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A Classification of Symmetric (1, 1)-Coherent Pairs of Linear Functionals

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Abstract: In this paper, we study a classification of symmetric (1, 1)-coherent pairs by using a symmetrization process. In particular, the positive-definite case is carefully described.

Keywords: symmetric (1,1)-coherent pairs; symmetric quasi-definite linear functionals; symmetrization process

1. Introduction

The concept of a coherent pair of measures on the real line was introduced by Iserles et al. [1] in the framework of the theory of polynomials orthogonal with respect to a Sobolev inner product associated with a pair of nontrivial positive measures (ν_0, ν_1) supported on the real line. This Sobolev inner product is defined by:

$$\langle p, q \rangle_S = \int_{\mathbb{R}} p(x)q(x)d\nu_0(x) + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\nu_1(x), \quad (1)$$

where p and q are polynomials with real coefficients and λ is a nonnegative real number.

The pair of measures (ν_0, ν_1) is said to be coherent if the corresponding sequences of monic orthogonal polynomials $\{P_n(\nu_0; x)\}_{n \geq 0}$ and $\{P_n(\nu_1; x)\}_{n \geq 0}$ satisfying:

$$nP_{n-1}(\nu_1; x) = P'_n(\nu_0; x) + a_n P'_{n-1}(\nu_0; x), \quad n \geq 1, \quad (2)$$

with $a_n \neq 0$ for $n \geq 2$. Assuming (2), if $\{S_n(\nu_0, \nu_1; \lambda; x)\}_{n \geq 0}$ denotes the sequence of monic orthogonal polynomials associated with the Sobolev inner product, then there exists a nice algebraic relation with the sequence of monic orthogonal polynomials $\{P_n(\nu_0; x)\}_{n \geq 0}$ with respect to the measure ν_0 . Indeed,

$$S_n(\nu_0, \nu_1; \lambda; x) + b_n(\lambda)S_{n-1}(\nu_0, \nu_1; \lambda; x) = P_n(\nu_0; x) + a_n P_{n-1}(\nu_0; x), \quad n \geq 1. \quad (3)$$

Meijer in [2] proved that if (ν_0, ν_1) is a coherent pair of positive measures supported on the real line, i.e., (2) holds, then one of the measures is classical (Laguerre or Jacobi), and its companion is the rational perturbation of it.

What was proven by Meijer [2] is slightly more general than what is stated above. He dealt with orthogonal polynomials with respect to a pair of quasi-definite linear functionals on the set

of polynomials with real coefficients, and he proved that one of such linear functionals must be classical, i.e., either a Laguerre, or Jacobi, or Bessel linear functional. Notice that positive definite linear functionals are associated with nontrivial probability measures supported on the real line (see [3]). Thus, Meijer [2] also determined all the possible coherent pairs of positive measures supported in the real line.

The relation (3) is very useful when we study analytic properties of the corresponding Sobolev orthogonal polynomials. In particular, outer relative asymptotics have been deeply analyzed in the literature (see [4,5], as well as the recent survey [6], where an updated list of references concerning this topic is presented).

In [7], the authors showed that there are Sobolev inner products of the type (1) where the pair of measures (ν_0, ν_1) is not coherent, but the relation (3) still holds ([7], Theorem 4.1), or in other words, a combination of Sobolev orthogonal polynomials as:

$$S_n(\nu_0, \nu_1; \lambda; x) + b_n(\lambda)S_{n-1}(\nu_0, \nu_1; \lambda; x), \tag{4}$$

can be written as a linear combination of orthogonal polynomials $P_n(\nu; x)$ and $P_{n-1}(\nu; x)$, where the measure ν is closely related to the measures ν_0 and ν_1 ([7], Theorem 3.1).

The results obtained in [7] can be covered by extending the concept of coherence (see [8]). It is important to observe that given the Sobolev inner product as (1), if the sequences $\{S_n(\nu_0, \nu_1; \lambda; x)\}_{n \geq 0}$ and $\{P_n(\nu_0; x)\}_{n \geq 0}$ satisfy (3), then:

$$nP_{n-1}(\nu_1; x) + c_nP_{n-2}(\nu_1; x) = P'_n(\nu_0; x) + a_nP'_{n-1}(\nu_0; x), \quad n \geq 1, \tag{5}$$

with $a_n \neq 0$, for $n \geq 2$. When (5) holds (see [9]), the pair (ν_0, ν_1) is referred to as a $(1, 1)$ -coherent pair. In this case, one of the measures must be semiclassical of class at most 1, and the other one is a rational perturbation of it. Semiclassical orthogonal polynomials have been introduced in [10,11]. A nice survey about this topic is [12]. In particular, the concept of the class of the corresponding linear functional plays a central role in the study of the algebraic properties of semiclassical orthogonal polynomials. The class $s = 0$ is constituted by the classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel). The classification of semiclassical orthogonal polynomials of Class 1 appears in [13].

