)U_4(z) + 1.5U_3(z) + 1.1U_2(z) + (2+1i)U_1(z)$.

Example 2. *Consider the polynomial*

$$P_2(z) = U_9(z) + 3U_8(z) - (1-2i)U_7(z) - (1-i)U_6(z) + iU_4(z) - U_2(z) + 3iU_1(z) - 2U_0(z),$$

In Figure 2, the black area is the disk obtained from Theorem 2, the blue area is the Cassini oval obtained from the Theorem 8. The red dots are the zeros of $P_2(z)$. Obviously, our result is tighter than Theorem 2.

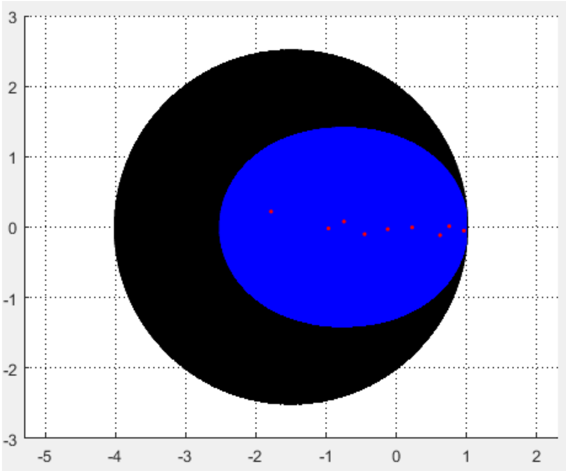

Figure 2. $P_2(z) = U_9(z) + 3U_8(z) - (1-2i)U_7(z) - (1-i)U_6(z) + iU_4(z) - U_2(z) + 3iU_1(z) - 2U_0(z)$.

4. Conclusions

In the paper, several new inclusion sets of zeros for Chebyshev polynomials are presented by applying Brauer theorem about the eigenvalues of the comrade matrix of Chebyshev polynomial and applying the properties of ovals of Cassini. Some examples are given to show that the new inclusion sets are tighter than those provided by Melman (2014) in some cases. We can generalize the results to other polynomials that satisfy three-term recurrence relation such as Newton polynomial. On the other hand, because the system matrix of leader-follower cooperative control in muliti-agentsystems can be viewed as a comrade matrix [14]. Therefore, the results presented in this paper can be used to estimate the consensus rate in these problems. These problems need to be studied in the future. We can generalize our results to other polynomials that satisfy three-term recurrence relation such as Newton polynomial, but that might increase the computation.

Author Contributions: Y.L. put forward the research questions and the framework. The specific calculation, derivation and the writing were completed by X.F.

Funding: This work was supported by National Natural Science Foundation of China (No. 11861077).

Acknowledgments: The authors are grateful to the referees for their useful and constructive suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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