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Article

Weak and Strong Convergence Theorems for the Inclusion Problem and the Fixed-Point Problem of Nonexpansive Mappings

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Abstract: In this work, we study the inclusion problem of the sum of two monotone operators and the fixed-point problem of nonexpansive mappings in Hilbert spaces. We prove the weak and strong convergence theorems under some weakened conditions. Some numerical experiments are also given to support our main theorem.

Keywords: strong convergence; weak convergence; fixed point; nonexpansive mappings; maximal monotone operator; inverse strongly monotone mapping; hilbert space

1. Introduction

Let H be a real Hilbert space. We study the following inclusion problem: find $\hat{x} \in H$ such that

$$0 \in A\hat{x} + B\hat{x} \tag{1}$$

where $A : H \to H$ is an operator and $B : H \to 2^H$ is a set-valued operator.

If $A := \nabla F$ and $B := \partial G$, where ∇F is the gradient of F and G is the subdifferential of G which is defined by

$$\partial G(x) = \{z \in H : \langle y - x, z\rangle + G(x) \leq G(y), \forall y \in H\}. \tag{2}$$

Then problem (1) becomes the following minimization problem:

$$\min_{x \in H} F(x) + G(x) \tag{3}$$

To solve the inclusion problem via fixed-point theory, let us define, for $r > 0$, the mapping $T_r : H \to H$ as follows:

$$T_r = (I + rB)^{-1}(I - rA). \tag{4}$$

It is known that solutions of the inclusion problem involving A and B can be characterized via the fixed-point equation:

$$\begin{aligned} T_r x = x \quad &\Leftrightarrow \quad x = (I + rB)^{-1}(x - rAx) \\ &\Leftrightarrow \quad x - rAx \in x + rBx \\ &\Leftrightarrow \quad 0 \in Ax + Bx, \end{aligned}$$

which suggests the following iteration process: $x_1 \in H$ and

$$x_{n+1} = (I + r_n B)^{-1}(x_n - r_n A x_n), n \geq 1, \tag{5}$$

where $\{r_n\} \subset (0, \infty)$.

Xu [1] and Kamimura-Takahashi [2] introduced the following inexact iteration process: $u, x_1 \in H$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n + e_n, n \geq 1, \tag{6}$$

where $\{\alpha_n\} \subset (0,1), \{r_n\} \subset (0, \infty), \{e_n\} \subset H$ and $J_{r_n} = (I + r_n B)^{-1}$. Strong convergence was proved under some mild conditions. This scheme was also investigated subsequently by [3–5] with different conditions. In [6], Yao-Noor proposed the generalized version of the scheme (6) as follows: $u, x_1 \in H$ and

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n) J_{r_n} x_n + e_n, n \geq 1, \tag{7}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ with $0 \leq \alpha_n + \beta_n \leq 1, \{r_n\} \subset (0, \infty)$ and $\{e_n\} \subset H$. The strong convergence is discussed with some suitable conditions. Recently, Wang-Cui [7] also studied the contraction-proximal point algorithm (7) by the relaxed conditions on parameters: $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty, \limsup_{n\to\infty} \beta_n < 1, \liminf_{n\to\infty} r_n > 0$, and either $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\frac{\|e_n\|}{\alpha_n} \to 0$.

Takahashi et al. [8] introduced the following Halpern-type iteration process: $u, x_1 \in H$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n}(x_n - r_n A x_n), n \geq 1, \tag{8}$$

where $\{\alpha_n\} \subset (0,1), \{r_n\} \subset (0, \infty)$, A is an α-inverse strongly monotone operator on H and B is a maximal monotone operator on H. They proved that $\{x_n\}$ defined by (8) strongly converges to zeroes of $A + B$ if the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(iv) $0 < a \leq r_n < 2\alpha$.

Takahashi et al. [8] also studied the following iterative scheme: $u, x_1 \in H$ and

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n) J_{r_n}(x_n - r_n A x_n)), n \geq 1, \tag{9}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0, \infty)$. They proved that $\{x_n\}$ defined by (9) strongly converges to zeroes of $A + B$ if the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \leq c < 1$;

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $0 < a \leq r_n < 2\alpha$.

There have been, in the literature, many methods constructed to solve the inclusion problem for maximal monotone operators in Hilbert or Banach spaces; see, for examples, in [9–11].

Let C be a nonempty, closed, and convex subset in a Hilbert space H and let T be a nonexpansive mapping of C into itself, that is,

$$\|Tx - Ty\| \leq \|x - y\| \tag{10}$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T.

The iteration procedure of Mann's type for approximating fixed points of a nonexpansive mapping T is the following: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \qquad n \geq 1 \tag{11}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

On the other hand, the iteration procedure of Halpern's type is the following: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n, \qquad n \geq 1. \tag{12}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Recently, Takahashi et al. [12] proved the following theorem for solving the inclusion problem and the fixed-point problem of nonexpansive mappings.

Theorem 1. *[12] Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let T be a nonexpansive mapping of C into itself such that $F(T) \cap (A + B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T(\alpha_n x + (1 - \alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n)) \tag{13}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < 2\alpha\,, \qquad 0 < c \leq \beta_n \leq d < 1,$$
$$\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0, \lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point of $F(T) \cap (A + B)^{-1}0$.

In this paper, motivated by Takahashi et al. [13] and Halpern [14], we introduce an iteration of finding a common point of the set of fixed points of nonexpansive mappings and the set of inclusion problems for inverse strongly monotone mappings and maximal monotone operators by using the inertial technique (see, [15,16]). We then prove strong and weak convergence theorems under suitable conditions. Finally, we provide some numerical examples to support our iterative methods.

2. Preliminaries

In this section, we provide some basic concepts, definitions, and lemmas which will be used in the sequel. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. When x_n is a sequence in H, $x_n \rightharpoonup x$ implies that $\{x_n\}$ converges weakly to x and $x_n \to x$ means the strong convergence. In a real Hilbert space, we have

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2, \tag{14}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

We know the following Opial's condition:

$$\liminf_{n\to\infty} \|x_n - u\| < \liminf_{n\to\infty} \|x_n - v\| \tag{15}$$

if $x_n \rightharpoonup u$ and $u \neq v$.

Let C be a nonempty, closed, and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. The operator P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, for all $x, y \in H$

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle. \tag{16}$$

or equivalently

$$\|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|(I - P_C)x - (I - P_C)y\|^2. \tag{17}$$

It is well known that P_C is characterized by the inequality, for all $x \in H$ and $y \in C$

$$\langle x - P_C x, y - P_C x \rangle \leq 0. \tag{18}$$

In a real Hilbert space H, we have the following equality:

$$\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2. \tag{19}$$

and the subdifferential inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \tag{20}$$

for all $x, y \in H$.

Let $\alpha > 0$. A mapping $A : C \to H$ is said to be α-inverse strongly monotone iff

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \tag{21}$$

for all $x, y \in C$.

A mapping $f : H \to H$ is said to be a contraction if there exists $a \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq a\|x - y\| \tag{22}$$

for all $x, y \in H$.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H iff $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in dom(B)$, $u \in Bx$ and $v \in By$. A monotone operator B on H is said to be maximal iff its graph is not strictly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we define a single-valued operator $J_r = (I + rB)^{-1} : H \to dom(B)$, which is called the resolvent of B for r.

Lemma 1. *[17] Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \ \forall n \geq 1 \tag{23}$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.

