mathematics

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A Subclass of Bi-Univalent Functions Based on the Faber Polynomial Expansions and the Fibonacci Numbers

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Received: 10 December 2018; Accepted: 9 January 2019; Published: 11 February 2019

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Abstract: In this investigation, by using the Komatu integral operator, we introduce the new class $\mathfrak{L}_{\Sigma,t}^{\eta,\rho}(\widetilde{\mathfrak{p}})$ of bi-univalent functions based on the rule of subordination. Moreover, we use the Faber polynomial expansions and Fibonacci numbers to derive bounds for the general coefficient $|a_n|$ of the bi-univalent function class.

Keywords: bi-univalent functions; subordination; Faber polynomials; Fibonacci numbers; Komatu integral operator

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1. Introduction and Preliminaries

Let $\mathbb{C}$ be the complex plane and $D = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Further, let $\mathcal{A}$ represent the class of functions analytic in D, thus satisfying the condition:

$$f(0) = f'(0) - 1 = 0.$$

Then, each of the functions f in $\mathcal{A}$ has the following Taylor series expansion:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Suppose S is a subclass of $\mathcal{A}$ consisting of univalent functions in D.

In the Koebe-One Quarter Theorem, every univalent function f in $\mathcal{A}$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z (z \in D)$ and $f(f^{-1}(\omega)) = \omega \left(|\omega| < r_0(f);\ r_0(f) \geq \frac{1}{4} \right)$, where

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \ldots \tag{2}$$

A function f in $\mathcal{A}$ is said to be bi-univalent in D if both f and f^{-1} are univalent in D. Let Σ indicate the class of bi-univalent functions in D given by Equation (1). For a brief historical account and for several notable investigations of functions in Σ, see the pioneering work on this subject by Srivastava et al. [1] (see also [2–5]). The interest on the estimates for the first two coefficients $|a_2|, |a_3|$ of the bi-univalent functions continues to attract many researchers (for examples, see [6–9]). However, determination of the bounds for a_n is a remarkable problem in Geometric Function Theory. The coefficient estimate problem for each of the general coefficients $|a_n| (n \in \mathbb{N}\backslash\{1, 2\};\ \mathbb{N} = \{1, 2, 3, \ldots\})$ is still an open problem.

The Faber polynomials established by Faber [10] play a crucial role in numerous areas of mathematical sciences, such as Geometric Function Theory. Grunsky [11] established some sufficient conditions for the univalency of a given function, and in these conditions, the expansions of the Faber polynomials play an important role.

By utilizing the Faber polynomial expansions for a function f in $\mathcal{A}$, the coefficients of its inverse map $g = f^{-1}$ can be stated by (see [12,13]):

$$g(\omega) = f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots)\, \omega^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] \\ &+ \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that $V_j (7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$. In the following, the first three terms of K_{n-1}^{-n} are stated by:

$$\begin{aligned} \tfrac{1}{2} K_1^{-2} &= -a_2, \\ \tfrac{1}{3} K_2^{-3} &= 2a_2^2 - a_3, \\ \tfrac{1}{4} K_3^{-4} &= -\left(5a_2^3 - 5a_2 a_3 + a_4\right). \end{aligned}$$

In general, the expansion of $K_n^p (p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\})$ is stated by:

$$K_n^p = p a_n + \frac{p(p-1)}{2} \mathcal{G}_n^2 + \frac{p\,!}{(p-3)\,!\,3\,!} \mathcal{G}_n^3 + \ldots + \frac{p\,!}{(p-n)\,!\,n\,!} \mathcal{G}_n^n,$$

where $\mathcal{G}_n^p = \mathcal{G}_n^p(a_1, a_2, \ldots)$ and by [14]:

$$\mathcal{G}_n^m(a_1, a_2, \ldots, a_n) = \sum_{n=1}^{\infty} \frac{m\,!(a_1)^{\delta_1} \ldots (a_n)^{\delta_n}}{\delta_1! \ldots \delta_n!},$$

while $a_1 = 1$, the sum is taken over all nonnegative integers $\delta_1, \ldots, \delta_n$ satisfying:

$$\begin{aligned} \delta_1 + \delta_2 + \ldots + \delta_n &= m, \\ \delta_1 + 2\delta_2 + \ldots + n\delta_n &= n. \end{aligned}$$

The first and the last polynomials are:

$$\mathcal{G}_n^1 = a_n, \mathcal{G}_n^n = a_1^n.$$

Several authors worked on using Faber polynomial expansions to find coefficient bounds for functions in Σ, see [15–18] for examples.

