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Some Remarks on Harmonic Functions in Minkowski Spaces

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Abstract: We prove that in Minkowski spaces, a harmonic function does not necessarily satisfy the mean value formula. Conversely, we also show that a function satisfying the mean value formula is not necessarily a harmonic function. Finally, we conclude that in a Minkowski space, if all harmonic functions have the mean value property or any function satisfying the mean value formula must be a harmonic function, then the Minkowski space is Euclidean.

Keywords: harmonic function; mean value formula; Minkowski spaces

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1. Introduction

Harmonic functions play a crucial role in many areas of mathematics, physics, and engineering. In mathematics, a harmonic function in a Euclidean space is a twice continuously-differentiable function that satisfies the following equation:

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial (x^i)^2} = 0.$$

It is well known that harmonic functions in Euclidean spaces admit the mean value property, and vice versa. Namely, f is a harmonic function in $\Omega(\subset \mathbb{R}^n)$, if and only if for any point $x_0 \in \Omega \subset \mathbb{R}^n$, we have (see [1,2]):

$$f(x_0) = \frac{1}{\mathrm{vol}(B_r(x_0))} \int_{B_r(x_0)} f(x) dx^1 \cdots dx^n,$$

where $B_r(x_0)(\subset \Omega)$ denotes the ball centered at x_0 of radius r. In this short note, we are able to discuss this issue in a Minkowski space.

Recall that in a Euclidean space $(\mathbb{R}^n, \|\cdot\|)$, the norm of a vector $y = (y^1, \cdots, y^n)$ is defined as:

$$\|y\| := \sqrt{\sum_{i=1}^{n} (y^i)^2}.$$

Thus, by computing the Hessian of $\|y\|^2$, the metric on a Euclidean space is given by:

$$g := \sum_{i=1}^{n} \delta_{ij} dx^i dx^j,$$

where $\delta_{ii} = 1$ for $1 \leq i \leq n$ and $\delta_{ij} = 0$ if $i \neq j$.

To compare it with the Minkowski metric, we now equip it with a norm $F(y)$ on $\mathbb{R}^n$ such that:

$$g := \sum_{i=1}^{n} g_{ij}(y)dx^i dx^j, \quad g_{ij} := \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}.$$

Here, the functions $g_{ij}(y)$ are smooth in $\mathbb{R}^n \setminus 0$, and $g_{ij}(cy) = cg_{ij}(y)$ for any $c > 0$. We also require the matrix (g_{ij}) to be positive definite. Then, $(\mathbb{R}^n, F)$ is called a Minkowski space. It is a special Finsler manifold with zero flag curvature (see [3]). Obviously, a Euclidean space is a Minkowski space.

Up to now, there have been many research works about harmonic functions on Riemannian manifolds. For more details, we refer the reader to [4–8], and so on. In the meantime, the study of harmonic functions has now been developed for Finsler manifolds. One can see Xia [9], Zhang [10], and the references therein. For a more general harmonic map on Finsler manifolds, we refer the reader to [11].

2. Preliminaries

In this section, we shall briefly review some fundamental concepts that are necessary for the present paper.

Let $(\mathbb{R}^n, F)$ be a Minkowski space. For a smooth function f, the gradient of f at x is defined by $\nabla f(x) := \mathcal{L}^{-1}(df)$, where $\mathcal{L} : T_x M \to T_x^* M, y \mapsto \sum_{i=1}^n F\frac{\partial F}{\partial y^i}dx^i$ is the Legendre transform. Specifically, ∇f can be written as:

$$\nabla f(x) = \begin{cases} \sum\limits_{i,j=1}^{n} g^{*ij}(x, df)\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}, & df(x) \neq 0; \\ 0, & df(x) = 0, \end{cases}$$

where $g^{*ij}(x,\eta) = \frac{1}{2}\frac{\partial^2(F^*(x,\eta))^2}{\partial \eta_i \partial \eta_j}$ and F^* is the dual metric of F. It holds that $g^{ij}(x, \nabla f) = g^{*ij}(x, df)$. However, ∇f may not be necessarily written explicitly. We note that if f is a smooth function, then the gradient ∇f is smooth in $\{x \in \mathbb{R}^n | df(x) \neq 0\}$ and only continuous at x where $df(x) = 0$.