Several generalizations of the concept of coherent pair have been extensively studied and documented. The more general case of coherence for standard orthogonal polynomials corresponds to the concept of (M, N) -coherence of order (m, n) , and it is defined as follows.

Definition 1. Let ν_0, ν_1 be a pair of positive Borel measures and $\{P_n(\nu_0; x)\}_{n \geq 0}$ and $\{P_n(\nu_1; x)\}_{n \geq 0}$ the corresponding SMOPs. $\{\nu_0, \nu_1\}$ is a (M, N) -coherent pair of measures of order (m, k) if the corresponding monic orthogonal polynomial sequence (MOPS) satisfies:

$$\sum_{i=0}^M a_{i,n}P_{n+m-i}^{[m]}(\nu_0; x) = \sum_{i=0}^N b_{i,n}P_{n+k-i}^{[k]}(\nu_1; x), \tag{6}$$

where $m, k, M, N \in \mathbb{N} \cup \{0\}$, $P_n^{[i]}(\nu_0; x) := \frac{P_{n+i}^{(i)}(\nu_0; x)}{(n+1)_i}$, and $\{a_{i,n}\}_{n \geq 0}, \{b_{j,n}\}_{n \geq 0}, 0 \leq i \leq M, 0 \leq j \leq N$, are sequences of numbers with $a_{0,n} = b_{0,n} = 1$. If $k = 0$, then we will say that $\{\nu_0, \nu_1\}$ is an (M, N) -coherent pair of measures of order m .

The notion of “ (M, N) -coherence” was introduced in [14] for order one, where the natural connection with Sobolev polynomials orthogonal with respect to the inner product (1) is presented.

In [15], the inverse problem of order (m, n) was studied. In the framework of order m , the connection with Sobolev polynomials orthogonal with respect to the inner product:

$$\langle p, q \rangle_S = \int_{\mathbb{R}} p(x)q(x)dv_0(x) + \lambda \int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x)dv_1(x), \tag{7}$$

was analyzed in [16] and for $(1, 1)$ -coherent pairs in [17]. With this terminology, if (5) holds, then the pair (ν_0, ν_1) is $(1, 1)$ -coherent of order one. The $(2, 0)$ -coherent pairs of order one were studied in [18]. The $(k + 1, 0)$ -coherence of order one was proposed in [19] through the so-called k -coherence. Of course, the zero-coherence is the coherence defined in [1], and the one-coherence was defined in [18]. The one-coherence was also studied in [20], but from a more general approach considering pairs of quasi-definite functionals and proving that if (u, v) is a one-coherent pair, then u and v must be semiclassical of classes at most 6 and 2, respectively. In [21,22], direct and inverse problems associated with $(1, 1)$ -coherence of order zero were studied. As for the solution of the inverse problem for the (M, N) -coherence of order one, in [23], a rational relation satisfied by the linear functional was obtained. In [24,25], a direct problem associated with the $(M, 0)$ -coherence of order zero was analyzed. Additional studies on the $(2, 0)$ -coherence of order zero appeared in [26]. Finally, an inverse problem associated with the $(2, 1)$ -coherence of order zero was studied in [27], where the interesting topic about when the $(2, 1)$ -coherence algebraic relation is **non-degenerate** was analyzed, i.e., conditions in such a way that the $(2, 1)$ -coherence relation cannot be reduced to a (M, N) -coherence relation with either $M < 2$ or $N < 1$ are given.

If the measures involved in (1) are symmetric, i.e., their odd moments are zero, in [1], the concept of a symmetrically-coherent pair of measures was introduced. Indeed, a pair of symmetric measures (ν_0, ν_1) is said to be symmetrically coherent if their corresponding sequences of monic orthogonal polynomials $\{P_n(\nu_0; x)\}_{n \geq 0}$ and $\{P_n(\nu_1; x)\}_{n \geq 0}$ satisfy:

$$nP_{n-1}(\nu_1; x) = P'_n(\nu_0; x) + c_n P'_{n-2}(\nu_0; x), \quad n \geq 2, \tag{8}$$

with $c_n \neq 0$ for $n \geq 2$.

