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Extended Local Convergence for the Combined Newton-Kurchatov Method Under the Generalized Lipschitz Conditions

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Abstract: We present a local convergence of the combined Newton-Kurchatov method for solving Banach space valued equations. The convergence criteria involve derivatives until the second and Lipschitz-type conditions are satisfied, as well as a new center-Lipschitz-type condition and the notion of the restricted convergence region. These modifications of earlier conditions result in a tighter convergence analysis and more precise information on the location of the solution. These advantages are obtained under the same computational effort. Using illuminating examples, we further justify the superiority of our new results over earlier ones.

Keywords: nonlinear equation; iterative process; non-differentiable operator; Lipschitz condition

MSC: 65H10; 65J15; 47H17

1. Introduction

Consider the nonlinear equation

$$F(x) + Q(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable nonlinear operator on an open convex subset D of a Banach space E_1 with values in a Banach space E_2, and $Q : D \to E_2$ is a continuous nonlinear operator.

Let x, y be two points of D. A linear operator from E_1 into E_2, denoted $Q(x, y)$, which satisfies the condition

$$Q(x, y)(x - y) = Q(x) - Q(y) \tag{2}$$

is called a divided difference of Q at points x and y.

Let x, y, z be three points of D. A operator $Q(x, y, z)$ will be called a divided difference of the second order of the operator Q at the points x, y and z, if it satisfies the condition

$$Q(x, y, z)(y - z) = Q(x, y) - Q(x, z). \tag{3}$$

A well-known simple difference method for solving nonlinear equations $F(x) = 0$ is the Secant method

$$x_{n+1} = x_n - (F(x_{n-1}, x_n))^{-1} F(x_n), \; n = 0,1,2,\ldots, \tag{4}$$

where $F(x_{n-1}, x_n)$ is a divided difference of the first order of $F(x)$ and x_0, x_{-1} are given.

Secant method for solving nonlinear operator equations in a Banach space was explored by the authors [1–6] under the condition that the divided differences of a nonlinear operator F satisfy the Lipschitz (Hölder) condition with constant L of type

$$\|F(x,y) - F(u,v)\| \le L(\|x-u\| + \|y-v\|).$$

In [7] a one-point iterative Secant-type method with memory was psoposed.

In [8,9] the Kurchatov method under the classical Lipschitz conditions for the divided differences of the first and second order was explored and its quadratic convergence of it was determined. The iterative formula of Kurchatov method has the form [1,8–11]

$$x_{n+1} = x_n - (F(2x_n - x_{n-1}, x_{n-1}))^{-1} F(x_n),\ n = 0,1,2,\ldots. \tag{5}$$

Related articles but with stronger convergence criteria exist; see works of Argyros, Ezquerro, Hernandez, Rubio, Gutierrez, Wang, Li [1,12–15] and references therein.

In [14] which dealt with the study of the Newton method, it was proposed that there are generalized Lipschitz conditions for the nonlinear operator, in which instead of constant L, some positive integrable function is used.

In our work [16], we introduced, for the first time, a similar generalized Lipschitz condition for the operator of the first order divided difference, and under this condition, the convergence of the Secant method was studied and it was found that its convergence order is $(1+\sqrt{5})/2$.

In [17], we introduced a generalized Lipschitz condition for the divided differences of the second order, and we have studied the local convergence of the Kurchatov method (5).

Note that in many papers, such as [3,18–21], the authors investigated the Secant and Secant-type methods under the generalized conditions for the first divided differences of the form

$$\|(F(x,y) - F(u,v)))\| \le \omega(\|x-y\|, \|u-v\|) \quad \forall x,y,u,v \in D, \tag{6}$$

where $\omega : \mathbf{R}_+ \times \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ is continuous nondecreasing function in their two arguments. Under these same conditions, in the work of Argyros [10], it was proven that there is a semi-local convergence of the Kurchatov method and in [22] of Ren and Argyros the semi-local convergence of a combined Kurchatov method and Secant method was demonstrated. In both cases, only the linear convergence of the methods is received.

We also refer the reader to the intersting paper by Donchev et al. [23], where several other relaxed Lipschitz conditions are used in the setting of fixed points for these conditions. Clearly, our results can be written in this setting too in an analogous way.