Next, we recall some definitions and lemmas used in this paper.

Definition 1. *Let the functions f, g be analytic in D. A function f is subordinate to g, indicated as $f \prec g$, if there exists a Schwarz function:*

$$\varpi(z) = \sum_{n=1}^{\infty} \mathfrak{c}_n z^n (\varpi(0) = 0,\ |\varpi(z)| < 1),$$

analytic in D such that:

$$f(z) = g(\varpi(z))(z \in D).$$

Lemma 1. *(See [19]) For two analytic functions* $\mathfrak{u}(z)$, $\mathfrak{v}(\omega)(\mathfrak{u}(0) = \mathfrak{v}(0) = 0, |\mathfrak{u}(z)| < 1, |\mathfrak{v}(\omega)| < 1)$, suppose that:

$$\mathfrak{u}(z) = \sum_{n=1}^{\infty} x_n z^n (z \in D),$$
$$\mathfrak{v}(\omega) = \sum_{n=1}^{\infty} y_n \omega^n (\omega \in D).$$

Then:

$$|x_1| \leq 1,\ |x_2| \leq 1 - |x_1|^2,\ |y_1| \leq 1,\ |y_2| \leq 1 - |y_1|^2. \tag{3}$$

Definition 2. *(See [20]) The Komatu integral operator of* $f \in \mathcal{A}$ *is denoted by* $\mathfrak{L}_{\mathfrak{t}}^{\eta} f(z)$ *and defined by:*

$$\begin{aligned} \mathfrak{L}_{\mathfrak{t}}^{\eta} f(z) &= z + \sum_{n=2}^{\infty} \left(\tfrac{t}{t+n-1}\right)^{\eta} a_n z^n (t > 0, \eta \geq 0, z \in D) \\ &= \tfrac{t^{\eta}}{\Gamma(\eta)} \int_0^1 \xi^{t-2} \left(\log \tfrac{1}{\xi}\right)^{\eta-1} f(z\xi) d\xi. \end{aligned}$$

It is easy to verify that:

$$\mathfrak{L}_{\mathfrak{t}}^{\eta}(zf'(z)) = z\left(\mathfrak{L}_{\mathfrak{t}}^{\eta} f(z)\right)'.$$

By suitably specializing the parameters η *and* t we obtain the following operators studied by various authors:

(i) $\mathfrak{L}_{\mathfrak{t}}^{0} f(z) = f(z)$;
(ii) $\mathfrak{L}_{2}^{1} f(z) = A[f](z)$ called Libera operator [21];
(iii) $\mathfrak{L}_{1}^{-k} f(z) = D^k f(z)$ $(k \in \mathbb{N}_0 = (0, 1, 2, \ldots))$ called Salagean differential operator [22];
(iv) $\mathfrak{L}_{2}^{-k} f(z) = \mathfrak{L} f(z)$ $(k \in \mathbb{N}_0 = (0, 1, 2, \ldots))$ was studied by Uralegaddi and Somanatha [23];
(v) $\mathfrak{L}_{2}^{\eta} f(z) = I^{\eta} f(z)$ was studied by Jung et al. [24].

Using the operator $\mathfrak{L}_{t}^{\eta}$, *we now establish the class* $\mathfrak{L}_{\Sigma,\mathfrak{t}}^{\eta,\rho}(\widetilde{\mathfrak{p}})$

Definition 3. *A function* $f \in \Sigma$ *is said to be in the class:*

$$\mathfrak{L}_{\Sigma,\mathfrak{t}}^{\eta,\rho}(\widetilde{\mathfrak{p}})(\rho \geq 1, t > 0,\ \eta \geq 0,\ z, \omega \in D)$$

if the following subordination relationships are satisfied:

$$\left[(1-\rho)\frac{\mathfrak{L}_{t}^{\eta} f(z)}{z} + \rho\left(\mathfrak{L}_{t}^{\eta} f(z)\right)'\right] \prec \widetilde{\mathfrak{p}}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$$

and

$$\left[(1-\rho)\frac{\mathfrak{L}_{t}^{\eta} g(\omega)}{\omega} + \rho\left(\mathfrak{L}_{t}^{\eta} g(\omega)\right)'\right] \prec \widetilde{\mathfrak{p}}(\omega) = \frac{1+\tau^2 \omega^2}{1-\tau \omega-\tau^2 \omega^2},$$

where the function g is given by (2) and $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

Remark 1. *The function* $\widetilde{\mathfrak{p}}(z)$ *is not univalent in D, but it is univalent in the disc* $|z| < \frac{3-\sqrt{5}}{2} \approx -0.38$. *For example,* $\widetilde{\mathfrak{p}}(0) = \mathfrak{p}\left(-\frac{1}{2\tau}\right)$ *and* $\widetilde{\mathfrak{p}}\left(e^{\pm i\arccos(1/4)}\right) = \frac{\sqrt{5}}{5}$. *Also, it can be written as:*

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which indicates that the number $|\tau|$ divides [0,1] so that it fulfills the golden section (see for details Dziok et al. [25]).