Let $V = V^i \frac{\partial}{\partial x^i}$ be a smooth vector field. The divergence of V with respect to an arbitrary volume form $d\mu$ is defined by:

$$\mathrm{div} V := \sum_{i=1}^{n} \left(\frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \log \sigma}{\partial x^i} \right),$$

where $d\mu = \sigma(x)dx^1 \cdots dx^n$. Then, the Finsler-Laplacian of f can be defined by:

$$\Delta f := \mathrm{div}(\nabla f).$$

There are many Finslerian Laplacians, as described in the Kluwer monograph edited by Peter Antonelli and Brad Lackey. In this article, we discuss the nonlinear Laplacian proposed by Shen [3].

For simplicity, we choose the Busemann–Hausdorff volume form $d\mu$, for which in a Minkowski space, the $\sigma(x)$ is constant. Then, in a Minkowski space $(\mathbb{R}^n, F)$, the Laplacian of f can be expressed as:

$$\Delta f(x) = \sum_{i,j=1}^{n} g^{ij}(\nabla f)\frac{\partial^2 f}{\partial x^i x^j}, \quad \text{if} \quad df(x) \neq 0.$$

In fact, setting $\nabla f(x) = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}$, $f_i = \frac{\partial f}{\partial x^i}$, and $f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$, we have $f^i = \sum_{j=1}^{n} g^{ij}(\nabla f) f_j$, and thus:

$$\begin{aligned}
\Delta f &= \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left(\sum_{j=1}^{n} g^{ij}(\nabla f) f_j\right) = \sum_{i,j=1}^{n} \frac{\partial g^{ij}(\nabla f)}{\partial x^i} f_j + \sum_{i,j=1}^{n} g^{ij}(\nabla f) f_{ij} \\
&= -\sum_{i,j,p,q=1}^{n} g^{ip}(\nabla f) \frac{\partial g_{pq}(\nabla f)}{\partial x^i} g^{jq}(\nabla f) f_j + \sum_{i,j=1}^{n} g^{ij}(\nabla f) f_{ij} \\
&= -\sum_{i,p,q=1}^{n} g^{ip}(\nabla f) \frac{\partial g_{pq}(\nabla f)}{\partial x^i} f^q + \sum_{i,j=1}^{n} g^{ij}(\nabla f) f_{ij} \\
&= -\sum_{i,p,q,r=1}^{n} g^{ip}(\nabla f) \frac{\partial g_{pq}(\nabla f)}{\partial f^r} \frac{\partial f^r}{\partial x^i} f^q + \sum_{i,j=1}^{n} g^{ij}(\nabla f) f_{ij} \\
&= -\sum_{i,p,r=1}^{n} g^{ip}(\nabla f) \left(\sum_{q=1}^{n} \frac{\partial g_{pr}(\nabla f)}{\partial f^q} f^q\right) \frac{\partial f^r}{\partial x^i} + \sum_{i,j=1}^{n} g^{ij}(\nabla f) f_{ij} \\
&= \sum_{i,j=1}^{n} g^{ij}(\nabla f) f_{ij}.
\end{aligned}$$

The sixth step is due to $\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j} = \frac{1}{2} \frac{\partial^3 (F^2)}{\partial y^i \partial y^j \partial y^k}$ for any $1 \leq i, j, k \leq n$, while the final step follows from the Euler lemma, which shows that $\sum_{i=1}^{n} \frac{\partial h(y)}{\partial y^i} y^i = \lambda h(y)$ if $h(cy) = c^{\lambda} h(y)$ for $c > 0$. The Laplacian in a Minkowski space is a nonlinear operator, and when $(\mathbb{R}^n, F)$ is a Euclidean space, it is just $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial (x^i)^2}$.

If a smooth function f satisfies:

$$\Delta f = 0 \quad \text{in} \quad \Omega(\subset \mathbb{R}^n), \tag{1}$$

then f is called a harmonic function in Ω. Since Δf has no definition at x where $df(x) = 0$, in general, (1) is viewed in the sense of the distribution:

$$\int_{\Omega} d\varphi(\nabla f) dx^1 \cdots dx^n = 0, \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

Apparently, in a Minkowski space, linear functions are harmonic functions. However, this is not necessarily true if the volume form is not Busemann–Hausdorff. In [9], Xia obtained a Liouville-type theorem on harmonic functions in a complete Finsler manifold. Based on this, we conclude that any positive or negative harmonic function in a Minkowski space is constant.

