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Common Fixed Point Theorems of Generalized Multivalued (ψ, ϕ)-Contractions in Complete Metric Spaces with Application

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Abstract: The purpose of this paper is to introduce the notion of generalized multivalued (ψ, ϕ)-type contractions and generalized multivalued (ψ, ϕ)-type Suzuki contractions and establish some new common fixed point theorems for such multivalued mappings in complete metric spaces. Our results are extension and improvement of the Suzuki and Nadler contraction theorems, Jleli and Samet, Piri and Kumam, Mizoguchi and Takahashi, and Liu et al. fixed point theorems. We provide an example for supporting our new results. Moreover, an application of our main result to the existence of solution of system of functional equations is also presented.

Keywords: fixed point; generalized multivalued (ψ, ϕ)-type contraction; generalized multivalued (ψ, ϕ)-type Suzuki contraction

1. Introduction and Preliminaries

In the fixed point theory of continuous mappings, a well-known theorem of Banach [1] states that if (X, d) is a complete metric space and if S is a self-mapping on X which satisfies the inequality $d(Sx, Sy) \leq kd(x, y)$ for some $k \in [0, 1)$ and all $x, y \in X$, then S has a unique fixed point x^* and the sequence of successive approximations $\{Sx_n\}$ converges to x^* for all $x \in X$, the Banach's theorem [1] has been extensively studied and generalized on many settings (see [2–15]).

Suzuki [16] proved the following fixed point theorem.

Theorem 1 ([16])**.** *Let (X, d) be a compact metric space and $S : X \to X$ be a self-mapping. If for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{2}d\left(x, Sx\right) < d(x, y) \Longrightarrow d\left(Sx, Sy\right) < d(x, y),$$

then S has a unique fixed point in X.

Wardowski [17] introduced the notion of F-contractions and proved fixed point theorems concerning F-contractions as follows.

Definition 1 ([17])**.** *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exist $F \in \digamma$ and $\tau > 0$ such that*

$$\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right),$$

where $\digamma$ is the set of functions $F : [0, \infty) \to (-\infty, \infty)$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $x, y \in \mathbb{R}^+$ such that $x \leq y$, $F(x) < F(y)$;
(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers,

$$\lim_{n\to\infty} F(\alpha_n) = -\infty \text{ if and only if } \lim_{n\to\infty} \alpha_n = 0;$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Theorem 2 ([17]). *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Piri and Kumam [18] modified the notion of F-contraction as follows.

Definition 2 ([18]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $\mathcal{F}$ is the set of functions $F : (0, \infty) \to (-\infty, \infty)$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $x, y \in \mathbb{R}^+$ such that $x < y$, $F(x) < F(y)$;
(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers,

$$\lim_{n\to\infty} F(\alpha_n) = -\infty \text{ if and only if } \lim_{n\to\infty} \alpha_n = 0;$$

(F3) F is continuous.

On the other hand, recently, Jleli and Samet [19,20] introduced the notion of θ-contraction.

Definition 3. *Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be a θ-contraction if there exist a constant $k \in (0, 1)$ and $\theta \in \Theta$ such that*

$$x, y \in X, d(Tx, Ty) \neq 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k,$$

where Θ is the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:

(Θ1) *θ is nondecreasing,*
(Θ2) *for each sequence $\{t_n\} \subset (0, \infty)$,*

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0,$$

(Θ3) *there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\longrightarrow 0^+} \frac{\theta(t)-1}{t^r} = \ell$,*
(Θ4) *θ is continuous.*

Jleli and Samet [20] established the following fixed point theorem as follows.

Theorem 3 ([20]). *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a θ-contraction. Then T has a unique fixed point.*

As in [21] we denote by Ξ the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:

(Θ1)$'$ θ is nondecreasing,

$(\Theta 2)'$ $\inf_{t\in(0,\infty)} \theta(t_n) = 1$,
$(\Theta 3)'$ θ is continuous.

Theorem 4 ([21]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Then the following assertions are equivalent.*

(i) T is a θ-contraction with $\theta \in \Xi$;

(ii) T is an F-contraction with $F \in \mathcal{F}$.

Very recently, Liu et al. [21] proved new fixed point theorems for (ψ, ϕ)-type Suzuki contractions in complete metric spaces as follows.

Definition 4. *Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be a (ψ, ϕ)-type Suzuki contraction if there exists a comparison function ψ and $\phi \in \Phi$ such that for all, $x, y \in X$ with $Tx \neq Ty$*

$$\frac{1}{2} d(x, Tx) < d(x, y) \Longrightarrow \phi(d(Tx, Ty)) \leq \psi[\phi(M(x, y))],$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \tag{1.2}$$

and Φ is the set of functions $\phi : (0, \infty) \longrightarrow (0, \infty)$ satisfying the following conditions:

$(\Phi 1)$ *ϕ is nondecreasing,*
$(\Phi 2)$ *for each sequence $\{t_n\} \subset (0, \infty)$,*

$$\lim_{n\to\infty} \phi(t_n) = 0 \text{ if and only if } \lim_{n\to\infty} t_n = 0,$$

$(\Phi 3)$ *ϕ is continuous on $(0, \infty)$.*

And as in [22], a function $\psi : (0, \infty) \longrightarrow (0, \infty)$ is called a comparison function if it satisfies the following conditions:

(1) ψ is monotone increasing, that is, $t_1 < t_2 \Longrightarrow \psi(t_1) \leq \psi(t_2)$,
(2) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where ψ^n stands for the n-th iterate of ψ.