(ii) If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} b_n/\delta_n \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2. *[17] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\psi(n)\}_{n\geq n_0}$ of integers as follows:*

$$\psi(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \tag{24}$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\psi(n_0) \leq \psi(n_0+1) \leq ...$ and $\psi(n) \to \infty$,

(ii) $\Gamma_{\psi(n)} \leq \Gamma_{\psi(n)+1}$ and $\Gamma_n \leq \Gamma_{\psi(n)+1}$, $\forall n \geq n_0$.

Lemma 3. *[18] Let H be a Hilbert space and $\{x_n\}$ a sequence in H such that there exists a nonempty set $S \subset H$ satisfying:*

(i) For every $\tilde{x} \in S$, $\lim_{n\to\infty} \|x_n - \tilde{x}\|$ exists.

(ii) Any weak cluster point of $\{x_n\}$ belongs to S.

Then, there exists $\tilde{x} \in S$ such that $\{x_n\}$ weakly converges to $\tilde{x}$.

Lemma 4. *[18] Let $\{\phi_n\} \subset [0,\infty)$ and $\{\delta_n\} \subset [0,\infty)$ verify:*

(i) $\phi_{n+1} - \phi_n \leq \theta_n(\phi_n - \phi_{n-1}) + \delta_n$,

(ii) $\sum_{n=1}^{\infty} \delta_n < \infty$,

(iii) $\{\theta_n\} \subset [0,\theta]$, where $\theta \in [0,1]$.

Then $\{\phi_n\}$ is a converging sequence and $\sum_{n=1}^{\infty} [\phi_{n+1} - \phi_n]_+ < \infty$, where $[t]_+ := max\{t,0\}$ $(for\ any\ t \in \mathbb{R})$.

3. Strong Convergence Theorem

In this section, we are now ready to prove the strong convergence theorem in Hilbert spaces.

Theorem 2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of H into itself and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping of C into itself such that $F(S) \cap (A+B)^{-1}0 \neq \emptyset$. Let $f : C \to C$ be a contraction. Let $x_0, x_1 \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)S(\alpha_n f(x_n) + (1-\alpha_n)J_{\lambda_n}(y_n - \lambda_n A y_n)) \end{aligned} \tag{25}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,2\alpha)$ and $\{\theta_n\} \subset [0,\theta]$, where $\theta \in [0,1)$ satisfy

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(C3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$;

(C4) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then $\{x_n\}$ *converges strongly to a point of* $F(S) \cap (A+B)^{-1}0$.

Proof. Let $z = P_{F(S)\cap(A+B)^{-1}0} f(z)$. Then $z = J_{\lambda_n}(z - \lambda_n Az)$ for all $n \geq 1$. It follows that by the firm nonexpansivity of J_{λ_n},

$$\begin{aligned}
\|J_{\lambda_n}(y_n - \lambda_n Ay_n) - z\|^2 &= \|J_{\lambda_n}(y_n - \lambda_n Ay_n) - J_{\lambda_n}(z - \lambda_n Az)\|^2 \\
&\leq \|(y_n - \lambda_n Ay_n) - (z - \lambda_n Az)\|^2 \\
&\quad - \|(I - J_{\lambda_n})(y_n - \lambda_n Ay_n) - (I - J_{\lambda_n})(z - \lambda_n Az)\|^2 \\
&= \|(y_n - z) - \lambda_n(Ay_n - Az)\|^2 \\
&\quad - \|y_n - \lambda_n Ay_n - J_{\lambda_n}(y_n - \lambda_n Ay_n) - z + \lambda_n Az + J_{\lambda_n}(z - \lambda_n Az)\|^2 \\
&= \|(y_n - z) - \lambda_n(Ay_n - Az)\|^2 \\
&\quad - \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\|^2 \\
&= \|y_n - z\|^2 - 2\lambda_n\langle y_n - z, Ay_n - Az\rangle + \lambda_n^2\|Ay_n - Az\|^2 \\
&\quad - \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\|^2 \\
&\leq \|y_n - z\|^2 - 2\lambda_n\alpha\|Ay_n - Az\|^2 + \lambda_n^2\|Ay_n - Az\|^2 \\
&\quad - \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\|^2 \\
&= \|y_n - z\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ay_n - Az\|^2 \\
&\quad - \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\|^2.
\end{aligned} \tag{26}$$

By (C3), we obtain

$$\|J_{\lambda_n}(y_n - \lambda_n Ay_n) - z\| \leq \|y_n - z\|. \tag{27}$$

On the other hand, since $y_n = x_n + \theta_n(x_n - x_{n-1})$, it follows that

$$\begin{aligned}
\|y_n - z\| &= \|x_n - z + \theta_n(x_n - x_{n-1})\| \\
&\leq \|x_n - z\| + \theta_n\|x_n - x_{n-1}\|.
\end{aligned} \tag{28}$$

Hence $\|J_{\lambda_n}(y_n - \lambda_n Ay_n) - z\| \leq \|x_n - z\| + \theta_n\|x_n - x_{n-1}\|$ by (27) and (28).
Let $w_n = \alpha_n f(x_n) + (1-\alpha_n)J_{\lambda_n}(y_n - \lambda_n Ay_n)$ for all $n \geq 1$. Then we obtain

$$\begin{aligned}
\|w_n - z\| &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)J_{\lambda_n}(y_n - \lambda_n Ay_n) - z\| \\
&\leq \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\
&\quad + \theta_n(1-\alpha_n)\|x_n - x_{n-1}\| \\
&\leq \alpha_n a\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\
&\quad + \theta_n(1-\alpha_n)\|x_n - x_{n-1}\| \\
&= (1 - \alpha_n(1-a))\|x_n - z\| + \alpha_n\|f(z) - z\| + \theta_n(1-\alpha_n)\|x_n - x_{n-1}\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_{n+1}-z\| &= \|\beta_n(x_n-z)+(1-\beta_n)(Sw_n-z)\| \\
&\le \beta_n\|x_n-z\|+(1-\beta_n)\|Sw_n-z\| \\
&\le \beta_n\|x_n-z\|+(1-\beta_n)\|w_n-z\| \\
&\le \beta_n\|x_n-z\|+(1-\beta_n)[(1-\alpha_n(1-a))\|x_n-z\|+\alpha_n\|f(z)-z\| \\
&\quad +\theta_n(1-\alpha_n)\|x_n-x_{n-1}\|] \\
&= (1-\alpha_n(1-\beta_n)(1-a))\|x_n-z\| \\
&\quad +\alpha_n(1-\beta_n)(1-a)\left[\frac{\|f(z)-z\|}{1-\alpha}+\frac{\theta_n(1-\alpha_n)}{\alpha_n(1-\alpha)}\|x_n-x_{n-1}\|\right].
\end{aligned}$$

By Lemma 1(i), we have that $\{x_n\}$ is bounded. We see that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\beta_n(x_n-z)+(1-\beta_n)(Sw_n-z)\|^2 \\
&= \beta_n\|x_n-z\|^2+(1-\beta_n)\|Sw_n-z\|^2-\beta_n(1-\beta_n)\|x_n-Sw_n\|^2 \\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|w_n-z\|^2-\beta_n(1-\beta_n)\|Sw_n-x_n\|^2. \qquad (29)
\end{aligned}$$