In the Finsler setting, the definition of the harmonic function is the same as the Riemannian case in form. Furthermore, it is easy to see that in a Finsler manifold $(M, F, d\mu)$, the harmonic functions are the local minimizers of the energy functional:

$$E(u) = \int_M F^2(x, \nabla u) d\mu.$$

Unlike the Laplacian on Riemannian manifolds, the Finsler Laplacian is a nonlinear operator. In fact, this is the major difference between Finsler and Riemannian manifolds. Therefore, all harmonic functions in Finsler manifolds cannot construct a vector space, while in the Riemannian situation, the kernel of the Laplacian is not the same.

3. Some Important Counterexamples

We know that, in Euclidean spaces, a function is harmonic if and only if it satisfies the mean value formula. In the following, we will show that this is not true in a Minkowski space. The proofs are simple and straightforward by giving some counterexamples.

Example 1. *Let* $(\mathbb{R}^n, F)$ *be a Minkowski space, where* $F = \sqrt{\sum_{i=1}^n (y^i)^2} + \frac{1}{2}y^1$. *Suppose that* $f(x) = \sum_{i=1}^n a_i x^i + b$. *Then,* $f(x)$ *is a harmonic function and* $f(0) = b$. *Notice that in a Minkowski space, the ball centered at* x_0 *of radius r is* $B_r(x_0) = \{x \in \mathbb{R}^n | F(x - x_0) \le r\}$. *A direct computation yields:*

$$
\begin{aligned}
&\int_{F(x)\le 1} f(x)dx^1 \cdots dx^n \\
=&\int_{\frac{3}{4}(x^1+\frac{2}{3})^2+\sum_{i=2}^n (x^i)^2 \le \frac{4}{3}} f(x)dx^1 \cdots dx^n \\
=&\int_{\frac{3}{4}(x^1+\frac{2}{3})^2+\sum_{i=2}^n (x^i)^2 \le \frac{4}{3}} (a_1 x^1 + b)dx^1 \cdots dx^n \\
=&\int_{\frac{3}{4}(\tilde{x}^1)^2+\sum_{i=2}^n (x^i)^2 \le \frac{4}{3}} (a_1 \tilde{x}^1 - \frac{2}{3}a_1 + b)d\tilde{x}^1 \cdots dx^n \quad (by \quad \tilde{x}^1 = x^1 + \frac{2}{3}) \\
=&\int_{\frac{3}{4}(\tilde{x}^1)^2+\sum_{i=2}^n (x^i)^2 \le \frac{4}{3}} (-\frac{2}{3}a_1 + b)d\tilde{x}^1 \cdots dx^n \\
=&(-\frac{2}{3}a_1 + b)\int_{\frac{3}{4}(x^1+\frac{2}{3})^2+\sum_{i=2}^n (x^i)^2 \le \frac{4}{3}} dx^1 \cdots dx^n \\
=&(-\frac{2}{3}a_1 + b)vol(F(x) \le 1).
\end{aligned}
$$

Clearly, when $a_1 \ne 0$, *one obtains:*

$$
f(0) \ne \frac{1}{vol(F(x) \le 1)} \int_{F(x)\le 1} f(x)dx^1 \cdots dx^n.
$$

From the discussion above, we have:

Theorem 1. *In a Minkowski space, a harmonic function does not necessarily satisfy the mean value property.*

In what follows, we will consider the functions satisfying the mean value formula.

Theorem 2. *In a Minkowski space, a function satisfying the mean value formula is not necessarily a harmonic function.*

Proof. Let $(\mathbb{R}^n, F)$ be a Minkowski space where F is defined as Example 1. Let $u(\bar{x}) = u(x^2, \cdots, x^n)$ be a harmonic function in a domain of the Euclidean space $(\mathbb{R}^{n-1}, \bar{\alpha})$ where $\bar{\alpha} = \sqrt{\sum_{i=2}^n (y^i)^2}$. In this case, one can easily check:

$$
\Delta_{(\mathbb{R}^{n-1},\bar{\alpha})} u = \sum_{i=2}^n \frac{\partial^2 u}{\partial (x^i)^2} = 0.
$$