In [24], we first proposed and studied the local convergence of the combined Newton-Kurchatov method

$$x_{n+1} = x_n - (F'(x_n) + Q(2x_n - x_{n-1}, x_{n-1}))^{-1}(F(x_n) + Q(x_n)),\ n = 0,1,2,\ldots, \tag{7}$$

where $F'(u)$ is a Fréchet derivative, $Q(u,v)$ is a divided difference of the first order, x_0, x_{-1} are given, which is built on the basis of the mentioned Newton and Kurchatov methods. Semi-local convergence of the method (7) under the classical Lipschitz conditions is studied in the mentioned article, but the convergence only with the order $(1+\sqrt{5})/2$ has been determined.

In [25], we studied the method (7) under relatively weak, generalized Lipschitz conditions for the derivatives and divided differences of nonlinear operators. Setting $Q(x) \equiv 0$, we receive the results for the Newton method [14], and when $F(x) \equiv 0$ we got the known results for Kurchatov method [9,17]. We proved the quadratic order of convergence of the method (7), which is higher than the convergence order $(1+\sqrt{5})/2$ for the Newton–Secant method [1,26–28]

$$x_{n+1} = x_n - (F'(x_n) + Q(x_{n-1}, x_n))^{-1}(F(x_n) + Q(x_n)),\ n = 0,1,2,\ldots,\ . \tag{8}$$

The results of the numerical study of the method (7) and other combined methods on the test problems are provided in our works [24,28].

In this work, we continue to study a combined method (7) for solving nonlinear Equation (1), but with optimization considerations resulting in a tighter analysis than in [25].

The rest of the article is structured as follows: In Section 2, we present the local convergence analysis of the method (7) and the uniquness ball for solution of the equation. Section 3 contains the Corollaries of Theorems from Section 2. In Section 4, we provide the numerical example. The article ends with some conclusions.

2. Local Convergence of Newton-Kurchatov Method (7)

Let us denote $B(x_0, r) = \{x : \|x - x_0\| < r\}$ an open ball of radius $r > 0$ with center at point $x_0 \in D$, $B(x_0, r) \subset D$.

Condition on the divided difference operator $Q(x, y)$

$$\|Q(x, y) - Q(u, v)\| \leq L(\|x - u\| + \|y - v\|) \quad \forall x, y, u, v \in D \tag{9}$$

is called Lipschitz condition in domain D with constant $L > 0$. If the condition is being fulfilled

$$\|Q(x, y) - Q'(x_0)\| \leq L(\|x - x_0\| + \|y - x_0\|) \quad \forall x, y \in B(x_0, r), \tag{10}$$

then we call it the center Lipschitz condition in the ball $B(x_0, r)$ with constant L.

However, L in Lipschitz conditions can be not a constant, and can be a positive integrable function. In this case, if for $x_* \in D$ inverse operator $[F'(x_*)]^{-1}$ exists, then the conditions (9) and (10) for $x_0 = x_*$ can be replaced respectively for

$$\|Q'(x_*)^{-1}(Q(x, y) - Q(u, v))\| \leq \int_0^{\|x-y\|+\|u-v\|} L(t)dt \quad \forall x, y, u, v \in D \tag{11}$$

and

$$\|Q'(x_*)^{-1}(Q(x, y) - Q'(x_*))\| \leq \int_0^{\|x-x_*\|+\|y-x_*\|} L(t)dt \quad \forall x, y \in B(x_*, r). \tag{12}$$

Simultaneously

Lipschitz conditions (11) and (12) are called generalized Lipschitz conditions or Lipschitz conditions with the L average.

Similarly, we introduce the generalized Lipschitz condition for the divided difference of the second order

$$\|Q'(x_*)^{-1}(Q(u, x, y) - Q(v, x, y))\| \leq \int_0^{\|u-v\|} N(t)dt \; \forall x, y, u, v \in B(x_*, r), \tag{13}$$

where N is a positive integrable function.

Remark 1. *Note than the operator F is Fréchet differentiable on D when the Lipschitz conditions (9) or (11) are fulfilled $\forall x, y, u, v \in D$ (the divided differences $F(x, y)$ are Lipschitz continuous on D) and $F(x, x) = F'(x)$ $\forall x \in D$ [29].*

Suppose that equation

$$\int_0^r L_1^0(u)du + \int_0^{2r} L_2^0(u)du + 2r\int_0^{2r} N_0(u)du = 1.$$

has at least one positive solution. Denote by r_0 the smallest such solution. Set $D_0 = D \cap B(x_*, r_0)$

The radius of the convergence ball and the convergence order of the combined Newton–Kurchatov method (7) are determined in next theorem.