Additionally, Dziok et al. [25] indicate a useful connection between the function $\widetilde{\mathfrak{p}}(z)$ and the Fibonacci numbers. Let $\{\Lambda_n\}$ be the sequence of Fibonacci numbers:

$$\Lambda_0 = 0,\ \Lambda_1 = 1,\ \Lambda_{n+2} = \Lambda_n + \Lambda_{n+1} (n \in \mathbb{N}_0 = \{0,1,2,\ldots\}),$$

then

$$\Lambda_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}},\ \tau = \frac{1-\sqrt{5}}{2}.$$

If we set:

$$\widetilde{\mathfrak{p}}(z) = 1 + \sum_{n=1}^{\infty} \widetilde{\mathfrak{p}}_{\mathfrak{n}} z^n = 1 + (\Lambda_0 + \Lambda_2)\tau z + (\Lambda_1 + \Lambda_3)\tau^2 z^2 + \sum_{n=3}^{\infty} (\Lambda_{n-3} + \Lambda_{n-2} + \Lambda_{n-1} + \Lambda_n)\tau^n z^n,$$

then the coefficients $\widetilde{\mathfrak{p}}_n$ satisfy:

$$\widetilde{\mathfrak{p}}_{\mathfrak{n}} = \begin{cases} \tau & (n=1) \\ 3\tau^2 & (n=2) \\ \tau\widetilde{\mathfrak{p}}_{n-1} + \tau^2\widetilde{\mathfrak{p}}_{n-2} & (n=3,\ 4,\ \ldots) \end{cases}. \tag{4}$$

In this paper, we study the new class $\mathfrak{L}^{\eta,\rho}_{\Sigma,\mathfrak{t}}(\widetilde{\mathfrak{p}})$ of bi-univalent functions established by using the Komatu integral operator. Furthermore, we use the Faber polynomial expansions and Fibonacci numbers to derive bounds for the general coefficient $|a_n|$ of the bi-univalent function class.

2. Main Result and Its Consequences

First, we get a bound for the general coefficients of functions in $\mathfrak{L}^{\eta,\rho}_{\Sigma,\mathfrak{t}}(\widetilde{\mathfrak{p}})$.

Theorem 1. *For $\rho \geq 1,\ t > 0$ and $\eta \geq 0$, let the function f given by (1) be in the function class $\mathfrak{L}^{\eta,\rho}_{\Sigma,\mathfrak{t}}(\widetilde{\mathfrak{p}})$. If $a_m = 0 (2 \leq m \leq n-1)$, then:*

$$|a_n| \leq \frac{|\tau|}{[1+(n-1)\rho]\ \left(\frac{t}{t+n-1}\right)^{\eta}} (n \geq 3).$$

Proof. By the definition of subordination yields:

$$\left[(1-\rho)\frac{\mathfrak{L}^{\eta}_t f(z)}{z} + \rho\left(\mathfrak{L}^{\eta}_t f(z)\right)'\right] \prec \widetilde{\mathfrak{p}}(\mathfrak{u}(z)) \tag{5}$$

and

$$\left[(1-\rho)\frac{\mathfrak{L}^{\eta}_t g(\omega)}{\omega} + \rho\left(\mathfrak{L}^{\eta}_t g(\omega)\right)'\right] \prec \widetilde{\mathfrak{p}}(\mathfrak{v}(\omega)). \tag{6}$$