In [2], H. G. Meijer proved that if (ν_0, ν_1) is a symmetrically-coherent pair of positive measures supported on the real line, i.e., (8) holds, then one of such measures is symmetric and classical (Hermite or Gegenbauer) and the other one is a rational perturbation of it. Analytic properties of the corresponding sequences of Sobolev orthogonal polynomials have been studied in the literature (see [6] and the references therein). Indeed, the main tool is the existence of an algebraic relation:

$$S_n(\nu_0, \nu_1; \lambda; x) + v_n(\lambda)S_{n-2}(\nu_0, \nu_1; \lambda; x) = P_n(\nu_0; x) + c_n P_{n-2}(\nu_0; x), \quad n \geq 2, \tag{9}$$

where $c_n \neq 0$, for $n \geq 2$ and $\{S_n(\nu_0, \nu_1; \lambda; x)\}_{n \geq 0}$, denotes the sequence of monic orthogonal polynomials associated with the Sobolev inner product (1), where ν_0 and ν_1 are symmetric measures. (9) is an important tool to study such Sobolev orthogonal polynomials. Indeed, in [28], it was proven that if (9) holds, then:

$$(n + 1)P_n(\nu_1; x) + r_n P_{n-2}(\nu_1; x) = P'_{n+1}(\nu_0; x) + c_n P'_{n-1}(\nu_0; x), \quad n \geq 2, \tag{10}$$

with $c_n \neq 0$, for $n \geq 2$. Some examples of symmetric measures whose sequences of orthogonal polynomials satisfy (10) have been studied in [28]. Asymptotic properties of the corresponding sequences of orthogonal polynomials and the location of their zeros were analyzed in [29,30] for the Gegenbauer case, as well as in [31,32] for the Hermite case. The aim of the present contribution is to find all the symmetric pairs of measures such that (10) holds.

Semiclassical symmetric linear functionals of order at most two are the natural framework of our study. They have been analyzed by many authors (see [33–37], among others). On the other hand, the so-called symmetrization process for linear functionals (see [3]) will play a central role in this

contribution. In [38], the class of the symmetrized linear functional associated with a semiclassical linear functional has been studied. Notice that this process can also be considered in the framework of Sobolev inner products (see [39]).

The structure of this manuscript is the following. In Section 2, the basic background about linear functionals and orthogonal polynomials is presented. A special emphasis on semiclassical linear functionals is given. The symmetrization process for linear functionals is also analyzed. Moreover, the main results about $(1, 1)$ -coherent pairs of measures are summarized. By using a symmetrization process, in Section 3, we study pairs (\tilde{u}, \tilde{v}) whose respective symmetrized pairs (u, v) are symmetric $(1, 1)$ -coherent pairs. Finally, in Section 4, we deduce all positive-definite symmetric $(1, 1)$ -coherent pairs (u, v) when either u or v are of class $s \leq 2$.

2. Preliminaries

Let \mathbf{P} be the linear space of polynomials with complex coefficients. Its topological dual space will be denoted by \mathbf{P}' . \mathbf{P}_n will represent the linear subspace of polynomials of degree at most n . If $U \in \mathbf{P}'$, then $\langle U, p \rangle$ will denote the action of the linear functional U on the polynomial $p \in \mathbf{P}$. $\{u_n\}_{n \geq 0}$, with $u_n = \langle U, x^n \rangle$, is said to be the sequence of moments associated with U .

Definition 2. For any polynomial q and $a \in \mathbb{C}$, we define the operator $\theta_a : \mathbf{P} \rightarrow \mathbf{P}$ as follows:

$$(\theta_a q)(x) = \frac{q(x) - q(a)}{x - a}. \tag{11}$$

If $U \in \mathbf{P}'$ and $a, b \in \mathbb{C}, b \neq 0$, a displacement of U , denoted by $(\tau_a \circ h_b)U$, is defined as follows:

$$\langle (\tau_a \circ h_b)U, p(x) \rangle = \langle U, (h_b \circ \tau_{-a})p(x) \rangle = \langle U, p(bx + a) \rangle, \tag{12}$$

for every $p \in \mathbf{P}$. If $q \in \mathbf{P}$, then the linear functional qU is defined by:

$$\langle qU, p \rangle = \langle U, qp \rangle, \quad p \in \mathbf{P}. \tag{13}$$