Theorem 1. *Let F and Q be continuous nonlinear operators defined in open convex domain D of a Banach space E_1 with values in the Banach space E_2. Let us suppose, that: (1) $H(x) \equiv F(x) + Q(x) = 0$ has a solution $x_* \in D$, for which there exists a Fréchet derivative $H'(x_*)$ and it is invertible; (2) F has the Fréchet derivative of the first order, and Q has divided differences of the first and second order on $B(x_*, 3r) \subset D$, so that for each $x, y, u, v \in D$*

$$\|H'(x_*)^{-1}(F'(x) - F'(x_*))\| \le \int_0^{\rho(x)} L_1^0(u)du, \tag{14}$$

$$\|H'(x_*)^{-1}(Q(x,y) - Q(x_*,x_*))\| \le \int_0^{\|x-x_*\|+\|y-x_*\|} L_2^0(t)dt, \tag{15}$$

$$\|H'(x_*)^{-1}(Q(u,x,y) - Q(v,x,y))\| \le \int_0^{\|u-v\|} N_0(t)dt, \tag{16}$$

and for each $x, y, u, v \in D_0$

$$\|H'(x_*)^{-1}(F'(x) - F'(x^\theta))\| \le \int_{\theta\rho(x)}^{\rho(x)} L_1(u)du, \; 0 \le \tau \le 1, \tag{17}$$

$$\|H'(x_*)^{-1}(Q(x,y) - Q(u,v))\| \le \int_0^{\|x-u\|+\|y-v\|} L_2(t)dt, \tag{18}$$

$$\|H'(x_*)^{-1}(Q(u,x,y) - Q(v,x,y))\| \le \int_0^{\|u-v\|} N(t)dt, \tag{19}$$

where $x^\theta = x_* + \theta(x - x_*)$, $\varrho(x) = \|x - x_*\|$, L_1^0, L_2^0, N_0 L_1, L_2 and N are positive nondecreasing integrable functions and $r > 0$ satisfies the equation

$$\frac{\frac{1}{r}\int_0^r L_1(u)udu + \int_0^r L_2(u)du + 2r\int_0^{2r} N(u)du}{1 - \left(\int_0^r L_1^0(u)du + \int_0^{2r} L_2^0(u)du + 2r\int_0^{2r} N_0(u)du\right)} = 1. \tag{20}$$

Then for all $x_0, x_{-1} \in B(x_*, r)$ the iterative method (7) is well defined and the generated by it sequence $\{x_n\}_{n\ge0}$, which belongs to $B(x_*, r)$, converges to x_* and satisfies the inequality

$$\|x_{n+1} - x_*\| \le e_n := \frac{\frac{1}{\rho(x_n)}\int_0^{\rho(x_n)} L_1(u)udu + \int_0^{\rho(x_n)} L_2(u)du + \int_0^{\|x_n-x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|}{1 - \left(\int_0^{\rho(x_n)} L_1^0(u)du + \int_0^{2\rho(x_n)} L_2^0(u)du + \int_0^{\|x_n-x_{n-1}\|} N_0(u)du\|x_n - x_{n-1}\|\right)}\|x_n - x_*\|. \tag{21}$$

Proof. First we show that $f(t) = \frac{1}{t^2}\int_0^t L_1(u)udu$, $g(t) = \frac{1}{t}\int_0^t L_2(u)du$, $h(t) = \frac{1}{t}\int_0^t N(u)du$, $f_0(t) = \frac{1}{t^2}\int_0^t L_1^0(u)udu$, $g_0(t) = \frac{1}{t}\int_0^t L_2^0(u)du$, $h_0(t) = \frac{1}{t}\int_0^t N_0(u)du$ monotonically nondecreasing with respect to t. Indeed, under the monotony of L_1, L_2, N we have

$$\left(\frac{1}{t_2^2}\int_0^{t_2} - \frac{1}{t_1^2}\int_0^{t_1}\right)L_1(u)udu = \left(\frac{1}{t_2^2}\int_{t_1}^{t_2} + \left(\frac{1}{t_2^2} - \frac{1}{t_1^2}\right)\int_0^{t_1}\right)L_1(u)udu \ge$$

$$\ge L(t_1)\left(\frac{1}{t_2^2}\int_{t_1}^{t_2} + \left(\frac{1}{t_2^2} - \frac{1}{t_1^2}\right)\int_0^{t_1}\right)udu = L_1(t_1)\left(\frac{1}{t_2^2}\int_0^{t_2} - \frac{1}{t_1^2}\int_0^{t_1}\right)udu = 0,$$