Lemma 1 ([21]). *Let $\phi : (0, \infty) \longrightarrow (0, \infty)$ be a nondecreasing and continuous function with $\inf_{t\in(0,\infty)} \phi(t) = 0$ and $\{t_k\}_k$ be a sequence in $(0, \infty)$. Then the following holds:*

$$\lim_{k\to\infty} \phi(t_k) = 0 \text{ if and only if } \lim_{k\to\infty} t_k = 0.$$

Example 1 ([22]). *The following functions $\psi : (0, \infty) \longrightarrow (0, \infty)$ are comparison functions:*

(1) $\psi(t) = at$, $0 < a < 1$ for all $t > 0$,
(2) $\psi(t) = \frac{t}{t+1}$ for all $t > 0$.

Example 2 ([21]). *Define some functions as follows: for all $t \in (0, \infty)$,*

(1) $\phi_1(t) = t$,
(2) $\phi_2(t) = \sqrt{t\sqrt{t}}$,
(3) $\phi_3(t) = te^t$.

Then $\phi_1, \phi_2, \phi_3 \in \Phi$.

Consider a metric space (X,d). By $CB(X)$, we will denote the family of all bounded and closed subsets of X. For $x \in X$ and $A, B \in CB(X)$, we define

$$D(x,B) = \inf_{y\in B} d(x,y).$$

Define a mapping $H : CB(X) \times CB(X) \to [0,1)$ by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in A} d(y,A)\right\}$$

for all $A, B \in CB(X)$. Then H is a metric on $CB(X)$, which is called the Pompeiu-Hausdorff metric induced by d.

Nadler [23] proved the following Banach contraction principle for multivalued mappings.

Theorem 5 ([23]). *Let (X,d) be a complete metric space and $S : X \longrightarrow CB(X)$ be a multivalued mapping, if there exists $\lambda \in [0,1)$ such that*

$$H(Sx, Sy) \leq \lambda d(x,y)$$

for all $x, y \in X$, then S has a fixed point $x^* \in X$ such that $x^* \in Sx^*$.

Theorem 6 ([24]). *Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ be a multivalued mapping. If there exists a function $\beta : (0,+\infty) \to [0,1)$ such that*

$$\lim_{t\longrightarrow s^+} \sup \beta(t) < 1, \text{ for all } s \in (0,\infty)$$

satisfying

$$H(T(x), T(y)) \leq \beta(d(x,y))d(x,y)$$

for all $x, y \in X$ with $x \neq y$, then T has a fixed point.

Definition 5 ([25]). *Let (X,d) be a metric space. Let $T : X \longrightarrow CB(X)$ be a multivalued mapping. Then T is said to be a generalized multivalued F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,*

$$H(Tx, Ty) > 0 \Longrightarrow \tau + F(H(Tx, Ty)) \leq F((M(x,y))),$$

where

$$M(x,y) = \max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2}\right\}.$$

HanÇer et al. [26] (see also [27]) extended the concept of θ-contraction to multivalued mappings as follows.

Definition 6 ([26]). *Let (X,d) be a metric space, $T : X \to CB(X)$ and $\theta \in \Theta$. Then, we say that T is a multivalued θ-contraction, if there exists $k \in [0,1)$ such that*

$$\theta(H(Tx, Ty)) \leq [\theta(d(x,y))]^k$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

Theorem 7. *Let (X,d) be a complete metric space and $T : X \to CB(X)$. Then the following are equivalent.*

(i) T is a multivalued θ-contraction with $\theta \in \Xi$;
(ii) T is a multivalued F-contraction with $F \in \mathcal{F}$.

Proof. The proof follows immediately from the proof of Theorem 4. □

Now we introduce the following definitions.