The linear functional $\delta(x - c)$ given by $\langle \delta(x - c), p \rangle = p(c), p \in \mathbf{P}, c \in \mathbb{C}$, is said to be the Dirac delta linear functional at c . Let $U \in \mathbf{P}'$, and let $\sigma \in \mathbf{P}$ be a polynomial of degree n with zeros $x_k \in \mathbb{C}, 1 \leq k \leq r$, of multiplicities n_k , respectively, i.e., $\sum_{k=1}^r n_k = n$. Then, for every $p \in \mathbf{P}$, we define $\sigma^{-1}(x)U \in \mathbf{P}'$ as follows:

$$\langle \sigma^{-1}(x)U, p(x) \rangle = \left\langle U, \frac{p(x) - L_\sigma(x; p)}{\sigma(x)} \right\rangle, \tag{14}$$

where $L_\sigma(x; p)$ is the interpolatory polynomial:

$$L_\sigma(x; p) = \sum_{i=1}^r \sum_{j=0}^{n_i-1} p^{(j)}(x_i) L_{i,j}(x). \tag{15}$$

There, $L_{i,j}(x)$ is the polynomial of degree at most $n - 1$ such that $L_{i,j}^{(k)}(x_i) = \delta_{i,l} \delta_{k,j}, i, l = 1, \dots, r$, and $0 \leq k, j \leq n_i - 1$. As an illustrative example, when $\sigma(x) = x^2 - \zeta$, with $\zeta > 0$, the zeros of σ are $\pm\sqrt{\zeta}$. Then, if $q \in \mathbf{P}$, we get:

$$L_{x^2-\zeta}(x; q) = \sum_{i=1}^2 q(x_i) \frac{x^2 - \zeta}{(x - x_i)2x_i} = \frac{x^2 - \zeta}{2\sqrt{\zeta}} \left(\frac{q(\sqrt{\zeta})}{(x - \sqrt{\zeta})} - \frac{q(-\sqrt{\zeta})}{(x + \sqrt{\zeta})} \right).$$

Furthermore, if $q(x) = p(x^2)$, we deduce:

$$L_{x^2-\zeta}(x; q) = p(\zeta) \frac{x^2 - \zeta}{-(x + \sqrt{\zeta})2\sqrt{\zeta}} + p(\zeta) \frac{x^2 - \zeta}{(x - \sqrt{\zeta})2\sqrt{\zeta}} = p(\zeta). \tag{16}$$

Besides, if $\sigma(x) = x - \zeta$, then $L_{x-\zeta}(x; p(x)) = p(\zeta)$, and we conclude that:

$$L_{x^2-\zeta}(x; p(x^2)) = L_{x-\zeta}(x; p(x)) = p(\zeta). \tag{17}$$

On the other hand, if $\sigma(x) = (x - \zeta)^n$, i.e., σ has a zero of multiplicity n , then for any linear functional U :

$$\langle (x - \zeta)^{-n}U, p(x) \rangle = \left\langle U, \frac{p(x) - T_{n-1}^\zeta(p)(x)}{(x - \zeta)^n} \right\rangle, \tag{18}$$

where $T_{n-1}^\zeta(p)$ denotes the Taylor polynomial of degree $n - 1$ of the polynomial p around $x = \zeta$. When $\zeta = 0$, we will write $T_{n-1}(p)$.

Definition 3. Given $a \in \mathbb{C}$, the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a + 1)(a + 2) \dots (a + n - 1)$, $n \geq 1$, and $(a)_0 = 1$.

Lemma 1. Let $p \in \mathbf{P}$ and $q(x) = p((x - \zeta)^2)$. Then, for $n \geq 0$, we get:

$$T_n(p)((x - \zeta)^2) = T_{2n}^\zeta(q)(x). \tag{19}$$

Remark 1. Since $T_{2n+1}^\zeta(q)(x) = T_{2n}^\zeta(q)(x)$, then also $T_n(p)((x - \zeta)^2) = T_{2n+1}^\zeta(q)(x)$.