We next estimate the following:

$$\begin{aligned}
\|w_n-z\|^2 &= \langle w_n-z,w_n-z\rangle \\
&= \langle \alpha_n(f(x_n)-z)+(1-\alpha_n)(J_{\lambda_n}(y_n-\lambda_nAy_n)-z),w_n-z\rangle \\
&= \alpha_n\langle f(x_n)-f(z),w_n-z\rangle+\alpha_n\langle f(z)-z,w_n-z\rangle \\
&\quad +(1-\alpha_n)\langle J_{\lambda_n}(y_n-\lambda_nAy_n)-z,w_n-z\rangle \\
&\le \alpha_n\|f(x_n)-f(z)\|\|w_n-z\| \\
&\quad +(1-\alpha_n)\|J_{\lambda_n}(y_n-\lambda_nAy_n)-z\|\|w_n-z\|+\alpha_n\langle f(z)-z,w_n-z\rangle \\
&\le \alpha_na\|x_n-z\|\|w_n-z\|+(1-\alpha_n)\|J_{\lambda_n}(y_n-\lambda_nAy_n)-z\|\|w_n-z\| \\
&\quad +\alpha_n\langle f(z)-z,w_n-z\rangle \\
&\le \frac{1}{2}\alpha_na\|x_n-z\|^2+\frac{1}{2}\alpha_na\|w_n-z\|^2+\frac{1}{2}(1-\alpha_n)\|J_{\lambda_n}(y_n-\lambda_nAy_n)-z\|^2 \\
&\quad +\frac{1}{2}(1-\alpha_n)\|w_n-z\|^2+\alpha_n\langle f(z)-z,w_n-z\rangle \\
&= \frac{1}{2}\alpha_na\|x_n-z\|^2+\frac{1}{2}(1-\alpha_n)\|J_{\lambda_n}(y_n-\lambda_nAy_n)-z\|^2 \\
&\quad +\frac{1}{2}(1-\alpha_n(1-a))\|w_n-z\|^2+\alpha_n\langle f(z)-z,w_n-z\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|w_n-z\|^2 &\le \frac{\alpha_na}{1-\alpha_n(a-1)}\|x_n-z\|^2+\frac{1-\alpha_n}{1-\alpha_n(a-1)}\|J_{\lambda_n}(y_n-\lambda_nAy_n)-z\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n(a-1)}\langle f(z)-z,w_n-z\rangle. \qquad (30)
\end{aligned}$$

We also have, using (19)

$$
\begin{aligned}
\|y_n - z\|^2 &= \|(x_n - z) + \theta_n(x_n - x_{n-1})\|^2 \\
&= \|x_n - z\|^2 + 2\theta_n\langle x_n - z, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - z\|^2 + 2\theta_n\left[\frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2 - \frac{1}{2}\|x_n - z - x_n + x_{n-1}\|^2\right] \\
&\quad +\theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - z\|^2 + \theta_n[\|x_n - z\|^2 + \|x_n - x_{n-1}\|^2 - \|x_{n-1} - z\|^2] \\
&\quad +\theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - z\|^2 + \theta_n[\|x_n - z\|^2 - \|x_{n-1} - z\|^2] + (\theta_n^2 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - z\|^2 + \theta_n[\|x_n - z\|^2 - \|x_{n-1} - z\|^2] + 2\theta_n\|x_n - x_{n-1}\|^2. \qquad (31)
\end{aligned}
$$

Combining (26) and (31), we get

$$
\begin{aligned}
\|J_{\lambda_n}(y_n - \lambda_n Ay_n) - z\|^2 &\leq \|x_n - z\|^2 + \theta_n[\|x_n - z\|^2 - \|x_{n-1} - z\|^2] + 2\theta_n\|x_n - x_{n-1}\|^2 \\
&\quad -\lambda_n(2\alpha - \lambda_n)\|Ay_n - Az\|^2 \\
&\quad -\|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\|^2. \qquad (32)
\end{aligned}
$$

Combining (30) and (32), we obtain

$$
\begin{aligned}
\|w_n - z\|^2 &\leq \frac{\alpha_n a}{1 - \alpha_n(a-1)}\|x_n - z\|^2 + \frac{1 - \alpha_n}{1 - \alpha_n(a-1)}[\|x_n - z\|^2 + \theta_n\|x_n - z\|^2 \\
&\quad -\theta_n\|x_{n-1} - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ay_n - Az\|^2 \\
&\quad -\|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\|^2] \\
&\quad +\frac{2\alpha_n}{1 - \alpha_n(a-1)}\langle f(z) - z, w_n - z\rangle \\
&= \frac{1 - \alpha_n(1-a)}{1 - \alpha_n(a-1)}\|x_n - z\|^2 + \frac{\theta_n(1 - \alpha_n)}{1 - \alpha_n(a-1)}[\|x_n - z\|^2 - \|x_{n-1} - z\|^2] \\
&\quad +\frac{2\theta_n(1 - \alpha_n)}{1 - \alpha_n(a-1)}\|x_n - x_{n-1}\|^2 - \frac{\lambda_n(1 - \alpha_n)(2\alpha - \lambda_n)}{1 - \alpha_n(a-1)}\|Ay_n - Az\|^2 \\
&\quad -\frac{1 - \alpha_n}{1 - \alpha_n(a-1)}\|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\|^2 \\
&\quad +\frac{2\alpha_n}{1 - \alpha_n(a-1)}\langle f(z) - z, w_n - z\rangle. \qquad (33)
\end{aligned}
$$

From (29) and (33), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 \le\;& \beta_n\|x_n-z\|^2+(1-\beta_n)\left[\frac{1-\alpha_n(1-\alpha)}{1-\alpha_n(\alpha-1)}\|x_n-z\|^2\right.\\
&+\frac{\theta_n(1-\alpha_n)}{1-\alpha_n(a-1)}(\|x_n-z\|^2-\|x_{n-1}-z\|^2)+\frac{2\theta_n(1-\alpha_n)}{1-\alpha_n(a-1)}\|x_n-x_{n-1}\|^2\\
&-\frac{\lambda_n(1-\alpha_n)(2\alpha-\lambda_n)}{1-\alpha_n(a-1)}\|Ay_n-Az\|^2\\
&-\frac{1-\alpha_n}{1-\alpha_n(a-1)}\|y_n-\lambda_n(Ay_n-Az)-J_{\lambda_n}(y_n-\lambda_nAy_n)\|^2\\
&\left.+\frac{2\alpha_n}{1-\alpha_n(a-1)}\langle f(z)-z,w_n-z\rangle\right]-\beta_n(1-\beta_n)\|Sw_n-x_n\|^2\\
=\;&\left(1-\frac{2\alpha_n(1-a)(1-\beta_n)}{1-\alpha_n(a-1)}\right)\|x_n-z\|^2\\
&+\frac{\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}(\|x_n-z\|^2-\|x_{n-1}-z\|^2)\\
&+\frac{2\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}\|x_n-x_{n-1}\|^2\\
&-\frac{\lambda_n(1-\alpha_n)(1-\beta_n)(2\alpha-\lambda_n)}{1-\alpha_n(a-1)}\|Ay_n-Az\|^2\\
&-\frac{(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}\|y_n-\lambda_n(Ay_n-Az)-J_{\lambda_n}(y_n-\lambda_nAy_n)\|^2\\
&+\frac{2\alpha_n(1-\beta_n)}{1-\alpha_n(a-1)}\langle f(z)-z,w_n-z\rangle-\beta_n(1-\beta_n)\|Sw_n-x_n\|^2. \qquad (34)
\end{aligned}$$