$$\Big(\frac{1}{t_2}\int_0^{t_2} - \frac{1}{t_1}\int_0^{t_1}\Big)L_2(u)du = \Big(\frac{1}{t_2}\int_{t_1}^{t_2} + \Big(\frac{1}{t_2} - \frac{1}{t_1}\Big)\int_0^{t_1}\Big)L_2(u)du \geq$$

$$\geq L_2(t_1)\Big(\frac{1}{t_2}\int_{t_1}^{t_2} + \Big(\frac{1}{t_2} - \frac{1}{t_1}\Big)\int_0^{t_1}\Big)du = L_2(t_1)\Big(\frac{t_2 - t_1}{t_2} + t_1\Big(\frac{1}{t_2} - \frac{1}{t_1}\Big)\Big) = 0$$

for $0 < t_1 < t_2$. So, $f(t)$, $g(t)$ are nondecreasing with respect to t. Similarly we get for $h(t)$,$f_0(t)$, $g_0(t)$ and $h_0(t)$.

We denote by A_n linear operator $A_n = F'(x_n) + Q(2x_n - x_{n-1}, x_{n-1})$. Easy to see that if $x_n, x_{n-1} \in B(x_*, r)$, then $2x_n - x_{n-1}, x_{n-1} \in B(x_*, 3r)$. Then A_n is invertible and the inequality holds

$$\|A_n^{-1}H'(x_*)\| = \|[I - (I - H'(x_*)^{-1}A_n)]^{-1}\| \leq$$
$$\leq \Big(1 - \Big(\int_0^{\rho(x_n)} L_1^0(u)du + \int_0^{2\rho(x_n)} L_2^0(u)du + \int_0^{\|x_n - x_{n-1}\|} N_0(u)du\|x_n - x_{n-1}\|\Big)\Big)^{-1}. \quad (22)$$

Indeed from the formulas (14)–(16) we get

$$\|I - H'(x_*)^{-1}A_n\| = \|H'(x_*)^{-1}(F'(x_*) - F'(x_n) + Q(x_*, x_*) - Q(x_n, x_n) +$$
$$+ Q(x_n, x_n) - Q(2x_n - x_{n-1}, x_{n-1})\|) \leq \int_0^{\rho(x_n)} L_1^0(u)du + \|H'(x_*)^{-1}(Q(x_*, x_*) -$$
$$- Q(x_n, x_n) + Q(x_n, x_n) - Q(x_n, x_{n-1}) + Q(x_n, x_{n-1}) - Q(2x_n - x_{n-1}, x_{n-1}))\| \leq$$
$$\leq \int_0^{\rho(x_n)} L_1^0(u)du + \int_0^{2\rho(x_n)} L_2^0(u)du +$$
$$+ \|H'(x_*)^{-1}(Q(x_n, x_{n-1}, x_n) - Q(2x_n - x_{n-1}, x_{n-1}, x_n))(x_n - x_{n-1})\| \leq$$
$$\leq \int_0^{\rho(x_n)} L_1^0(u)du + \int_0^{2\rho(x_n)} L_2^0(u)du + \int_0^{\|x_n - x_{n-1}\|} N_0(u)du\|x_n - x_{n-1}\|.$$

From the definition r_0 (20), we get

$$\int_0^{r_0} L_1(u)du + \int_0^{2r_0} L_2(u)du + 2r\int_0^{2r_0} N(u)du < 1, \quad (23)$$

since $r < r_0$.

Using the Banach theorem on inverse operator [30], we get formula (22). Then we can write

$$\|x_{n+1} - x_*\| = \|x_n - x_* - A_n^{-1}(F(x_n) - F(x_*) + Q(x_n) - Q(x_*))\| =$$
$$= \| - A_n^{-1}\Big(\int_0^1 (F'(x_n^\tau) - F'(x_n))d\tau + Q(x_n, x_*) - Q(2x_n - x_{n-1}, x_{n-1}))(x_n - x_*)\| \leq$$
$$\leq \|A_n^{-1}H'(x_*)\|(\|H'(x_*)^{-1}\int_0^1 \int_{\tau\rho(x_n)}^{\rho(x_n)} L_1(u)dud\tau + \|H'(x_*)^{-1}(+Q(x_n, x_*) - \quad (24)$$
$$- Q(2x_n - x_{n-1}, x_{n-1}))\|)\|x_n - x_*\|.$$