Definition 7. *Let (X,d) be a metric space. Let $S,T : X \longrightarrow CB(X)$. Then the pair (T,S) is said to be a generalized multivalued (ψ,ϕ)-type contraction if there exist a comparison function ψ and $\phi \in \Phi$ such that for all $x,y \in X$,*

$$H(Sx,Ty) > 0 \Longrightarrow \phi(H(Sx,Ty)) \le \psi(\phi(M(x,y))),$$

where

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2}\right\}.$$

Definition 8. *Let (X,d) be a metric space. Let $S,T : X \longrightarrow CB(X)$. Then the pair (T,S) is said to be a generalized multivalued (ψ,ϕ)-type Suzuki contraction if there exist a comparison function ψ and $\phi \in \Phi$ such that for all $x,y \in X$ with $Sx \neq Ty$,*

$$\frac{1}{2}\min\{D(x,Sx), D(y,Ty)\} < d(x,y) \Longrightarrow \phi(H(Sx,Ty)) \le \psi(\phi(M(x,y))). \quad (1)$$

2. Main Results

Theorem 8. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow CB(X)$ be generalized multivalued (ψ,ϕ)-type Suzuki contractions. If ψ is continuous, then S and T have a common fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.*

Proof. Let $x_0 \in X$. Choose $x_1 \in Sx_0$. Assume that $D(x_0,Sx_0)$ and $D(x_1,Tx_1) > 0$. Then

$$\frac{1}{2}\min\{D(x_0,Sx_0), D(x_1,Tx_1)\} < d(x_0,x_1).$$

By the definition of Hausdorff metric, there exists $x_2 \in Tx_1$.

$$0 < d(x_1,x_2) = D(x_1,Tx_1) \le H(Sx_0,Tx_1).$$

Since ϕ is nondecreasing, we have

$$\phi(d(x_1,x_2)) \le \phi(H(Sx_0,Tx_1)).$$

Hence from (1)

$$\begin{aligned} 0 &\le \phi(d(x_1,x_2)) \le \phi(H(Sx_0,Tx_1)) \\ &\le \psi(\phi(M(x_0,x_1))), \end{aligned} \quad (2)$$

where

$$\begin{aligned} M(x_0,x_1) &= \max\left\{\begin{array}{c} d(x_0,x_1), D(x_0,Sx_0), D(x_1,Tx_1), \\ \frac{D(x_0,Tx_1)+D(x_1,Sx_0)}{2} \end{array}\right\} \\ &\le \max\left\{d(x_0,x_1), D(x_1,Tx_1), \frac{D(x_0,Tx_1)}{2}\right\} \\ &\le \max\{d(x_0,x_1), D(x_1,Tx_1)\}. \end{aligned}$$

If $\max\{d(x_0,x_1), D(x_1,Tx_1)\} = D(x_1,Tx_1)$, then from (2), we have

$$\phi(d(x_1,x_2)) \le \psi(\phi(d(x_1,x_2))) < \phi(d(x_1,x_2)),$$

which is a contradiction. Thus, we conclude that $\max\{d(x_0,x_1), D(x_1,Tx_1)\} = d(x_0,x_1)$. By (2), we get that

$$\phi(d(x_1,x_2)) \le \psi(\phi(d(x_0,x_1))).$$

Similarly, for $x_2 \in Tx_1$ and $x_3 \in Sx_2$, we have

$$\begin{aligned}\phi(d(x_2,x_3)) &= \phi(D(x_2,Sx_2)) \\ &\le \phi(H(Tx_1,Sx_2)) \\ &\le \psi(\phi(M(x_1,x_2))) \\ &\le \psi(\phi(d(x_1,x_2))),\end{aligned}$$

which implies

$$\phi(d(x_2,x_3)) \le \psi(\phi(d(x_1,x_2))).$$

By continuing this process, we construct a sequence $\{x_n\}$ in X such that $x_{2i+1} \in Sx_{2i}$ and $x_{2i+2} \in Tx_{2i+1}$, $i = 0,1,2,\ldots$ and

$$\frac{1}{2}\min\left\{D(x_{2i},Sx_{2i}), D\left(x_{2i+1},Tx_{2i+1}\right)\right\} < d(x_{2i},x_{2i+1}).$$

Hence from (1), we have

$$\begin{aligned}0 &< \phi(d(x_{2i+1},x_{2i+2})) \le \phi(H(Sx_{2i},Tx_{2i+1})) \\ &\le \psi(\phi(M(x_{2i},x_{2i+1}))),\end{aligned} \tag{3}$$

where

$$\begin{aligned}M(x_{2i},x_{2i+1}) &= \max\left\{\begin{array}{c} d(x_{2i},x_{2i+1}), D(x_{2i},Sx_{2i}), D(x_{2i+1},Tx_{2i+1}), \\ \frac{D(x_{2i},Tx_{2i+1})+D(x_{2i+1},Sx_{2i})}{2}\end{array}\right\} \\ &\le \max\left\{\begin{array}{c} d(x_{2i},x_{2i+1}), d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2}), \\ \frac{d(x_{2i},x_{2i+2})}{2}\end{array}\right\} \\ &\le \max\{d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2})\}.\end{aligned}$$

If $\max\{d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2})\} = d(x_{2i+1},x_{2i+2})$, then from (3) we have

$$\begin{aligned}\phi(d(x_{2i+1},x_{2i+2})) &\le \psi(\phi(d(x_{2i+1},x_{2i+2}))) \\ &< \phi(d(x_{2i+1},x_{2i+2})),\end{aligned}$$

which is a contradiction. Thus,

$$\max\{d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2})\} = d(x_{2i},x_{2i+1}).$$