Set $\Gamma_n=\|x_n-z\|^2, \forall n\ge 1$. We next consider two cases.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1}\le\Gamma_n$ for all $n\ge N$. In this case, $\{\Gamma_n\}$ is convergent. From (34) we obtain

$$\begin{aligned}
\Gamma_{n+1} \le\;& \left[1-\frac{2\alpha_n(1-a)(1-\beta_n)}{1-\alpha_n(a-1)}\right]\Gamma_n+\frac{\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}(\Gamma_n-\Gamma_{n-1})\\
&+\frac{2\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}\|x_n-x_{n-1}\|^2-\frac{\lambda_n(1-\alpha_n)(1-\beta_n)(2\alpha-\lambda_n)}{1-\alpha_n(a-1)}\|Ay_n-Az\|^2\\
&-\frac{(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}\|y_n-\lambda_n(Ay_n-Az)-J_{\lambda_n}(y_n-\lambda_nAy_n)\|^2\\
&+\frac{2\alpha_n(1-\beta_n)}{1-\alpha_n(a-1)}\langle f(z)-z,w_n-z\rangle-\beta_n(1-\beta_n)\|Sw_n-x_n\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{\lambda_n(1-\alpha_n)(1-\beta_n)(2\alpha-\lambda_n)}{1-\alpha_n(a-1)}\|Ay_n-Az\|^2\\
&\le\Gamma_n-\Gamma_{n+1}+\frac{\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}(\Gamma_n-\Gamma_{n-1})+\frac{2\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}\|x_n-x_{n-1}\|^2\\
&\quad+\frac{2\alpha_n(1-\beta_n)}{1-\alpha_n(a-1)}\langle f(z)-z,w_n-z\rangle.
\end{aligned}$$

Also, we obtain

$$\frac{(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}\|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\|^2$$
$$\leq \Gamma_n - \Gamma_{n+1} + \frac{\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}(\Gamma_n - \Gamma_{n-1}) + \frac{2\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}\|x_n - x_{n-1}\|^2$$
$$+\frac{2\alpha_n(1-\beta_n)}{1-\alpha_n(a-1)}\langle f(z) - z, w_n - z\rangle.$$

We also have

$$\begin{aligned}\beta_n(1-\beta_n)\|Sw_n - x_n\|^2 &\leq \Gamma_n - \Gamma_{n+1} + \frac{\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}(\Gamma_n - \Gamma_{n-1}) \\ &\quad + \frac{2\theta_n(1-\beta_n)(1-\alpha_n)}{1-\alpha_n(a-1)}\|x_n - x_{n-1}\|^2 \\ &\quad + \frac{2\alpha_n(1-\beta_n)}{1-\alpha_n(a-1)}\langle f(z) - z, w_n - z\rangle.\end{aligned}$$

Since $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, $\lim_{n\to\infty}\alpha_n = 0$ and $\{\Gamma_n\}$ converges, we have

$$\|Ay_n - Az\| \to 0,$$

$$\|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\| \to 0,$$

and

$$\|Sw_n - x_n\| \to 0$$

as $n \to \infty$. We next show that $\|J_{\lambda_n}(y_n - \lambda_n Ay_n) - y_n\| \to 0$ as $n \to \infty$. We see that

$$\begin{aligned}\|J_{\lambda_n}(y_n - \lambda_n Ay_n) - y_n\| &= \|J_{\lambda_n}(y_n - \lambda_n Ay_n) - \lambda_n(Ay_n - Az) + \lambda_n(Ay_n - Az) - y_n\| \\ &\leq \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\| + \lambda_n\|Ay_n - Az\| \\ &\to 0, \quad as \quad n \to \infty.\end{aligned}$$

We also have

$$\begin{aligned}\|w_n - x_n\| &= \|\alpha_n(f(x_n) - x_n) + (1-\alpha_n)(J_{\lambda_n}(y_n - \lambda_n Ay_n) - x_n)\| \\ &\leq \alpha_n\|f(x_n) - x_n\| + \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\| \\ &\quad +\lambda_n\|Ay_n - Az\| + \|x_n - y_n\| + \alpha_n\|J_{\lambda_n}(y_n - \lambda_n Ay_n) - x_n\| \\ &= \alpha_n\|f(x_n) - x_n\| + \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\| \\ &\quad +\lambda_n\|Ay_n - Az\| + \theta_n\|x_n - x_{n-1}\| + \alpha_n\|J_{\lambda_n}(y_n - \lambda_n Ay_n) - x_n\| \\ &\to 0, \quad as \quad n \to \infty.\end{aligned}$$

We next show that $\|Sx_n - x_n\| \to 0$ as $n \to \infty$. We see that

$$\begin{aligned}\|Sx_n - x_n\| &\leq \|Sx_n - Sw_n\| + \|Sw_n - x_n\| \\ &\leq \|x_n - w_n\| + \|Sw_n - x_n\| \\ &\to 0, \quad as \quad n \to \infty.\end{aligned}$$

Since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a point $x^* \in C$. Suppose that $x^* \neq Sx^*$. Then by Opial's Condition we obtain

$$\begin{aligned}
\liminf_{i\to\infty} \|x_{n_i} - x^*\| &< \liminf_{i\to\infty} \|x_{n_i} - Sx^*\| \\
&= \liminf_{i\to\infty} \|x_{n_i} - Sx_{n_i} + Sx_{n_i} - Sx^*\| \\
&\leq \liminf_{i\to\infty} \|x_{n_i} - Sx_{n_i}\| + \liminf_{i\to\infty} \|Sx_{n_i} - Sx^*\| \\
&\leq \liminf_{i\to\infty} \|x_{n_i} - x^*\|.
\end{aligned}$$

This is a contradiction. Hence $x^* \in F(S)$. From $w_n = \alpha_n f(x_n) + (1-\alpha_n)J_{\lambda_n}(y_n - \lambda_n A y_n)$, we have

$$\frac{w_n - \alpha_n f(x_n)}{1-\alpha_n} = J_{\lambda_n}(y_n - \lambda_n A y_n).$$

From $J_{\lambda_n} = (I + \lambda_n B)^{-1}$, we also have

$$\frac{w_n - \alpha_n f(x_n)}{1-\alpha_n} = (I + \lambda_n B)^{-1})(y_n - \lambda_n A y_n).$$

This gives

$$y_n - \lambda_n A y_n \in \frac{w_n - \alpha_n f(x_n)}{1-\alpha_n} + \lambda_n B \frac{w_n - \alpha_n f(x_n)}{1-\alpha_n}.$$

So, we obtain

$$\frac{y_n}{\lambda_n} - A y_n - \frac{w_n + \alpha_n f(x_n)}{\lambda_n(1-\alpha_n)} \in B \frac{w_n - \alpha_n f(x_n)}{1-\alpha_n}.$$

Since B is monotone, we have for $(p,q) \in B$

$$\left\langle \frac{w_n - \alpha_n f(x_n)}{1-\alpha_n} - p, \frac{y_n}{\lambda_n} - A y_n - \frac{w_n + \alpha_n f(x_n)}{\lambda_n(1-\alpha_n)} - q \right\rangle \geq 0.$$

So, we have
$\langle \lambda_n(w_n - \alpha_n f(x_n)) - p\lambda_n(1-\alpha_n), y_n(1-\alpha_n) - Ay_n\lambda_n(1-\alpha_n) - w_n + \alpha_n f(x_n) - q\lambda_n(1-\alpha_n)\rangle \geq 0$,
which implies

$$\langle \lambda_n w_n - p\lambda_n - \lambda_n\alpha_n(f(x_n) - p), y_n - w_n - \alpha_n(y_n - f(x_n)) - \lambda_n(1-\alpha_n)(Ay_n + q)\rangle \geq 0. \quad (35)$$

Since $\langle y_n - x^*, Ay_n - Ax^*\rangle \geq \alpha\|Ay_n - Ax^*\|^2$, $Ay_n \to Az$ and $y_{n_i} \rightharpoonup x^*$(since $\|x_n - y_n\| \to 0$), we have $\alpha\|Ay_n - Ax^*\|^2 \leq 0$ and thus $Az = Ax^*$. From (35), we have $\langle x^* - p, -Ax^* - q\rangle \geq 0$. Since B is maximal monotone, we have $-Ax^* \in Bx^*$. Hence $0 \in (A+B)x^*$ and thus we have $x^* \in F(S) \cap (A+B)^{-1}0$.