According to the condition (17)–(19) of the theorem we get

$$\|H'(x_*)^{-1}(\int_0^1 \int_{\tau\rho(x_n)}^{\rho(x_n)} L_1(u)dud\tau + Q(x_n, x_*) - A_n)\| =$$

$$= \frac{1}{\rho(x_n)} \int_0^{\rho(x_n)} L_1(u)udu + \|H'(x_*)^{-1}(Q(x_n, x_*) - Q(x_n, x_n) +$$

$$+Q(x_n, x_n) - Q(x_n, x_{n-1}) + Q(x_n, x_{n-1}) - Q(2x_n - x_{n-1}, x_{n-1}))\| \le$$

$$\le \frac{1}{\rho(x_n)} \int_0^{\rho(x_n)} L_1(u)udu + \|H'(x_*)^{-1}(Q(x_n, x_*) - Q(x_n, x_n))\| +$$

$$+\|H'(x_*)^{-1}(Q(x_n, x_{n-1}, x_n) - Q(2x_n - x_{n-1}, x_{n-1}, x_n))(x_n - x_{n-1})\| \le$$

$$\le \frac{1}{\rho(x_n)} \int_0^{\rho(x_n)} L_1(u)udu + \int_0^{\rho(x_n)} L_2(u)du + \int_0^{\|x_n - x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|.$$

From (22) and (24) shows that fulfills (21). Then from (21) and (20) we get

$$\|x_{n+1} - x_*\| < \|x_n - x_*\| < ... < \max\{\|x_0 - x_*\|, \|x_{-1} - x_*\|\} < r.$$

Therefore, the iterative process (5) is correctly defined and the sequence that it generates belongs to $B(x_*, r)$. From the last inequality and estimates (21) we get $\lim_{n\to\infty} \|x_n - x_*\| = 0$. Since the sequence $\{x_n\}_{n\ge0}$ converges to x_*, then

$$\|x_n - x_{n-1}\| \le \|x_n - x_*\| + \|x_{n-1} - x_*\| \le 2\|x_{n-1} - x_*\|$$

and $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$. □

Corollary 1. *The order of convergence of the iterative procedure (7) is quadratic.*

Proof. Let us denote $\rho_{\max} = \max\{\rho(x_0), \rho(x_{-1})\}$. Since $g(t)$ and $h(t)$ are monotonically nondecreasing, then with taking into account the expressions

$$\frac{1}{\rho(x_n)} \int_0^{\rho(x_n)} L_1(u)udu = \frac{\int_0^{\rho(x_n)} L_1(u)udu\rho(x_n))}{(\rho(x_n))^2} \le \frac{\int_0^{\rho_{\max}} L_1(u)udu\rho(x_n)}{(\rho_{\max})^2} =: A_1\rho(x_n),$$

$$\int_0^{\rho(x_n)} L_2(u)du = \frac{\int_0^{\rho(x_n)} L_2(u)du\rho(x_n)}{\rho(x_n)} \le \frac{\int_0^{\rho_{\max}} L_2(u)du\rho(x_n)}{\rho_{\max}} =: A_2\rho(x_n),$$

$$\int_0^{\|x_n - x_{n-1}\|} N(u)du = \frac{\int_0^{\|x_n - x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|}{\|x_n - x_{n-1}\|} <$$

$$< \frac{\int_0^{\|x_0 - x_{-1}\|} N(u)du\|x_n - x_{n-1}\|}{\|x_0 - x_{-1}\|} =: A_3\|x_n - x_{n-1}\|$$

and

$$\Big(1 - \Big(\int_0^{\rho(x_n)} L_1^0(u)du + 2\int_0^{\rho(x_n)} L_2^0(u)du + \int_0^{\|x_n - x_{n-1}\|} N_0(u)du\|x_n - x_{n-1}\|\Big)\Big)^{-1} <$$

$$< \Big(1 - \Big(\int_0^{\rho_{\max}} L_1^0(u)du + 2\int_0^{\rho_{\max}} L_2^0(u)du + \int_0^{\|x_0 - x_{-1}\|} N_0(u)du\|x_0 - x_{-1}\|\Big)\Big)^{-1} =: A_4,$$

from the inequality (21) follows

$$\|x_{n+1} - x_*\| \leq A_4(A_1\rho(x_n) + A_2\rho(x_n) + A_3\|x_n - x_{n-1}\|^2)\|x_n - x_*\|.$$

or

$$\|x_{n+1} - x_*\| \leq C_3\|x_n - x_*\|^2 + C_4\|x_n - x_{n-1}\|^2\|x_n - x_*\|. \tag{25}$$

Here $A_k, k = 1, ..., 4, C_3, C_4$ are some positive constants.