By (3), we get that

$$\phi(d(x_{2i+1},x_{2i+2})) < \psi(\phi(d(x_{2i},x_{2i+1}))).$$

This implies that

$$\frac{1}{2}\min\left\{D(x_n,Sx_n), D\left(x_{n+1},Tx_{n+1}\right)\right\} < d(x_n,x_{n+1}).$$

Hence

$$\phi(d(x_{2n+1},x_{2n+2})) < \psi(\phi(d(x_{2n},x_{2n+1}))),$$

which implies that

$$\begin{aligned}\phi\left(d\left(x_{2n+1},x_{2n+2}\right)\right) &\leq \psi\left(\phi\left(d\left(x_{2n},x_{2n+1}\right)\right)\right)\leq\psi^{2}\left(\phi\left(d\left(x_{2n-1},x_{2n}\right)\right)\right)\\ &\leq \cdots\leq\psi^{n}\left(\phi\left(d\left(x_{0},x_{1}\right)\right)\right).\end{aligned}$$

Letting $n\longrightarrow\infty$ in the above inequality, we get

$$0\leq\lim_{n\longrightarrow\infty}\phi\left(d\left(x_{2n+1},x_{2n+2}\right)\right)\leq\lim_{n\longrightarrow\infty}\psi^{n}\left(\phi\left(d\left(x_{0},x_{1}\right)\right)\right)=0,$$

which implies that

$$\lim_{n\longrightarrow\infty}\phi\left(d\left(x_{2n+1},x_{2n+2}\right)\right)=0.$$

This together with $(\Phi2)$ and Lemma 1 gives

$$\lim_{n\longrightarrow\infty}d\left(x_{2n+1},x_{2n+2}\right)=0. \tag{4}$$

Now, we prove that the sequence $\{x_n\}$ is Cauchy. Arguing by contradiction, we assume that there exist $\varepsilon>0$ and sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ of positive integers such for all $n\in\mathbb{N}$, $p_n>q_n>n$ with $d\left(x_{p(n)},x_{q(n)}\right)\geq\varepsilon$, $d\left(x_{p(n)-1},x_{q(n)}\right)<\varepsilon$. Therefore,

$$\begin{aligned}\varepsilon\leq d\left(x_{p(n)},x_{q(n)}\right)&\leq d\left(x_{p(n)},x_{p(n)-1}\right)+d\left(x_{p(n)-1},x_{q(n)}\right)\\ &<\varepsilon+d\left(x_{p(n)},x_{p(n)-1}\right).\end{aligned} \tag{5}$$

By taking the limit as $n\to\infty$ in (5), we get

$$\lim_{n\to\infty}d\left(x_{p(n)},x_{q(n)}\right)=\varepsilon. \tag{6}$$

From (4) and (5) we can choose a positive integer $n_0\geq 1$ such that

$$\frac{1}{2}\min\left\{D\left(x_{p(n)},Sx_{p(n)}\right),D\left(x_{q(n)},Tx_{q(n)}\right)\right\}<\frac{1}{2}\varepsilon<d\left(x_{p(n)},x_{q(n)}\right)$$

and hence, from (1), we get

$$\begin{aligned}0 &< \phi\left(d\left(x_{p(n)+1},x_{q(n)+1}\right)\right)\leq\phi\left(H\left(Sx_{p(n)},Tx_{q(n)}\right)\right)\\ &\leq \psi\left(\phi\left(M\left(x_{p(n)},x_{q(n)}\right)\right)\right),\end{aligned}$$

where

$$\begin{aligned}M\left(x_{p(n)},x_{q(n)}\right) &= \max\left\{\begin{array}{c} d\left(x_{p(n)},x_{q(n)}\right),D\left(x_{p(n)},Sx_{p(n)}\right),D\left(x_{q(n)},Tx_{q(n)}\right),\\ \frac{D\left(x_{p(n)},Tx_{q(n)}\right)+D\left(x_{q(n)},Sx_{p(n)}\right)}{2}\end{array}\right\}\\ &\leq \max\left\{\begin{array}{c} d\left(x_{p(n)},x_{q(n)}\right),d\left(x_{p(n)},x_{p(n)+1}\right),d\left(x_{q(n)},x_{q(n)+1}\right),\\ \frac{d\left(x_{p(n)},x_{q(n)+1}\right)+d\left(x_{q(n)},x_{p(n)+1}\right)}{2}\end{array}\right\}\\ &\leq \max\left\{\begin{array}{c} d\left(x_{p(n)},x_{q(n)}\right),d\left(x_{p(n)},x_{p(n)+1}\right),d\left(x_{q(n)},x_{q(n)+1}\right),\\ \frac{d\left(x_{p(n)},x_{p(n)+1}\right)+d\left(x_{p(n)+1},x_{q(n)+1}\right)+d\left(x_{q(n)},x_{q(n)+1}\right)}{2}\end{array}\right\}.\end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality and by using (Φ2), (4), (6), we get

$$\begin{aligned} \phi(\varepsilon) &= \lim_{n\to\infty} \phi\left(d\left(x_{p(n)+1}, x_{q(n)+1}\right)\right) \leq \lim_{n\to\infty} \phi\left(\psi\left(d\left(x_{p(n)+1}, x_{q(n)+1}\right)\right)\right) \\ &= \psi\left(\phi(\varepsilon)\right) < \phi(\varepsilon). \end{aligned}$$