If $U \in \mathbf{P}'$, then the (distributional) derivative of U , denoted by DU , is the linear functional such that:

$$\langle DU, p \rangle = \langle U, -p' \rangle, \quad p \in \mathbf{P}.$$

Given $U \in \mathbf{P}'$, U is said to be *quasi-definite* or *regular* (see [3,12]) if the leading principal submatrices of the Hankel matrix $(u_{i+j})_{i,j=0}^\infty$ are non-singular. If all of them have a positive determinant, then U is said to be a *positive definite linear functional*. In this case, there exists a positive Borel measure μ supported on an infinite set $E \subseteq \mathbb{R}$ such that:

$$\langle U, p \rangle = \int_E p(x) d\mu(x), \quad p \in \mathbf{P}.$$

Proposition 1 ([3]). Let $U \in \mathbf{P}'$. U is quasi-definite if and only if there exists a sequence of monic polynomials $\{P_n\}_{n \geq 0}$, with $\deg P_n = n$, such that $\langle U, P_n P_m \rangle = 0$, for $n \neq m$, and $\langle U, P_n^2 \rangle \neq 0$, for every $n \in \mathbb{N}$. Such a sequence is said to be a *monic orthogonal polynomial sequence (MOPS)* with respect to the functional U .

Proposition 2 ([3]). Let $U \in \mathbf{P}'$ be a quasi-definite linear functional, and let $\{P_n\}_{n \geq 0}$ be the corresponding MOPS. If $P_n(0) \neq 0$, for every $n \geq 1$, then $xU \in \mathbf{P}'$ is quasi-definite. Furthermore, if $\{\tilde{P}_n\}_{n \geq 0}$ is the corresponding MOPS, then:

$$\tilde{P}_n(x) = x^{-1} \left(P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)} P_n(x) \right).$$

Moreover, if U is positive-definite in $[a, b]$, then xU is also positive-definite on $[a, b]$ if and only if $a \geq 0$. The polynomial \tilde{P}_n is called the $n^{\text{textit{th}}}$ monic Kernel polynomial corresponding to U with κ -parameter zero.

The above proposition defines a mapping in the linear space of quasi-definite linear functionals. A natural question can be posed. Is this mapping one-to-one? The answer is no. It is well known

that there exist infinitely many MOPS generating the same sequence of Kernel polynomials of the κ -parameter. The next result gives the answer to this question.

Theorem 1 ([40]). *Let $u \in \mathbf{P}'$ be a quasi-definite linear functional and $\{P_n\}_{n \geq 0}$ its corresponding MOPS. Let $v \in \mathbf{P}'$ be the linear functional $v = u + M\delta(x - a)$, with $M \in \mathbb{C}, a \in \mathbb{R}$. Then, v is quasi-definite if and only if $d_n := 1 + MK_n(a, a) \neq 0$, where $K_n(x, y)$ is the n^{th} Kernel polynomial associated with u . Besides $\{R_n\}_{n \geq 0}$, the MOPS associated with v , satisfies:*

$$R_n(x) = P_n(x) - M \frac{P_n(a)}{d_{n-1}} K_{n-1}(x, a), \quad n \geq 0,$$

with $d_1 = 1$ and $K_{-1}(x, y) = 0$.

To conclude this section, we state a lemma that will be needed later on.

Lemma 2. *If $u, v \in \mathbf{P}'$ are related by $xv = u + M\delta(x - a) + N\delta(x)$, $M, a \neq 0$, then $v = x^{-1}u + \frac{M}{a}\delta(x - a) + (\langle v, 1 \rangle - \frac{M}{a})\delta(x) - N\delta'(x)$.*

Proof. For any polynomial p , it is enough to consider the action of the linear functional xv , defined as above, on $q(x) = \frac{p(x) - p(0)}{x}$. \square

2.1. Semiclassical and Classical Linear Functionals

Let ϕ and ψ be two nonzero polynomials such that $\deg(\phi) = m \geq 0$ and $\deg(\psi) = n \geq 1$ with leading coefficients a_m and b_n , respectively. (ϕ, ψ) is said to be an *admissible pair* if either $m - 1 \neq n$ or if $m - 1 = n$, then $ka_{n+1} - b_n \neq 0$ for every $k \in \mathbb{N}$. $U \in \mathbf{P}'$ is said to be a *semiclassical linear functional* if there exists an admissible pair (ϕ, ψ) , where ϕ is monic, such that the following differential relation holds,

$$D(\phi U) + \psi U = 0, \quad (\text{Pearson equation}). \tag{20}$$

If $U \in \mathbf{P}'$ is a semiclassical linear functional, then the nonnegative integer number:

$$s = \min_{\Phi} \max\{\deg \phi - 2, \deg \psi - 1\},$$

is said to be the class of U . Here, Φ denotes the set of all admissible pairs of nonzero polynomials (ϕ, σ) such that (20) holds. With respect to the class of a semiclassical linear functional, we describe the next irreducibility condition.