We will show that $\limsup_{n\to\infty}\langle f(z) - z, w_n - z\rangle \leq 0$. Sine $\{w_n\}$ is bounded and $\|x_n - w_n\| \to 0$, there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(z) - z, w_n - z\rangle &= \lim_{i\to\infty} \langle f(z) - z, w_{n_i} - z\rangle \\
&= \langle f(z) - z, x^* - z\rangle \leq 0.
\end{aligned}$$

We know that

$$
\begin{aligned}
\Gamma_{n+1} \quad \leq \quad & \left[1-\frac{2\alpha_n(1-a)(1-\beta_n)}{1-\alpha_n(a-1)}\right]\Gamma_n \\
& +\left[\frac{2\alpha_n(1-a)(1-\beta_n)}{1-\alpha_n(a-1)}\right]\left[\frac{\theta_n(1-\alpha_n)}{\alpha_n(1-a)}\|x_n-x_{n-1}\|^2+\frac{1}{(1-a)}\langle f(z)-z,w_n-z\rangle\right].
\end{aligned}
$$

Since $\limsup_{n\to\infty}\left[\frac{\theta_n(1-\alpha_n)}{\alpha_n(1-a)}\|x_n-x_{n-1}\|^2 + \frac{1}{1-a}\langle f(z)-z,w_n-z\rangle\right] \leq 0$, by Lemma 1(ii) $\lim_{n\to\infty}\Gamma_n = 0$. So $x_n \to z$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\varphi : \mathbb{N} \to \mathbb{N}$ as in Lemma 2. Then, by Lemma 2, we have $\Gamma_{\varphi(n)} \leq \Gamma_{\varphi(n)+1}$. We see that

$$
\begin{aligned}
\|x_{\varphi(n)+1}-x_{\varphi(n)}\| \quad & \leq \quad (1-\beta_{\varphi(n)})\|Sw_{\varphi(n)}-x_{\varphi(n)}\| \\
& \to \quad 0, \qquad as \quad n\to\infty.
\end{aligned}
$$

From (34) we have

$$
\begin{aligned}
\Gamma_{\varphi(n)+1} \quad \leq \quad & \left[1-\frac{2\alpha_{\varphi(n)}(1-a)(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}\right]\Gamma_{\varphi(n)}+\frac{\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1}) \\
& +\frac{2\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2 \\
& -\frac{\lambda_{\varphi(n)}(1-\alpha_{\varphi(n)})(1-\beta_{\varphi(n)})(2\alpha-\lambda_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}\|Ay_{\varphi(n)}-Az\|^2 \\
& -\frac{(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}\|y_{\varphi(n)}-\lambda_{\varphi(n)}(Ay_{\varphi(n)}-Az)-J_{\lambda_{\varphi(n)}}(y_{\varphi(n)}-\lambda_{\varphi(n)}Ay_{\varphi(n)})\|^2 \\
& +\frac{2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}\langle f(z)-z,w_{\varphi(n)}-z\rangle-\beta_{\varphi(n)}(1-\beta_{\varphi(n)})\|Sw_{\varphi(n)}-x_{\varphi(n)}\|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \frac{\lambda_{\varphi(n)}(1-\alpha_{\varphi(n)})(1-\beta_{\varphi(n)})(2a-\lambda_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\|Ay_{\varphi(n)}-Az\|^2 \\
& \quad \leq \quad \Gamma_{\varphi(n)}-\Gamma_{\varphi(n)+1}-\left[1-\frac{2\alpha_{\varphi(n)}(1-\alpha)(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\right]\Gamma_{\varphi(n)} \\
& \qquad +\frac{\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1})+\frac{2\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2 \\
& \qquad +\frac{2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\langle f(z)-z,w_{\varphi(n)}-z\rangle \\
& \quad \leq \quad \frac{\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1})+\frac{2\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2 \\
& \qquad +\frac{2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\langle f(z)-z,w_{\varphi(n)}-z\rangle. \qquad (36)
\end{aligned}
$$

We also have

$$\begin{aligned}
&\beta_{\varphi(n)}(1-\beta_{\varphi(n)})\|Sw_{\varphi(n)}-x_{\varphi(n)}\|^2\\
&\quad\le \Gamma_{\varphi(n)}-\Gamma_{\varphi(n)+1}-\left[1-\frac{2\alpha_{\varphi(n)}(1-\alpha)(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\right]\Gamma_{\varphi(n)}\\
&\qquad+\frac{\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1})+\frac{2\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2\\
&\qquad+\frac{2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\langle f(z)-z,w_{\varphi(n)}-z\rangle\\
&\quad\le \frac{\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1})+\frac{2\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2\\
&\qquad+\frac{2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\langle f(z)-z,w_{\varphi(n)}-z\rangle. \qquad (37)
\end{aligned}$$

We also have

$$\begin{aligned}
&\frac{(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\|y_{\varphi(n)}-\lambda_{\varphi(n)}(Ay_{\varphi(n)}-Az)-J_{\lambda_{\varphi(n)}}(y_{\varphi(n)}-\lambda_{\varphi(n)}Ay_{\varphi(n)})\|^2\\
&\quad\le \Gamma_{\varphi(n)}-\Gamma_{\varphi(n)+1}-\left[1-\frac{2\alpha_{\varphi(n)}(1-\alpha)(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\right]\Gamma_{\varphi(n)}\\
&\qquad+\frac{\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1})\\
&\qquad+\frac{2\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2+\frac{2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\langle f(z)-z,w_{\varphi(n)}-z\rangle\\
&\quad\le \frac{\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1})\\
&\qquad+\frac{2\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2\\
&\qquad+\frac{2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(\alpha-1)}\langle f(z)-z,w_{\varphi(n)}-z\rangle. \qquad (38)
\end{aligned}$$

We know that

$$\begin{aligned}
\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1} &= \|x_{\varphi(n)}-z\|^2-\|x_{\varphi(n)-1}-z\|^2\\
&= [\|x_{\varphi(n)}-z\|-\|x_{\varphi(n)-1}-z\|][\|x_{\varphi(n)}-z\|+\|x_{\varphi(n)-1}-z\|]\\
&\le \|x_{\varphi(n)}-x_{\varphi(n)-1}\|[\|x_{\varphi(n)}-z\|+\|x_{\varphi(n)-1}-z\|]\\
&\to 0, \qquad as \quad n\to\infty.
\end{aligned}$$