This is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, we can ensure that $\{x_n\}$ converges to some point $x^* \in X$, that is, $\lim\limits_{n\to\infty} d\left(x_n, x^*\right) = 0$ and so

$$\lim_{n\to\infty} d\left(x_n, x^*\right) = \lim_{n\to\infty} d\left(x_{2n}, x^*\right) = \lim_{n\to\infty} d\left(x_{2n+1}, x^*\right) = 0. \tag{7}$$

Now we claim that

$$\frac{1}{2}\min\left\{D\left(x_n, Sx_n\right), D\left(x^*, Tx^*\right)\right\} < d\left(x_n, x^*\right) \tag{8}$$

or

$$\frac{1}{2}\min\left\{D\left(x^*, Sx^*\right), D\left(x_{n+1}, Tx_{n+1}\right)\right\} < d\left(x_{n+1}, x^*\right)$$

for all $n \in \mathbb{N}$.

Suppose that it is not the case. Then there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}\min\left\{D\left(x_m, Sx_m\right), D\left(x^*, Tx^*\right)\right\} \geq d\left(x_m, x^*\right) \tag{9}$$

and

$$\frac{1}{2}\min\left\{D\left(x^*, Sx^*\right), D\left(x_{m+1}, Tx_{m+1}\right)\right\} \geq d\left(x_{m+1}, x^*\right). \tag{10}$$

Therefore,

$$\begin{aligned} 2d\left(x_m, x^*\right) &\leq \min\left\{D\left(x_m, Sx_m\right), D\left(x^*, Tx^*\right)\right\} \\ &\leq \min\left\{d\left(x_m, x^*\right) + D\left(x^*, Sx_m\right), D\left(x^*, Tx^*\right)\right\} \\ &\leq d\left(x_m, x^*\right) + D\left(x^*, Sx_m\right) \leq d\left(x_m, x^*\right) + d\left(x^*, x_{m+1}\right), \end{aligned}$$

which implies

$$d\left(x_m, x^*\right) \leq d\left(x^*, x_{m+1}\right).$$

This together with (9) shows that

$$d\left(x_m, x^*\right) \leq d\left(x^*, x_{m+1}\right) \leq \frac{1}{2}\min\left\{D\left(x^*, Sx^*\right), D\left(x_{m+1}, Tx_{m+1}\right)\right\}. \tag{11}$$

Since $\frac{1}{2}\min\left\{D\left(x_m, Sx_m\right), D\left(x^*, Tx^*\right)\right\} < d\left(x_m, x_{m+1}\right)$, by (1), we have

$$\begin{aligned} 0 &< \phi\left(d\left(x_{m+1}, x_{m+2}\right)\right) \leq \phi\left(H\left(Sx_m, Tx_{m+1}\right)\right) \\ &\leq \psi\left(\phi\left(M\left(x_m, x_{m+1}\right)\right)\right), \end{aligned} \tag{12}$$

where

$$\begin{aligned} M\left(x_m, x_{m+1}\right) &= \max\left\{\begin{array}{c} d\left(x_m, x_{m+1}\right), D\left(x_m, Sx_m\right), D\left(x_{m+1}, Tx_{m+1}\right), \\ \frac{D(x_m, Tx_{m+1}) + D(x_{m+1}, Sx_m)}{2} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} d\left(x_m, x_{m+1}\right), d\left(x_m, x_{m+1}\right), d\left(x_{m+1}, x_{m+2}\right), \\ \frac{D(x_m, x_{m+2})}{2} \end{array}\right\} \\ &\leq \max\left\{d\left(x_m, x_{m+1}\right), d\left(x_{m+1}, x_{m+2}\right)\right\}. \end{aligned}$$

If $\max\{d(x_m, x_{m+1}), d(x_{m+1}, x_{m+2})\} = d(x_{m+1}, x_{m+2})$, then from (12) we have

$$\begin{aligned} \phi(d(x_{m+1}, x_{m+2})) &\leq \psi(\phi(d(x_{m+1}, x_{m+2}))) \\ &< \phi(d(x_{m+1}, x_{m+2})), \end{aligned}$$

which is a contradiction. Thus, we conclude that

$$\max\{d(x_m, x_{m+1}), d(x_{m+1}, x_{m+2})\} = d(x_m, x_{m+1}).$$

By (11), we get that

$$\begin{aligned} \phi(d(x_{m+1}, x_{m+2})) &\leq \psi(\phi(d(x_m, x_{m+1}))) \\ &< \phi(d(x_{2m}, x_{2m+1})). \end{aligned}$$