Now, an application of Faber polynomial expansion to the power series $\mathfrak{L}^{\eta,\rho}_{\Sigma,\mathfrak{t}}(\widetilde{\mathfrak{p}})$ (for examples, see [12,13]) yields:

$$(1-\rho)\frac{\mathfrak{L}_t^{\eta} f(z)}{z}+\rho\Big(\mathfrak{L}_t^{\eta} f(z)\Big)'=1+\sum_{n=2}^{\infty}\mathcal{F}_{n-1}(a_2,a_3,\ldots,a_n)z^{n-1},$$

where

$$\mathcal{F}_{n-1}(a_2,a_3,\ldots,a_n)=[1+(n-1)\rho]\ \left(\tfrac{t}{t+n-1}\right)^{\eta} \times \sum_{i_1+2i_2+\ldots+(n-1)i_{n-1}=n-1}\frac{(1-(i_1+i_2+\ldots+i_{n-1}))!\Big[(a_2)^{i_1}(a_3)^{i_2}\ldots(a_n)^{i_{n-1}}\Big]}{(i_1!)(i_2!)\ldots(i_{n-1}!)}.$$

In particular, the first two terms are $\mathcal{F}_1=(1+\rho)\ \left(\frac{t}{t+1}\right)^{\eta}a_2,\ \mathcal{F}_2=(1+2\rho)\ \left(\frac{t}{t+2}\right)^{\eta}a_3$.

By the same token, for its inverse map $g=f^{-1}$, it is seen that:

$$\begin{aligned}(1-\rho)\frac{\mathfrak{L}_t^{\eta} g(\omega)}{\omega}+\rho\Big(\mathfrak{L}_t^{\eta} g(\omega)\Big)' &=1+\sum_{n=2}^{\infty}[1+(n-1)\rho]\ \left(\tfrac{t}{t+n-1}\right)^{\eta}\times\tfrac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots)\omega^{n-1}\\ &=1+\sum_{n=2}^{\infty}\mathcal{F}_{n-1}(b_2,b_3,\ldots,b_n)\omega^{n-1}.\end{aligned}$$

Next, the Equations (5) and (6) lead to:

$$\begin{aligned}\widetilde{\mathfrak{p}}(\mathfrak{u}(z)) &=1+\widetilde{\mathfrak{p}}_1(\mathfrak{u}(z))+\widetilde{\mathfrak{p}}_2(\mathfrak{u}(z))^2+\ldots\\ &=1+\widetilde{\mathfrak{p}}_1x_1z+\left(\widetilde{\mathfrak{p}}_1x_2+\widetilde{\mathfrak{p}}_2x_1^2\right)z^2+\ldots\\ &=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\widetilde{\mathfrak{p}}_k\mathcal{G}_n^k(x_1,x_2,\ldots,x_n)z^n,\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathfrak{p}}(\mathfrak{v}(\omega)) &=1+\widetilde{\mathfrak{p}}_1(\mathfrak{v}(\omega))+\widetilde{\mathfrak{p}}_2(\mathfrak{v}(\omega))^2+\ldots\\ &=1+\widetilde{\mathfrak{p}}_1x_1\omega+\left(\widetilde{\mathfrak{p}}_1y_2+\widetilde{\mathfrak{p}}_2y_1^2\right)\omega^2+\ldots\\ &=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\widetilde{\mathfrak{p}}_k\mathcal{G}_n^k(y_1,y_2,\ldots,y_n)\omega^n.\end{aligned}$$

Comparing the corresponding coefficients of (5) and (6) yields:

$$\begin{aligned}&[1+(n-1)\rho]\ \left(\tfrac{t}{t+n-1}\right)^{\eta}a_n=\widetilde{\mathfrak{p}}_1x_{n-1},\\ &[1+(n-1)\rho]\ \left(\tfrac{t}{t+n-1}\right)^{\eta}b_n=\widetilde{\mathfrak{p}}_1y_{n-1}.\end{aligned}$$

For $a_m=0\ (2\leq m\leq n-1)$, we get $b_n=-a_n$ and thus:

$$[1+(n-1)\rho]\ \left(\frac{t}{t+n-1}\right)^{\eta}a_n=\widetilde{\mathfrak{p}}_1x_{n-1} \tag{7}$$

and

$$-\,[1+(n-1)\rho]\ \left(\frac{t}{t+n-1}\right)^{\eta}a_n=\widetilde{\mathfrak{p}}_1y_{n-1}. \tag{8}$$

Now, taking the absolute values of either of the two equations written above and from (4), we obtain:

$$|a_n|\leq\frac{|\tau|}{[1+(n-1)\rho]\ \left(\frac{t}{t+n-1}\right)^{\eta}}.$$

Relaxing the coefficient restrictions imposed in Theorem 1, we obtain the following initial coefficient bounds for functions in $\mathfrak{L}_{\Sigma,\mathfrak{t}}^{\eta,\rho}(\widetilde{\mathfrak{p}})$. □