From (36)–(38), we have

$$\|Ay_{\varphi(n)}-Az\|\to 0,\ \ \|y_{\varphi(n)}-\lambda_{\varphi(n)}(Ay_{\varphi(n)}-Az)-J_{\lambda_{\varphi(n)}}(y_{\varphi(n)}-\lambda_{\varphi(n)}Ay_{\varphi(n)})\|\to 0$$

and $\|Sw_{\varphi(n)} - x_{\varphi(n)}\| \to 0$. Now repeating the argument of the proof in Case 1, we obtain $\limsup_{n\to\infty}\langle f(z) - z, w_{\varphi(n)} - z\rangle \le 0$. We note that

$$
\begin{aligned}
\frac{2\alpha_{\varphi(n)}(1-a)(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}\Gamma_{\varphi(n)} \le\ & \frac{\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}(\Gamma_{\varphi(n)} - \Gamma_{\varphi(n)-1}) \\
&+\frac{2\theta_{\varphi(n)}(1-\beta_{\varphi(n)})(1-\alpha_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}\|x_{\varphi(n)} - x_{\varphi(n)-1}\|^2 \\
&+\frac{2\alpha_{\varphi(n)}(1-\beta_{\varphi(n)})}{1-\alpha_{\varphi(n)}(a-1)}\langle f(z) - z, w_{\varphi(n)} - z\rangle.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\Gamma_{\varphi(n)} \le\ & \frac{\theta_{\varphi(n)}(1-\alpha_{\varphi(n)})}{2\alpha_{\varphi(n)}(1-a)}[\Gamma_{\varphi(n)} - \Gamma_{\varphi(n)-1}] + \frac{\theta_{\varphi(n)}(1-\alpha_{\varphi(n)})}{\alpha_{\varphi(n)}(1-a)}\|x_{\varphi(n)} - x_{\varphi(n)-1}\|^2 \\
&+\frac{1}{1-a}\langle f(z) - z, w_{\varphi(n)} - z\rangle.
\end{aligned}
$$

So $\limsup_{n\to\infty}\Gamma_{\varphi(n)} \le 0$. This means $\lim_{n\to\infty}\Gamma_{\varphi(n)} = \lim_{n\to\infty}\|x_{\varphi(n)} - z\|^2 = 0$. Hence $x_{\varphi_n} \to z$. It follows that

$$
\begin{aligned}
\|x_{\varphi(n)+1} - z\| &\le \|x_{\varphi(n)+1} - x_{\varphi(n)}\| + \|x_{\varphi(n)} - z\| \\
&\to 0, \quad as \quad n \to \infty.
\end{aligned}
$$

By Lemma 2, we have $\Gamma_n \le \Gamma_{\varphi(n)+1}$. Thus, we obtain

$$
\begin{aligned}
\Gamma_n &= \|x_n - z\|^2 \\
&\le \|x_{\varphi(n)+1} - z\|^2 \\
&\to 0, \quad as \quad n \to \infty.
\end{aligned}
$$

Hence $\Gamma_n \to 0$ and thus $x_n \to z$. This completes the proof. □

Remark 1. *It is noted that the condition*

$$\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$$

is removed from Theorem TTT of Takahashi et al. [12].

Remark 2. *[17] We remark here that the conditions (C4) is easily implemented in numerical computation since the valued of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. Indeed, the parameter θ_n can be chosen such that $0 \le \theta_n \le \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\omega_n}{\|x_n - x_{n-1}\|}, \theta\right\} & if\ x_n \ne x_{n-1}, \\ \theta & otherwise, \end{cases}$$

where $\{\omega_n\}$ is a positive sequence such that $\omega_n = o(\alpha_n)$.

4. Weak Convergence Theorem

In this section, we prove the weak convergence theorem.

Theorem 3. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of H into itself and let B be a maximal monotone operator on H such that the* *domain of B is included in C. Let* $J_\lambda = (I + \lambda B)^{-1}$ *be the resolvent of B for* $\lambda > 0$ *and let S be a nonexpansive mapping of C into itself such that* $F(S) \cap (A+B)^{-1}0 \neq \emptyset$. *Let* $x_0, x_1 \in C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by*

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)S(J_{\lambda_n}(y_n - \lambda_n A y_n)) \end{aligned} \tag{39}$$

for all $n \in \mathbb{N}$, *where* $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\beta_n\} \subset (0,1)$ *and* $\{\theta_n\} \subset [0, \theta]$, *where* $\theta \in [0,1)$ *satisfy*

(C1) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(C2) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$;
(C3) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$.

Then $\{x_n\}$ *converges weakly to a point of* $F(S) \cap (A+B)^{-1}0$.

Proof. Let $z \in F(S) \cap (A+B)^{-1}0$ and $w_n = J_{\lambda_n}(y_n - \lambda_n A y_n) \forall n \geq 1$. Then $z = J_{\lambda_n}(z - \lambda_n A z)$. From Theorem 2 we have

$$\|x_{n+1} - z\|^2 \leq \beta_n \|x_n - z\|^2 + (1-\beta_n)\|w_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - Sw_n\|^2, \tag{40}$$

$$\begin{aligned} \|w_n - z\|^2 &= \|y_n - z\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ay_n - Az\|^2 \\ &\quad - \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n A y_n)\|^2 \end{aligned} \tag{41}$$

and

$$\|y_n - z\|^2 \leq (1+\theta_n)\|x_n - z\|^2 + 2\theta_n \|x_n - x_{n-1}\|^2 - \theta_n \|x_{n-1} - z\|^2. \tag{42}$$

Combining (42) and (41), we obtain

$$\begin{aligned} \|w_n - z\|^2 \leq\ & (1+\theta_n)\|x_n - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 - \theta_n\|x_{n-1} - z\|^2 \\ & - \lambda_n(2\alpha - \lambda_n)\|Ay_n - Az\|^2 - \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n A y_n)\|^2. \end{aligned} \tag{43}$$

Combining (40) and (43), we also have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[(1+\theta_n)\|x_n - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 \\ &\quad - \theta_n\|x_{n-1} - z\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ay_n - Az\|^2 \\ &\quad - \|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n A y_n)\|] - \beta_n(1-\beta_n)\|x_n - Sw_n\|^2 \\ &= \beta_n\|x_n - z\|^2 + (1-\beta_n)(1+\theta_n)\|x_n - z\|^2 + 2\theta_n(1-\beta_n)\|x_n - x_{n-1}\|^2 \\ &\quad - \theta_n(1-\beta_n)\|x_{n-1} - z\|^2 - \lambda_n(2\alpha - \lambda_n)(1-\beta_n)\|Ay_n - Az\|^2 \\ &\quad - (1-\beta_n)\|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n A y_n)\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - Sw_n\|^2 \\ &\leq \|x_n - z\|^2 + \theta_n(1-\beta_n)\|x_n - z\|^2 \\ &\quad + 2\theta_n(1-\beta_n)\|x_n - x_{n-1}\|^2 - \theta_n(1-\beta_n)\|x_{n-1} - z\|^2. \end{aligned} \tag{44}$$

This shows that

$$\begin{aligned} \|x_{n-1} - z\|^2 - \|x_n - z\|^2 &\leq \theta_n(1-\beta_n)[\|x_n - z\|^2 - \|x_{n-1} - z\|^2] \\ &\quad + 2\theta_n(1-\beta_n)\|x_n - x_{n-1}\|^2. \end{aligned}$$