It follows from ($\Phi 1$) that

$$d(x_{m+1}, x_{m+2}) < d(x_m, x_{m+1}). \tag{13}$$

From (10), (11) and (13), we get

$$\begin{aligned} d(x_{m+1}, x_{m+2}) &< d(x_m, x_{m+1}) \\ &\leq d(x_m, x^*) + d(x^*, x_{m+1}) \\ &\leq \frac{1}{2}\min\left\{D(x^*, Sx^*), D\left(x_{m+1}, Tx_{_{m+1}}\right)\right\} \\ &\quad + \frac{1}{2}\min\left\{D(x^*, Sx^*), D\left(x_{m+1}, Tx_{_{m+1}}\right)\right\} \\ &= \min\left\{D(x^*, Sx^*), d\left(x_{m+1}, x_{_{m+2}}\right)\right\} \\ &\leq d\left(x_{m+1}, x_{_{m+2}}\right). \end{aligned}$$

This is a contradiction. Hence (8) holds, that is, for every $n \geq 2$

$$\frac{1}{2}\min\{D(x_n, Sx_{_n}), D(x^*, Tx^*)\} < d(x_n, x^*)$$

holds. From (1) it follows that for every $n \geq 2$

$$\begin{aligned} 0 &< \phi(D(x_{n+1}, Tx^*)) \leq \phi(H(Sx_n, Tx^*)) \\ &\leq \psi(\phi(M(x_n, x^*))), \end{aligned} \tag{14}$$

where

$$M(x_n, x^*) = \max\left\{ \begin{array}{c} d(x_n, x^*), d(x_n, x_{n+1}), D(x^*, Tx^*), \\ \frac{D(x_n, Tx^*) + d(x_{n+1}, x_{n+1})}{2} \end{array} \right\}.$$

Now we prove that $x^* \in Tx^*$. Suppose on the contrary, $D(x^*, Tx^*) > 0$. Letting $n \longrightarrow \infty$ in (14) and by using (7) and ($\Phi 3$), we obtain

$$\begin{aligned} D(x^*, Tx^*) &= \lim_{n\to\infty} \phi(D(x_{n+1}, Tx^*)) \leq \lim_{n\to\infty} \psi(\phi(M(x_n, x^*))) \\ &= \psi(\phi(M(x^*, Tx^*))) < D(x^*, Tx^*), \end{aligned}$$

which is a contradiction. Therefore, $x^* \in Tx^*$.

Similarly, we can show that $x^* \in Sx^*$. Thus, S and T have a common fixed point. □

Corollary 1. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow CB(X)$ be generalized multivalued (ψ,ϕ)-type contractions. If ψ is continuous, then S and T have a common fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.*

Example 3. *Let $X = [0,1]$. Define a function $d : X \times X \to [0,+\infty)$ by $d(x,y) = |x-y|$. Clearly, (X,d) is a complete metric space. Define $\phi : (0,\infty) \longrightarrow (0,\infty)$ by $\phi(t) = t$ for all $t > 0$. Then $\phi \in \Phi$. Also define $\psi : (0,\infty) \longrightarrow (0,\infty)$ by $\psi(t) = \frac{98t}{100}$ for all $t > 0$. Then ψ is a continuos comparison function. Define the mappings $S,T : X \longrightarrow CB(X)$ by*

$$Sx = \left[0, \frac{x}{5}\right] \text{ and } Tx = \left[0, \frac{x}{3}\right].$$

Suppose, without any loss of generality, that all x,y are nonzero and $x < y$ and $H(Sx,Ty) > 0$. Then

$$\begin{aligned} \phi(H(Sx,Ty)) &= \phi\left(H\left(\left[0,\frac{x}{5}\right],\left[0,\frac{y}{3}\right]\right)\right) = \phi\left(\left|\frac{x}{5}-\frac{y}{3}\right|\right) \\ &= \left|\frac{x}{5}-\frac{y}{3}\right| \leq \frac{98}{100}|x-y| \leq \frac{98}{100}M(x,y) \\ &= \frac{98}{100}\phi(M(x,y)) = \psi(\phi(M(x,y))). \end{aligned}$$

Hence all the conditions of Corollary 1 are satisfied and 0 is a common fixed point of S and T.

In Theorem 8, if we set $S = T$ and

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Sy), \frac{D(x,Sy)+D(y,Sx)}{2}\right\},$$

then we obtain the following results.