Thus, $u(\bar{x})$ satisfies the mean value formula in $(\mathbb{R}^{n-1}, \bar{\alpha})$. Set:

$$
f(x^1, x^2, \cdots, x^n) = u(x^2, \cdots, x^n).
$$

Then $f(0) = u(0)$, and by the mean value formula in Euclid spaces, we have:

$$\begin{aligned}
&\int_{F(x)\leq 1} f(x)dx^1 \cdots dx^n \\
=&\int_{\frac{3}{4}(x^1+\frac{2}{3})^2+\sum_{i=2}^n (x^i)^2 \leq \frac{4}{3}} f(x)dx^1 \cdots dx^n \\
=&\int_{\frac{3}{4}(x^1+\frac{2}{3})^2+\sum_{i=2}^n (x^i)^2 \leq \frac{4}{3}} u(x^2, \cdots, x^n)dx^1 \cdots dx^n \\
=&\int_{-2}^{\frac{2}{3}} dx^1 \int_{\sum_{i=2}^n (x^i)^2 \leq \frac{4}{3} - \frac{3}{4}(x^1+\frac{2}{3})^2} u(x^2, \cdots, x^n)dx^2 \cdots dx^n \\
=&u(0, \cdots, 0) \int_{-2}^{\frac{2}{3}} dx^1 \int_{\sum_{i=2}^n (x^i)^2 \leq \frac{4}{3} - \frac{3}{4}(x^1+\frac{2}{3})^2} dx^2 \cdots dx^n \\
=&f(0) \int_{-2}^{\frac{2}{3}} dx^1 \int_{\sum_{i=2}^n (x^i)^2 \leq \frac{4}{3} - \frac{3}{4}(x^1+\frac{2}{3})^2} dx^2 \cdots dx^n \\
=&f(0) \int_{\frac{3}{4}(x^1+\frac{2}{3})^2+\sum_{i=2}^n (x^i)^2 \leq \frac{4}{3}} dx^1 \cdots dx^n \\
=&f(0)\mathrm{vol}(F(x) \leq 1).
\end{aligned}$$

Therefore, the function $f(x)$ satisfies the mean value formula in $(\mathbb{R}^n, F)$. However, by a direct calculation, we see that:

$$\Delta f(x) = \sum_{i,j=1}^{n} g^{ij}(\nabla f) \frac{\partial^2 f}{\partial x^i \partial x^j},$$

is not necessarily equal to zero everywhere. Indeed, one can take $f(x) = (x^2 + 1) + x^2 x^3$ in $(\mathbb{R}^n, F)$ to check. This completes the proof. □

4. Further Discussion and Conclusions

To give a cleaner path to the question, we list some examples above that are special cases of a Randers–Minkowski space. From the discussions, we can conclude that in a Minkowski space, a harmonic function has no relationship with the mean value formula, and vice versa. One can see that the method of the proof on the mean value formula in Euclidean spaces does not work here. Indeed, if $\Delta f = 0$, then we have:

$$0 = \int_{F(x)\leq 1} \Delta f dx^1 \cdots dx^n = \int_{F(x)=1} g_{\mathbf{n}}(\mathbf{n}, \nabla f) d\nu \neq \int_{F(x)=1} \frac{\partial f}{\partial \mathbf{n}} d\nu,$$

where $\mathbf{n}$ is the normal vector that points outwards $\{F(x) = 1\}$, and $d\nu$ denotes the induced volume form.

By the definition of harmonic functions, we see that it depends on both the metric and the volume form. Maybe a Minkowski metric is not so "good" as the Euclidean one, and some properties of harmonic functions cannot hold. In the following, we show that the mean value property has its own unique feature of harmonic function in Euclidean spaces. In fact, we prove the following two theorems.

Theorem 3. *If all harmonic functions defined on a Minkowski space satisfy the mean value property, then the Minkowski space is Euclidean.*

Proof. Let f be a harmonic function in a Minkowski space (M, F). That is $\Delta f = 0$, and thus:

$$0 = \int_{F(x-x_0)\leq C} \Delta f dx^1 \cdots dx^n = \int_{F(x-x_0)=C} g_{\frac{\partial}{\partial r}}(\frac{\partial}{\partial r}, \nabla f) d\nu, \tag{2}$$

where $\frac{\partial}{\partial r} = \frac{x-x_0}{F(x-x_0)}$ is the normal vector that points outwards $\{F(x-x_0)=C\}$. This is because, for any curve $x=x(t)$ with $w=x'(t)$ on $\{F(x-x_0)=C\}$, one has:

$$0 = \sum_{i=1}^{n} F_{x^i} \frac{dx^i}{dt} = \frac{g_{x-x_0}(x-x_0, w)}{C}.$$

Since f satisfies the mean value formula, we also have:

$$0 = \frac{\partial}{\partial r} \int_{F(x-x_0)=C} f dv = \int_{F(x-x_0)=C} \frac{\partial f}{\partial r} dv, \quad \forall C. \tag{3}$$