Theorem 2. *Let* $f\in\mathfrak{L}_{\Sigma,\mathfrak{t}}^{\eta,\rho}(\widetilde{\mathfrak{p}})$. *Then:*

$$|a_2| \le \min\left\{ \frac{|\tau|}{\sqrt{\left|(1+2\rho)\,\left(\frac{t}{t+2}\right)^{\eta} - 3(1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}\right||\tau| + (1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}}}, |\tau|\sqrt{\frac{3}{(1+2\rho)\,\left(\frac{t}{t+2}\right)^{\eta}}} \right\}$$

and

$$\begin{aligned} |a_3| \quad &\le \min\left\{ \frac{3\tau^2}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}, \right. \\ &\left. \frac{|\tau|}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}\left[1 + \frac{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}|\tau| - (1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}}{\left|(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} - 3(1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}\right||\tau| + (1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}}\right]\right\} \end{aligned}$$

Proof. Substituting n by 2 and 3 in (7) and (8), respectively, we find that:

$$(1+\rho)\left(\frac{t}{t+1}\right)^{\eta} a_2 = \widetilde{\mathfrak{p}}_1 x_1, \tag{9}$$

$$(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} a_3 = \widetilde{\mathfrak{p}}_1 x_2 + \widetilde{\mathfrak{p}}_2 x_1^2, \tag{10}$$

$$-(1+\rho)\left(\frac{t}{t+1}\right)^{\eta} a_2 = \widetilde{\mathfrak{p}}_1 y_1, \tag{11}$$

$$(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}\left(2a_2^2 - a_3\right) = \widetilde{\mathfrak{p}}_1 y_2 + \widetilde{\mathfrak{p}}_2 y_1^2. \tag{12}$$

Obviously, we obtain:

$$x_1 = -y_1. \tag{13}$$

If we add the Equation (12) to (10) and use (13), we get:

$$2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} a_2^2 = \widetilde{\mathfrak{p}}_1(x_1 + y_2) + 2\widetilde{\mathfrak{p}}_2 x_1^2. \tag{14}$$

Using the value of x_1^2 from (9), we get:

$$\left[2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}\widetilde{\mathfrak{p}}_1^2 - 2(1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}\widetilde{\mathfrak{p}}_2\right] a_2^2 = \widetilde{\mathfrak{p}}_1^3(x_2 + y_2). \tag{15}$$

Combining (15) and (3), we obtain:

$$\begin{aligned} 2\left|(1+2\rho)\,\left(\tfrac{t}{t+2}\right)^{\eta}\widetilde{\mathfrak{p}}_1^2 - (1+\rho)^2\,\left(\tfrac{t}{t+1}\right)^{2\eta}\widetilde{\mathfrak{p}}_2\right||a_2|^2 \quad &\le |\widetilde{\mathfrak{p}}_1|^3(|x_2| + |y_2|) \\ &\le 2|\widetilde{\mathfrak{p}}_1|^3\left(1 - |x_1|^2\right) \\ &= 2|\widetilde{\mathfrak{p}}_1|^3 - 2|\widetilde{\mathfrak{p}}_1|^3|x_1|^2. \end{aligned}$$

It follows from (9) that:

$$|a_2| \le \frac{|\tau|}{\sqrt{\left|(1+2\rho)\,\left(\frac{t}{t+2}\right)^{\eta} - 3(1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}\right||\tau| + (1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}}}. \tag{16}$$

Additionally, by (3) and (14):

$$\begin{aligned} 2(1+2\rho)\,\left(\tfrac{t}{t+2}\right)^{\eta}|a_2|^2 \quad &\le |\widetilde{\mathfrak{p}}_1|(|x_2| + |y_2|) + 2|\widetilde{\mathfrak{p}}_2||x_1|^2 \\ &\le 2|\widetilde{\mathfrak{p}}_1|\left(1 - |x_1|^2\right) + 2|\widetilde{\mathfrak{p}}_2||x_1|^2 \\ &= 2|\widetilde{\mathfrak{p}}_1| + 2|x_1|^2(|\widetilde{\mathfrak{p}}_2| - |\widetilde{\mathfrak{p}}_1|). \end{aligned}$$

Since $|\widetilde{\mathfrak{p}}_2| > |\widetilde{\mathfrak{p}}_1|$, we get:

$$|a_2| \leq |\tau| \sqrt{\frac{3}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}}.$$