Corollary 2. *Let (X,d) be a complete metric space and $S : X \longrightarrow CB(X)$ be a (ψ,ϕ)-type Suzuki contraction. If ψ is continuous, then S has a fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.*

Corollary 3 ([21]). *Let (X,d) be a complete metric space and $S : X \longrightarrow X$ be a generalized (ψ,ϕ)-type Suzuki contraction. If ψ is continuous, then S has a unique fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.*

Remark 1. *Theorem 8 is an improvement and a generalization and of the main results given by Suzuki [16] and the recent result given by Liu [21].*

Remark 2. *Corollary 1 is a generalization and improvement of Nadler [23] and the recent results by Jleli et al. [19,20], HanÇer et al. [26] and Vetro [27].*

3. Some Consequences

Corollary 4. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow CB(X)$ be multivalued mappings. If there exists $\lambda \in (0,1)$ such that for all $x,y \in X$,*

$$H(Sx,Ty) \leq \lambda M(x,y),$$

where

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2}\right\}.$$

Then S and T have a common fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

Proof. The result follows from Corollary 1 by taking $\psi(t) = \lambda t$ and $\phi(t) = t$, where $\phi : (0,\infty) \longrightarrow (0,\infty)$. $\square$

Corollary 5. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow CB(X)$ be multivalued mappings. Suppose that there exist $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that for all $x, y \in X$,*

$$H(Sx,Ty) \leq a_1 d(x,y) + a_2 D(x,Sx) + a_3 D(y,Ty) + \frac{1}{2} a_4 D(y,Sx) + a_5 D(x,Ty).$$

Then S and T have a common fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

Corollary 6. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow CB(X)$ be multivalued θ-contractions, that is, there exist $\theta \in \widetilde{\Theta}$ and $k \in (0,1)$ such that*

$$\forall x,y \in X,\ H(Sx,Ty) > 0 \Longrightarrow \theta(H(Sx,Ty)) \leq [\theta(M(x,y))]^k,$$

where

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty) + D(y,Sx)}{2}\right\}.$$

Then S and T have a common fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

Proof. The result follows from Corollary 1 by taking $\psi(t) := (\ln k)\, t$ and $\phi(t) = \ln t$, where $\phi : (0,\infty) \longrightarrow (0,\infty)$. $\square$

Corollary 7. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow CB(X)$ be multivalued F-contractions, that is, there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\forall x,y \in X,\ H(Sx,Ty) > 0 \Longrightarrow \tau + F(H(Sx,Ty)) \leq F(M(x,y)),$$

where

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty) + D(y,Sx)}{2}\right\}.$$

Then S and T have a common fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

Proof. The result follows from Corollary 1 by taking $\psi(t) = e^{-\tau} t$ and $\phi(t) = e^t$, where $\phi : (0,\infty) \longrightarrow (0,\infty)$. $\square$

Corollary 8. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow CB(X)$ be multivalued mappings. Suppose that*

$$H(Sx,Ty) \leq \frac{M(x,y)}{1 + M(x,y)}, \text{ for all } x,y \in X,\ Sx \neq Ty,$$

where

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty) + D(y,Sx)}{2}\right\}.$$

Then S and T have a common fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

Proof. It follows from Corollary 1 by taking $\psi(t) := \frac{t}{1+t}$, $t > 0$ and $\phi(t) = t$, where $\phi : (0,\infty) \longrightarrow (0,\infty)$. $\square$

Corollary 9. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow CB(X)$ be multivalued mappings. Suppose that, for all $x,y \in X$, $Sx \neq Ty$,*

$$H(Sx,Ty) \leq \beta(M(x,y))\, M(x,y),$$

where

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2}\right\}$$

and β is a function from $[0,\infty)$ into $[0,\infty)$ such that $\lim\limits_{r\longrightarrow t^+} \beta(r) < 1$ for each $t \in (0,\infty)$. Then S and T have a common fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

Proof. It follows from Corollary 1 by taking $\psi(t) := \beta(t)\,t$ and $\phi(t) = t$, where $\phi : (0,\infty) \longrightarrow (0,\infty)$. □

4. Application

In this section, we present an application of our result in solving functional equations arising in dynamic programming.

For more details on dynamic programming, we refer to [28–33]. Suppose that W and D represent the state and decision spaces, respectively. The problem of related dynamic programming is reduced to solve the functional equations.

$$p(x) = \sup_{y\in D}\{g(x,y) + \Gamma(x,y,p(\xi(x,y)))\}, \text{ for } x \in W, \tag{15}$$

$$q(x) = \sup_{y\in D}\{u(x,y) + \Psi(x,y,q(\xi(x,y)))\}, \text{ for } x \in W. \tag{16}$$

These settings allow us to formulate many problems, where U and V are Banach spaces, $W \subseteq U$, $D \subseteq V$ and

$$\begin{aligned} \xi &: W \times D \longrightarrow W, \\ g,u &: W \times D \longrightarrow \mathbb{R}, \\ \Gamma,\Psi &: W \times D \times \mathbb{R} \longrightarrow \mathbb{R}. \end{aligned}$$