Assume that the order of convergence of the iterative process (7) is not lower 2, therefore there exist $C_5 \geq 0$ and $N > 0$, that for all $n \geq N$ the inequality holds

$$\|x_n - x_*\| \geq C_5\|x_{n-1} - x_*\|^2.$$

Since

$$\|x_n - x_{n-1}\|^2 \leq (\|x_n - x_*\| + \|x_{n-1} - x_*\|)^2 \leq 4\|x_{n-1} - x_*\|^2,$$

then from (44) we get

$$\begin{aligned} &\|x_{n+1} - x_*\| \leq C_3\|x_n - x_*\|^2 + 4C_4\|x_{n-1} - x_*\|^2\|x_n - x_*\| \\ &\leq (C_3 + 4C_4/C_5)\|x_n - x_*\|^2 = C_6\|x_n - x_*\|^2. \end{aligned} \tag{26}$$

inequality (26) means that the order of convergence is not lower than 2. Thus, the convergence rate of sequence $\{x_n\}_{n \geq 0}$ to x_* is quadratic. □

3. Uniqueness Ball of the Solution

The next theorem determines the ball of uniqueness of the solution x_* of (1) in $B(x_*, r)$.

Theorem 2. *Let us assume that: (1) $H(x) \equiv F(x) + Q(x) = 0$ has a solution $x_* \in D$, in which there exists a Fréchet derivative $H'(x_*)$ and it is invertible; (2) F has a continuous Frećhet derivative in $B(x_*, r)$, F' satisfies the generalized Lipschitz condition*

$$\|H'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \int_0^{\rho(x)} L_1^0(u)du \quad \forall x \in B(x_*, r),$$

the divided difference $Q(x, y)$ satisfies the generalized Lipschitz condition

$$\|H'(x_*)^{-1}(Q(x, x_*) - G'(x_*))\| \leq \int_0^{\rho(x)} L_2^0(u)du \quad \forall x \in B(x_*, r),$$

where L_1 and L_2 are positive integrable functions. Let $r > 0$ satisfy

$$\frac{1}{r}\int_0^r (r - u)L_1^0(u)du + \int_0^r L_2^0(u)du \leq 1.$$

Then the equation $H(x) = 0$ has a unique solution x_* in $B(x_*, r)$.

Proof analogous to [27,31].

4. Corollaries

In the study of iterative methods, the traditional assumption is that the derivatives and/or the divided differences satisfy the classical Lipschitz conditions. Assuming that L_1, L_2 and N are constants, we get from Theorems 1 and 2 important corollaries, which are of interest.

Corollary 2. *Let us assume that: (1) $H(x) \equiv F(x) + Q(x) = 0$ has a solution $x_* \in D$, in which there exists Fréchet derivative $H'(x_*)$ and it is invertible; (2) F has a continuous Fréchet derivative and Q has divided differences of the first and second order $Q(x,y)$ and $Q(x,y,z)$ in $B(x_*, 3r) \subset D$, which satisfy the Lipschitz conditions for each $x, y, u, v \in D$*

$$\|H'(x_*)^{-1}(F'(x) - F'(x_*)\| \leq L_1^0 \|x - x_*\|,$$

$$\|H'(x_*)^{-1}(Q(x,y) - Q(u,v))\| \leq L_2^0(\|x - u\| + \|y - v\|),$$

for $x, y, u, v \in D_0$

$$\|H'(x_*)^{-1}(Q(u,x,y) - Q(v,x,y))\| \leq N_0 \|u - v\|,$$

$$\|H'(x_*)^{-1}(F'(x) - F'(x_* + \tau(x - x_*))\| \leq (1 - \tau) L_1 \|x - x_*\|,$$

$$\|H'(x_*)^{-1}(Q(x,y) - Q(u,v))\| \leq L_2(\|x - u\| + \|y - v\|),$$

$$\|H'(x_*)^{-1}(Q(u,x,y) - Q(v,x,y))\| \leq N \|u - v\|,$$

where L_1^0, L_2^0, N_0, L_1, L_2 and N are positive numbers,

$$r_0 = \frac{2}{L_1^0 + 2L_2^0 + \sqrt{(L_1^0 + 2L_2^0)^2 + 16N_0}},$$

and r is the positive root of the equation

$$\frac{L_1 r/2 + L_2 r + 4Nr^2}{1 - L_1^0 r - 2L_2^0 r - 4N_0 r^2} = 1.$$

Then Newton-Kurchatov method (5) converges for all $x_{-1}, x_0 \in B(x_*, r)$ and there fulfills

$$\|x_{n+1} - x_*\| \leq \frac{(L_1/2 + L_2)\|x_n - x_*\| + N\|x_n - x_{n-1}\|^2}{1 - \left(L_1^0 + 2L_2^0 \|x_n - x_*\| + N_0 \|x_n - x_{n-1}\|^2\right)}.$$

Moreover, r is the best of all possible.