Combining (2) and (3), we obtain that, for any harmonic function f, any x_0, and any constant C,

$$\int_{F(x-x_0)=C} \left(g_{\frac{\partial}{\partial r}}(\frac{\partial}{\partial r}, \nabla f) - \frac{\partial f}{\partial r} \right) dv = 0.$$

The arbitrary choice of f, x_0, and C gives that:

$$g_{\frac{\partial}{\partial r}}(\frac{\partial}{\partial r}, \nabla f) - \frac{\partial f}{\partial r} = 0.$$

Given $x_0 = 0$, a direct computation yields:

$$\sum_{i,j,k=1}^{n} g_{ij}(x) x^i g^{jk}(\nabla f) \frac{\partial f}{\partial x^k} = \sum_{k=1}^{n} x^k \frac{\partial f}{\partial x^k}.$$

Let the harmonic function $f = x^p$. Then:

$$\sum_{i,j=1}^{n} g_{ij}(x) x^i g^{jp}(\nabla x^p) = x^p.$$

Notice that ∇x^p is a constant vector. Taking the derivation by x^q in both sides above, we reach:

$$2 \sum_{i,j=1}^{n} C_{ijq}(x) x^i g^{jp}(\nabla x^p) + \sum_{j=1}^{n} g_{qj}(x) g^{jp}(\nabla x^p) = \delta_q^p.$$

Since $\sum_{i=1}^{n} C_{ijq}(x) x^i = 0$, it follows that:

$$\sum_{j=1}^{n} g_{qj}(x) g^{jp}(\nabla x^p) = \delta_q^p.$$

Taking the derivation by x^s again, we deduce:

$$\sum_{j=1}^{n} C_{qjs}(x) g^{jp}(\nabla x^p) = 0.$$

Now that the matrix $(g^{jp}(\nabla x^p))$ is positive, the Cartan tensor $C_{qjs}(x) = 0$ for all q, j and s, which implies that F is Euclidean. This finishes the proof. □

From the proof above, we can also obtain the following.

Theorem 4. *If any function defined on a Minkowski space admitting the mean value property is harmonic, then the Minkowski space is Euclidean.*

In Section 2, we have shown that using the BH volume form, any linear function is harmonic, and also, any positive or negative harmonic function in a Minkowski space is constant. At the end of this section, we again give a property of the harmonic function in Minkowski spaces.

Proposition 1. *Let Ω be a bounded domain in a Minkowski space. Assume that f is a harmonic function in Ω and $f|_{\partial\Omega} = C$. Then, f is constant in $\overline{\Omega}$.*

Remark 1. *If the Minkowski space is Euclidean, the result is obvious from the maximum principle. However, there is no such maximum principle in Finsler manifolds since Δf has no definition at extreme points.*

Proof. We may as well suppose that $f|_{\partial\Omega} = 0$. Otherwise, we replace it by $\tilde{f} = f - C$. Since $\Delta f = 0$, we have:

$$\frac{1}{2}\Delta^{\nabla f} f^2 = f\Delta f + F(\nabla f)^2 = F(\nabla f)^2.$$

By the divergence theorem, it follows that:

$$\begin{aligned}\int_\Omega \frac{1}{2}\Delta^{\nabla f} f^2 dx^1 \cdots dx^n &= \int_\Omega \mathrm{div}(f\nabla f) dx^1 \cdots dx^n \\ &= \int_{\partial\Omega} g_{\mathbf{n}}(\mathbf{n}, f\nabla f) dv = 0,\end{aligned}$$

where $\mathbf{n}$ is the normal vector that points outwards $\partial\Omega$. This gives that:

$$\int_\Omega F(\nabla f)^2 dx^1 \cdots dx^n = 0,$$

which yields $\nabla f = 0$, and thus, f is constant. □

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