Our aim is to give the existence and uniqueness of common and bounded solution of functional equations given in (15) and (16). Let B(W) denote the set of all bounded real-valued functions on W. Consider,

$$d(h,k) = \sup_{x\in W} |hx - kx|.$$

Then (B(W), d) is a complete metric space. Suppose that the following hold:

$(B1)$: Γ, Ψ, g, and u are bounded and continuous.
$(B2)$: For $x \in W$, $h \in B(W)$ and $b > 0$, define $E, A : B(W) \longrightarrow B(W)$ by

$$\begin{aligned} Eh(x) &= \sup\nolimits_{y\in D}\{g(x,y) + \Gamma(x,y,h(\xi(x,y)))\}, \\ Ah(x) &= \sup\nolimits_{y\in D}\{u(x,y) + \Psi(x,y,h(\xi(x,y)))\}. \end{aligned}$$

Moreover, for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$|\Gamma(x, y, h(t)) - \Psi(x, y, k(t))| \leq \frac{M(h(t), k(t))}{M(h(t), k(t)) + 1}, \tag{17}$$

where

$$\begin{aligned} M((h(t), k(t)) &= \max\{d(h(t), k(t)), d(h(t), Eh(t)), d(k(t), Ak(t)), \\ &\quad \frac{d(h(t), Ak(t)) + d(k(t), Eh(t))}{2}\}. \end{aligned}$$

Theorem 9. *Assume that the conditions $(B1) - (B2)$ are satisfied. Then the system of functional Equations (15) and (16) has a unique common and bounded solution in $B(W)$.*

Proof. Note that $(B(W), d)$ is a complete metric space. By $(B1)$, E, A are self-mappings of $B(W)$. Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Choose $x \in W$ and $y_1, y_2 \in D$ such that

$$Eh_1 < g(x, y_1) + \Gamma(x, y_1, h_1(\xi(x, y_1)) + \lambda, \tag{18}$$

$$Ah_2 < g(x, y_2) + \Psi(x, y_2, h_2(\xi(x, y_2)) + \lambda. \tag{19}$$

Further from (18) and (19), we have

$$Eh_1 \geq g(x, y_2) + \Gamma(x, y_2, h_1(\xi(x, y_2)), \tag{20}$$

$$Ah_2 \geq g(x, y_1) + \Psi(x, y_1, h_2(\xi(x, y_1)). \tag{21}$$

Then (18) and (21) together with (17) imply

$$\begin{aligned} Eh_1(x) - Ah_2(x) &< \Gamma(x, y_1, h_1(\xi(x, y_1))) - \Psi(x, y_1, h_2(\xi(x, y_1))) + \lambda \\ &\leq |\Gamma(x, y_1, h_1(\xi(x, y_1))) - \Psi(x, y_1, h_2(\xi(x, y_1)))| + \lambda \\ &\leq \frac{M(h_1(x), h_2(x))}{M(h_1(x), h_2(x) + 1} + \lambda. \end{aligned} \tag{22}$$

Then (19) and (20) together with (17) imply

$$\begin{aligned} Ah_2(x) - Eh_1(x) &\leq \Gamma(x, y_2, h_1(\xi(x, y_2)) - \Psi(x, y_2, h_2(\xi(x, y_2)) + \lambda \\ &\leq |\Gamma(x, y_2, h_1(\xi(x, y_2)) - \Psi(x, y_2, h_2(\xi(x, y_2))| + \lambda \\ &\leq \frac{M(h_1(x), h_2(x))}{M(h_1(x), h_2(x) + 1} + \lambda, \end{aligned} \tag{23}$$

where

$$\begin{aligned} M((h_1(x), h_2(x)) &= \max\{d(h_1(x), h_2(x)), d(h_1(x), Eh_1(x)), d(h_2(x), Ah_2(x)), \\ &\quad \frac{d(h_1(t), Ah_2(t)) + d(h_2(t), Eh_1(t))}{2}\}. \end{aligned}$$

From (22) and (23), we obtain

$$|Eh_1(x) - Ah_2(x)| \leq \frac{M(h_1(x), h_2(x))}{M(h_1(x), h_2(x) + 1} \tag{24}$$

since $\lambda > 0$ was taken as an arbitrary number. The inequality (24) implies

$$d(Eh_1(x), Ah_2(x)) \leq \frac{M(h_1(x), h_2(x))}{M(h_1(x), h_2(x) + 1}.$$

Taking $\phi(t) = t, t > 0$ and $\psi(t) = \frac{t}{t+1}, t > 0$, we get

$$\phi(d(Eh_1(x), Ah_2(x))) \leq \psi(\phi(M(h_1(x), h_2(x)))).$$

Therefore, all the conditions of Corollary 1 are immediately satisfied. Thus, E and A have a common fixed point $h^* \in B(W)$, that is, $h^*(x)$ is a unique, bounded and common solution of the system of functional Equations (15) and (16). $\square$

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