Note that value of $r = \dfrac{2}{3L}$ improves $\bar{r} = \dfrac{2}{3L_1^1}$ for Newton method for solving equation $F(x) = 0$ [14,32,33], and with $r = 2/(3L_2 + \sqrt{9L_2^2 + 32N})$ improves $\bar{r} = 2/(3L_2^1 + \sqrt{9(L_2^1)^2 + 32N_1})$ for Kurchatov method for solving the equation $Q(x) = 0$, as derived in [8].

Corollary 3. *Suppose that: (1) $H(x) \equiv F(x) + Q(x) = 0$ has a solution $x_* \in D$, in which there exists the Fréchet derivative $H'(x_*)$ and it is invertible; (2) F has continuous derivative and Q has divided difference $Q(x, x_*)$ in $B(x_*, r) \subset D$, which satisfy the Lipschitz conditions*

$$\|H'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq L_1^0 \|x - x_*\|,$$

$$\|H'(x_*)^{-1}(Q(x, x_*) - G'(x_*))\| \leq L_2^0 \|x - x_*\|$$

for all $x \in B(x_*, r)$, where L_1^0 and L_2^0 are positive numbers and $r = \dfrac{2}{L_1^0 + 2L_2^0}$. Then x_* is the only solution in $B(x_*, r)$ of $H(x) = 0$, r does not depend on F and Q and is the best choice.

Note that the resulting radius of the uniqueness ball of the solution $r = \dfrac{2}{L_1}$ improves $\bar{r} = \dfrac{2}{L_1^1}$ for Newton method for solving the equation $F(x) = 0$ [14] and $r = \dfrac{1}{L_2}$ improves $\bar{r} = \dfrac{1}{L_2^1}$ for Kurchatov method for solving the equation $Q(x) = 0$ [8]. (See also the numerical examples).

Remark 2. *We compare the results in [25] with the new results in this article. In order to do this, let us consider the conditions given in [25] corresponding to our conditions (15)–(17):*

For each $x, y, u, v \in D$

$$\|H'(x_*)^{-1}(F'(x) - F'(x^\theta))\| \leq \int_{\theta\rho(x)}^{\rho(x)} L_1^1(u)du, \; 0 \leq \theta \leq 1, \tag{27}$$

$$\|H'(x_*)^{-1}(Q(x,y) - Q(u,v))\| \leq \int_0^{\|x-u\|+\|y-v\|} L_2^1(t)dt, \tag{28}$$

$$\|H'(x_*)^{-1}(Q(u,x,y) - Q(v,x,y))\| \leq \int_0^{\|u-v\|} N^1(t)dt, \tag{29}$$

$$\frac{\frac{1}{\bar{r}}\int_0^{\bar{r}} L_1^1(u)udu + \int_0^{\bar{r}} L_2^1(u)du + 2\bar{r}\int_0^{2\bar{r}} N^1(u)du}{1 - \left(\int_0^{\bar{r}} L_1^1(u)du + \int_0^{2\bar{r}} L_2^1(u)du + 2\bar{r}\int_0^{2\bar{r}} N^1 1(u)du\right)} = 1, \tag{30}$$

$$\|x_{n+1} - x_n\| \leq \bar{e}_n. \tag{31}$$

It follows from (14)–(16), (17)–(19), (27)–(29), that

$$L_1^0(t) \leq L_1^1(t), \tag{32}$$

$$L_1(t) \leq L_1^1(t), \tag{33}$$

$$L_2^0(t) \leq L_2^1(t), \tag{34}$$

$$L_2(t) \leq L_2^1(t), \tag{35}$$

$$N_0(t) \leq N^1(t), \tag{36}$$

$$N(t) \leq N^1(t), \tag{37}$$

leading to

$$\bar{r} \leq r, \tag{38}$$

$$e_n \leq \bar{e}_n, \tag{39}$$

$$A_l \leq \bar{A}_l, \; l = 1,2,3,4, \tag{40}$$

$$C_l \leq \bar{C}_l, \; l = 1,2,3,4,6 \tag{41}$$

and

$$C_5 \geq \bar{C}_5, \tag{42}$$

$$\bar{e}_n := \frac{\frac{1}{\rho(x_n)}\int_0^{\rho(x_n)} L_1^1(u)udu + \int_0^{\rho(x_n)} L_2^1(u)du + \int_0^{\|x_n - x_{n-1}\|} N^1(u)du\|x_n - x_{n-1}\|}{1 - \left(\int_0^{\rho(x_n)} L_1^1(u)du + \int_0^{2\rho(x_n)} L_2^1(u)du + \int_0^{\|x_n - x_{n-1}\|} N^1(u)du\|x_n - x_{n-1}\|\right)}\|x_n - x_*\|, \tag{43}$$