Next, in order to derive the bounds on $|a_3|$, by subtracting (12) from (10), we may obtain:

$$2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} a_3 = 2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} a_2^2 + \widetilde{\mathfrak{p}}_1(x_2 - y_2). \tag{17}$$

Evidently, from (14), we state that:

$$a_3 = \frac{\widetilde{\mathfrak{p}}_1(x_2+y_2) + 2\widetilde{\mathfrak{p}}_2 x_1^2}{2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}} + \frac{\widetilde{\mathfrak{p}}_1(x_2-y_2)}{2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}} = \frac{\widetilde{\mathfrak{p}}_1 x_2 + \widetilde{\mathfrak{p}}_2 x_1^2}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}$$

and consequently

$$|a_3| \leq \frac{|\widetilde{\mathfrak{p}}_1||x_2| + |\widetilde{\mathfrak{p}}_2||x_1|^2}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}} \leq \frac{|\widetilde{\mathfrak{p}}_1|\left(1-|x_1|^2\right) + |\widetilde{\mathfrak{p}}_2||x_1|^2}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}} = \frac{|\widetilde{\mathfrak{p}}_1| + |x_1|^2(|\widetilde{\mathfrak{p}}_2| - |\widetilde{\mathfrak{p}}_1|)}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}$$

Since $|\widetilde{\mathfrak{p}}_2| > |\widetilde{\mathfrak{p}}_1|$, we must write:

$$|a_3| \leq \frac{3\tau^2}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}.$$

On the other hand, by (3) and (17), we have:

$$\begin{aligned} 2(1+2\rho)\left(\tfrac{t}{t+2}\right)^{\eta}|a_3| &\leq 2(1+2\rho)\left(\tfrac{t}{t+2}\right)^{\eta}|a_2|^2 + |\widetilde{\mathfrak{p}}_1|(|x_2|+|y_2|) \\ &\leq 2(1+2\rho)\left(\tfrac{t}{t+2}\right)^{\eta}|a_2|^2 + 2|\widetilde{\mathfrak{p}}_1|\left(1-|x_1|^2\right). \end{aligned}$$

Then, with the help of (9), we have:

$$(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}|a_3| \leq \left[(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} - \frac{(1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}}{|\widetilde{\mathfrak{p}}_1|}\right]|a_2|^2 + |\widetilde{\mathfrak{p}}_1|.$$

By considering (16), we deduce that:

$$|a_3| \leq \frac{|\tau|}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}\left\{1 + \frac{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}|\tau| - (1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}}{\left|(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} - 3(1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}\right||\tau| + (1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}}\right\}.$$

□

3. Conclusions

Our motivation is to get many interesting and fruitful usages of a wide variety of Fibonacci numbers in Geometric Function Theory. We introduced and investigated a new subclass of bi-univalent functions related to Komatu integral operator connected with Fibonacci numbers to obtain the estimates of the general coefficient $|a_n|$ of the bi-univalent function class $\mathfrak{L}_{\Sigma,t}^{\eta,\rho}(\widetilde{\mathfrak{p}})$. Furthermore, we obtained second and third Taylor-Maclaurin coefficients of functions in this class. These results were an improvement on the estimates obtained in the recent studies. Some interesting remarks of the results presented here were also discussed.

The geometric properties of the function class $\mathfrak{L}_{\Sigma,t}^{\eta,\rho}(\widetilde{\mathfrak{p}})$ vary according to the values assigned to the parameters. However, some results for the special cases of the parameters included could be expressed as illustrative examples. The image of the unit circle $|z| = 1$ under $\widetilde{\mathfrak{p}}(z)$ is a curve identified by the following equation:

$$\left(10x - \sqrt{5}\right)y^2 = \left(\sqrt{5} - 2x\right)\left(\sqrt{5}x - 1\right)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\widetilde{\mathfrak{p}}\left(re^{it}\right)$ is a closed curve without any loops for $0 < r \leq r_0 = \left(3 - \sqrt{5}\right)/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $\tau = 1$, it has a vertical asymptote. Since τ fulfills the Equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be solved all the way down to a linear combination of τ and 1. This recurrence relationships yield Fibonacci numbers Λ_n:

$$\tau^n = \Lambda_n \tau + \Lambda_{n-1}.$$

In this way, one can introduce and study different subclasses of the function class $\mathfrak{L}_{\Sigma,t}^{\eta,\rho}(\widetilde{\mathfrak{p}})$, which we studied in this paper.

Author Contributions: All three authors contributed equally to this work. All authors read and approved the final version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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