By Lemma 4, we have $\|x_n - z\|^2$ converges. Thus, $\lim_{n\to\infty} \|x_n - z\|^2$ exists. So, by (44) we have

$$\begin{aligned}\lambda_n(2\alpha - \lambda_n)(1-\beta_n)\|Ay_n - Az\|^2 &\leq \theta_n(1-\beta_n)[\|x_n - z\|^2 - \|x_{n-1} - z\|^2] \\ &\quad +2\theta_n(1-\beta_n)\|x_n - x_{n-1}\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\to 0, \qquad as \quad n \to \infty.\end{aligned}$$

We also have

$$\begin{aligned}(1-\beta_n)\|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\|^2 &\leq \theta_n(1-\beta_n)[\|x_n - z\|^2 - \|x_{n-1} - z\|^2] \\ &\quad +2\theta_n(1-\beta_n)\|x_n - x_{n-1}\|^2 + \|x_n - z\|^2 \\ &\quad -\|x_{n+1} - z\|^2 \\ &\to 0, \qquad as \quad n \to \infty.\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\beta_n(1-\beta_n)\|x_n - Sw_n\|^2 &\leq \theta_n(1-\beta_n)[\|x_n - z\|^2 - \|x_{n-1} - z\|^2] \\ &\quad +2\theta_n(1-\beta_n)\|x_n - x_{n-1}\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\to 0, \qquad as \quad n \to \infty.\end{aligned}$$

It follows that

$\|Ay_n - Az\| \to 0$, $\|y_n - \lambda_n(Ay_n - Az) - J_{\lambda_n}(y_n - \lambda_n Ay_n)\| \to 0$ and $\|x_n - Sw_n\| \to 0$.

By a similar proof as in Theorem 2, we can show that if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \rightharpoonup x^*$, then $x^* \in F(S) \cap (A+B)^{-1}0$. By Lemma 3, we conclude that $\{x_n\}$ weakly converges to a point in $F(S) \cap (A+B)^{-1}0$. We thus complete the proof. $\square$

Remark 3. *[18] We remark here that the conditions (C3) is easily implemented in numerical computation. Indeed, once x_n and x_{n-1} are given, it is just sufficient to compute the update x_{n+1} with (39) by choosing θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|^2}, \theta\right\} & if \ x_n \neq x_{n-1}, \\ \theta & otherwise, \end{cases}$$

where $\{\varepsilon_n\} \subset [0,\infty)$ is such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.

5. Numerical Examples

In this section, we give some numerical experiments to show the efficiency and the comparison with other methods.

Example 1. *Solve the following minimization problem:*

$$\min_{x\in\mathbb{R}^3} \| x \|_2^2 + (3,5,-1)x + 9 + \| x \|_1,$$

where $x = (y_1, y_2, y_3) \in \mathbb{R}^3$ and the fixed-point problem of $S : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$S(x) = (-2 - y_1, -4 - y_2, -y_3).$$

For each $x \in \mathbb{R}^3$, we set $F(x) = \| x \|_2^2 + (3,5,-1)x + 9$ and $G(x) = \| x \|_1$. Put $A = \nabla F$ and $B = \partial G$ in Theorem 2. We can check that F is convex and differentiable on $\mathbb{R}^3$ with 2-Lipschitz continuous gradient. Moreover, G is convex and lower semi-continuous but not differentiable on $\mathbb{R}^3$. We know that for $r > 0$

$$(I + rB)^{-1}(x) = (max\{|\ y_1\ | - r, 0\}sign(y_1), max\{|\ y_2\ | - r, 0\}sign(y_2), max\{|\ y_3\ | - r, 0\}sign(y_3)). \quad (45)$$

We choose $\alpha_n = \frac{1}{100n+1}$, $\beta_n = \frac{3n}{5n+1}$, $\lambda_n = 0.0001$ for all $n \in \mathbb{N}$ and $\theta = 0.5$. For each $n \in \mathbb{N}$, let $\omega_n = \frac{1}{(n+1)^3}$ and define $\theta_n = \bar{\theta_n}$ as in Remark 2. The stopping criterion is defined by

$$E_n = \|x_n - J_{\lambda_n}(I - \nabla F)x_n\| + \|x_n - Sx_n\| < 10^{-3}.$$

We now study the effect (in terms of convergence and the CPU time) and consider different choices of x_0 and x_1 as following, see Table 1.

Choice 1: $x_0 = (1, 2, -1)$ and $x_1 = (1, 5, 1)$;
Choice 2: $x_0 = (0, -2, 2)$ and $x_1 = (2, 0, -3)$;
Choice 3: $x_0 = (-5, 4, 6)$ and $x_1 = (3, -5, -9)$;
Choice 4: $x_0 = (1, 2, 3)$ and $x_1 = (8, 7, 3)$.

Table 1. Using Equations (13) and (25) with different choices of x_0 and x_1.

			Equation (13)	**Equation (25)**
Choice 1	$x_0 = (1, 2, -1)$ $x_1 = (1, 5, 1)$	No. of Iter. CPU (Time)	92 0.045106	6 0.016301
Choice 2	$x_0 = (0, -2, 2)$ $x_1 = (2, 0, -3)$	No. of Iter. CPU (Time)	92 0.039239	14 0.014759
Choice 3	$x_0 = (-5, 4, 6)$ $x_1 = (3, -5, -9)$	No. of Iter. CPU (Time)	92 0.064943	14 0.010813
Choice 4	$x_0 = (1, 2, 3)$ $x_1 = (8, 7, 3)$	No. of Iter. CPU (Time)	92 0.066736	14 0.047984

The error plotting of Equations (13) and (25) for each choice is shown in Figures 1–4, respectively.

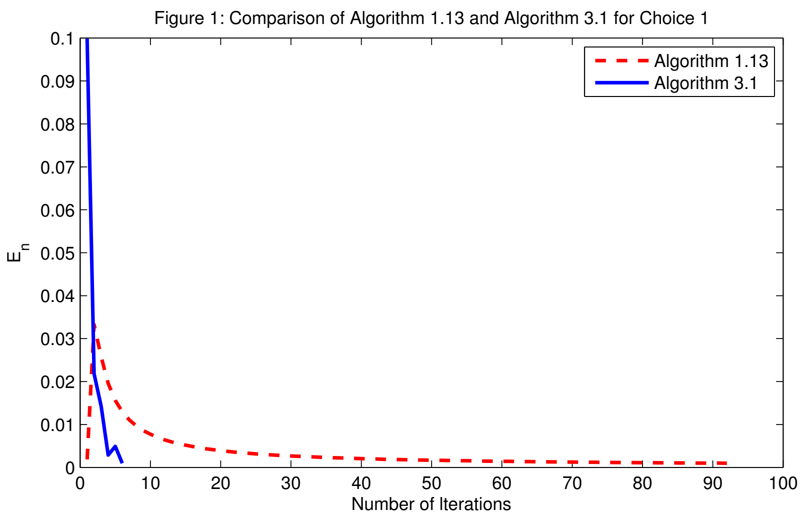

Figure 1. Comparison of Equations (13) and (25) for each choice 1.