$$\|x_{n+1} - x_*\| \leq \bar{A}_4(\bar{A}_1\rho(x_n) + \bar{A}_2\rho(x_n) + \bar{A}_3\|x_n - x_{n-1}\|^2)\|x_n - x_*\|,$$

or

$$\|x_{n+1} - x_*\| \leq \bar{C}_3 \|x_n - x_*\|^2 + \bar{C}_4 \|x_n - x_{n-1}\|^2 \|x_n - x_*\|, \tag{44}$$

or

$$\|x_{n+1} - x_*\| \leq \bar{C}_6 \|x_n - x_*\|^2$$

with

$$\bar{C}_6 = (\bar{C}_3 + 4\bar{C}_4 / \bar{C}_5)$$

for some

$$\|x_n - x_*\| \geq \bar{C}_5 \|x_{n-1} - x_*\|^2.$$

Hence, we obtain the impovements:

(1) At least as many initial choices x_{-1}, x_0 as before.
(2) At least as few iterations than before to obtain a predetermined error accuracy.
(3) At least as precice information on the location of the solution as before.

Moreover, if any of (32)–(37) holds as a strict inequality, then so do (38)–(42). Furthermore, we notice that these improvements are found using the same information, since the functions L_1^0, L_2^0, N_0, L_1, L_2, N are special cases of functions L_1^1, L_2^1, N^1 used in [25]. Finally, if $G = 0$ or $F = 0$, we obtain the results for Newton's method or the Kurchatov method as special cases. Clearly, the results for these methods are also improved. Our technique can also be used to improve the results of other iterative methods in an analogous way.

5. Numerical Examples

Example 1. *Let $E_1 = E_2 = R^3$ and $\Omega = S(x_*, 1)$. Define functions F and Q for $v = (v_1, v_2, v_3)^T$ on Ω by*

$$\begin{aligned} F(v) &= \left(e^{v_1} - 1,\ \tfrac{e-1}{2} v_2^2 + v_2,\ v_3\right)^T, \\ Q(v) &= \left(|v_1|,\ |v_2|,\ |v_2|, |\sin(v_3)|\right)^T \end{aligned} \tag{45}$$

$$\begin{aligned} F'(v) &= \operatorname{diag}\Big(e^{v_1},\ (e-1)v_2 + 1, 1\Big), \\ Q(v, \bar{v}) &= \operatorname{diag}\left(\tfrac{|\bar{v}_1| - |v_1|}{\bar{v}_1 - v_1},\ \tfrac{|\bar{v}_2| - |v_2|}{\bar{v}_2 - v_2},\ \tfrac{|\sin(\bar{v}_3)| - |\sin(v_3)|}{\bar{v}_3 - v_3}\right) \end{aligned} \tag{46}$$

Choose:

$$H(x) = F(x) + Q(x),$$

$$\|H'(x_*)^{-1}\| = 1, L_1^0 = e - 1, L_2^0 = 1, N_0 = \frac{1}{2},$$

$$L_1 = e^{\frac{1}{e-1}}, L_2 = 1, N = \frac{1}{2},$$

$$L_1^1 = e, L_2^1 = 1, N^1 = \frac{1}{2}.$$

Then compute:
r using (20), $r = 0.1599$;
$\bar{r}$ using (30), $\bar{r} = 0.1315$.
Also, $\bar{r} < r$.
Notice that $L_1^0 < L_1 < L_1^1$, so the improve ments stated in Remark 1 hold.

6. Conclusions

In [1,8,34], we studied the local convergence of Secant and Kurchatov methods in the case of fulfilment of Lipschitz conditions for the divided differences, which hold for some Lipschitz constants. In [14], the convergence of the Newton method is shown for the generalized Lipschitz conditions for the Fréchet derivative of the first order. We explored the local convergence of the Newton-Kurchatov method under the generalized Lipschitz conditions for Fréchet derivative of a differentiable part of the operator and the divided differences of the nondifferentiable part. Our results contain known parts as partial cases.

By using our idea of restricted convergence regions, we find tighter Lipschitz constants leading to a finer local convergence analysis of method (7) and its special cases compared to in [25].

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