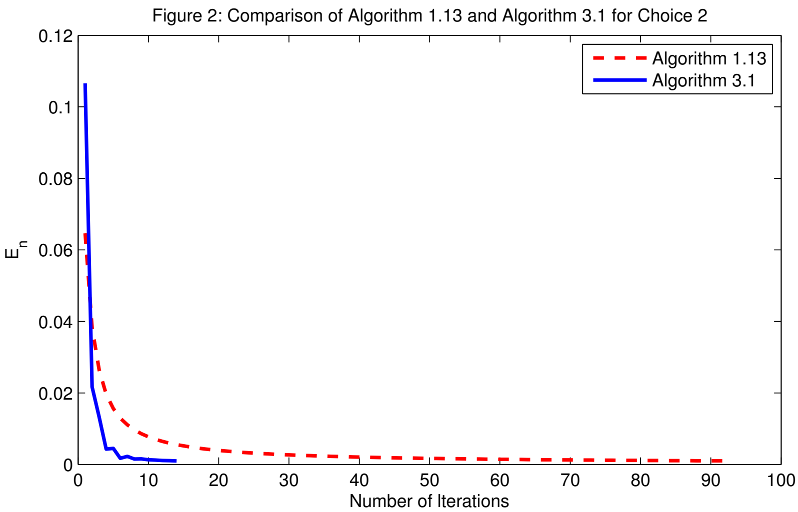

Figure 2. Comparison of Equations (13) and (25) for each choice 2.

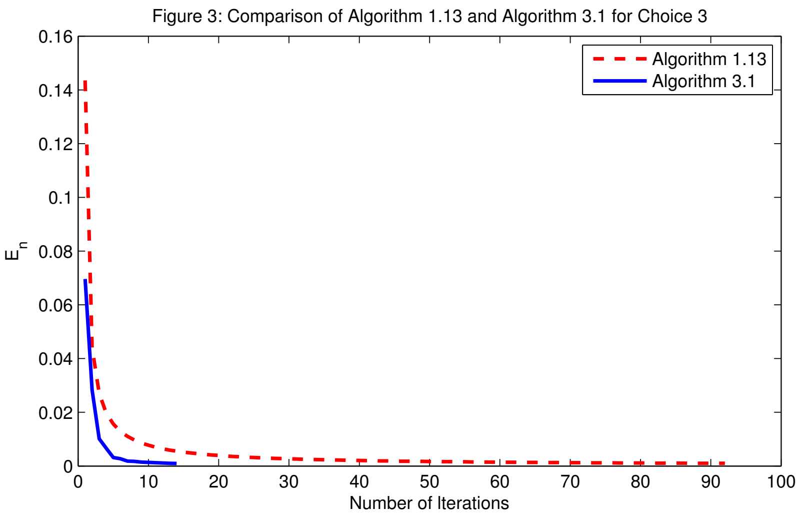

Figure 3. Comparison of Equations (13) and (25) for each choice 3.

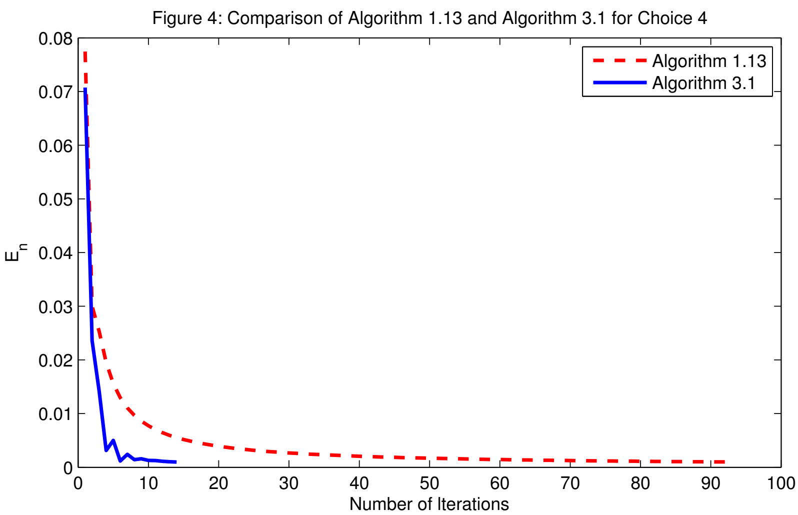

Figure 4. Comparison of Equations (13) and (25) for each choice 4.

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References

1. Xu, H.K. A regularization method for the proximal point algorithm. *J. Glob. Optim.* **2006**, *36*, 115–125. [CrossRef]
2. Kamimura, S.; Takahashi, W. Approximating solutions of maximal monotone operators in Hilbert spaces. *J. Approx. Theory* **2000**, *106*, 226–240. [CrossRef]
3. Boikanyo, O.A.; Morosanu, G. A proximal point algorithm converging strongly for general errors. *Optim. Lett.* **2010**, *4*, 635–641. [CrossRef]
4. Boikanyo, O.A.; Morosanu, G. Strong convergence of a proximal point algorithm with bounded errorsequence. *Optim. Lett.* **2013**, *7*, 415–420. [CrossRef]
5. Marino, G.; Xu, H.K. Convergence of generalized proximal point algorithm. *Commun. Pure Appl. Anal.* **2004**, *3*, 791–808.
6. Yao, Y.; Noor, M.A. On convergence criteria of generalized proximal point algorithms. *J. Comput. Appl. Math.* **2008**, *217*, 46–55. [CrossRef]
7. Wang, F.; Cui, H. On the contraction-proximal point algorithms with multi-parameters. *J. Glob. Optim.* **2012**, *54*, 485–491. [CrossRef]
8. Takahashi, W. Viscosity approximation methods for resolvents of accretive operators in Banach spaces. *J. Fixed Point Theory Appl.* **2007**, *1*, 135–147. [CrossRef]
9. Combettes, P.L. Iterative construction of the resolvent of a sum of maximal monotone operators. *J. Convex Anal.* **2009**, *16*, 727–748.
10. Lopez, G.; Martín-Márquez, V.; Wang, F.; Xu, H.K. Forward-Backward splitting methods for accretive operators in Banach spaces. *Abstr. Appl. Anal.* **2012**, *2012*, 109–236. [CrossRef]
11. Lehdili, N.; Moudafi, A. Combining the proximal algorithm and Tikhonov regularization. *Optimization* **1996**, *37*, 239–252. [CrossRef]
12. Takahashi, S.; Takahashi, W.; Toyoda, M. Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces. *J. Optim. Theory Appl.* **2010**, *147*, 27–41. [CrossRef]
13. Takahashi, W.; Tamura, T. Convergence theorems for a pair of nonexpansive mappings. *J. Convex Anal.* **1998**, *5*, 45–56.
14. Halpern, B. Fixed points of nonexpanding maps. *Bull. Am. Math. Soc.* **1967**, *73*, 957–961. [CrossRef]
15. Dong, Q.L.; Cho, Y.J.; Zhong, L.L.; Rassias, Th.M. Inertial projection and contraction algorithms for variational inequalities. *J. Glob. Optim.* **2018**, *70*, 687–704. [CrossRef]
16. Dong, Q.L.; Yuan, H.B.; Cho, Y.J.; Rassias, Th.M. Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings. *Optim. Lett.* **2018**, *12*, 87–102. [CrossRef]
17. Suantai, S.; Pholasa, N.; Cholamjiak, P. The modified inertial relaxed CQ algorithm for solving the split feasibility problems. *J. Ind. Manag. Optim.* **2018**, *14*, 1595–1615. [CrossRef]
18. Maingé, P.E. Convergence theorems for inertial KM-type algorithms. *J. Comput. Appl. Math.* **2008**, *219*, 223–236. [CrossRef]