Proposition 3 ([12]). *Suppose that $U \in \mathbf{P}'$ is semiclassical and $D(\phi U) + \psi(x)U = 0$. The class of U is a non-negative real number $s = \max\{\deg \phi - 2, \deg \psi - 1\}$ if and only if:*

$$|\phi'(c) + \psi(c)| + \left| \langle U, \theta_c \psi + \theta_c^2 \phi \rangle \right| > 0, \tag{21}$$

for every zero c of ϕ .

Next, we summarize some characterizations of semiclassical linear functionals.

Theorem 2 (see [12]). *Let u be a quasi-definite linear functional and $\{P_n\}_{n \geq 0}$ the corresponding MOPS. u is semiclassical of class s if and only if one of the next equivalent conditions holds.*

(A) *There exists a polynomial $\tilde{\phi}$, with $\deg(\tilde{\phi}) = t \leq s + 2$, such that the MOPS $\{P_n\}_{n \geq 0}$ satisfies:*

$$\tilde{\phi}(x) \frac{P'_{n+1}(x)}{n+1} = \sum_{k=n-s}^{n+t} a_{n,k} P_k(x), \quad n \geq s, \tag{22}$$

with $a_{n,n-s} \neq 0$ and $n \geq s + 1$.

(B) There exists a monic polynomial $\tilde{\phi}$ such that the sequence $\left\{ \frac{P'_{n+1}}{n+1} \right\}_{n \geq 0}$ is quasi-orthogonal of order s with respect to $\tilde{\phi}u$, i.e.,

$$\left\langle \tilde{\phi}u, x^k \frac{P'_{n+1}(x)}{n+1} \right\rangle = 0, \quad k \leq n - s - 1,$$

and:

$$\left\langle \tilde{\phi}u, x^{n-s} \frac{P'_{n+1}(x)}{n+1} \right\rangle \neq 0.$$

Remark 2. Notice that the classification of semiclassical quasi-definite linear functionals of class $s = 1$ is given in [13]. The semiclassical linear functionals of Class 2 are described in [41].

$U \in \mathbf{P}'$ is said to be classical if its class is $s = 0$, i.e., there exist nonzero polynomials ϕ and ψ , with $\deg(\phi) \leq 2$ and $\deg(\psi) = 1$, such that (20) holds. In this case, the MOPS associated with U is called a classical MOPS. Up to an affine transformation on the variable, the Hermite, Laguerre, Bessel, and Jacobi polynomials are the classical MOPS (see Table 1). Besides, except the Bessel polynomials, if U is classical, then, under certain restrictions on the parameters, it is positive-definite, and it has an integral representation with respect to a weight function ω on an interval (a, b) , as described in Table 2.

Table 1. Quasi-definite classical orthogonal polynomials.

Linear Functional	$\phi; \psi$	Restriction on the Parameters
\mathcal{H} , (Hermite)	$1; 2x$	-
$\mathcal{L}^{(\alpha)}$, (Laguerre)	$x; -x - \alpha - 1,$	$-\alpha \notin \mathbb{N},$
$\mathcal{B}^{(\alpha)}$, (Bessel)	$x^2; -2(\alpha x + 1),$	$-\alpha \notin \mathbb{N}$
$\mathcal{J}^{(\alpha, \beta)}$, (Jacobi)	$x^2 - 1;$ $-(\alpha + \beta + 2)x + \alpha - \beta$	$-\alpha, -\beta \notin \mathbb{N},$ $-\alpha - \beta \notin \mathbb{N} \setminus \{1\}$

Table 2. Positive-definite classical orthogonal polynomials.

Linear Functional	(a, b)	$\omega(x)$	Restriction on the Parameters
\mathcal{H}	$(-\infty, \infty)$	e^{-x^2}	-
$\mathcal{L}^{(\alpha)}$	$[0, \infty)$	$x^\alpha e^{-x},$	$\alpha > -1$
$\mathcal{J}^{(\alpha, \beta)}$	$[-1, 1]$	$(1 - x)^\alpha (1 + x)^\beta,$	$\alpha, \beta > -1$

The shifted Jacobi functional on a finite interval $[a, b]$ will be denoted by $\mathcal{J}_{[a,b]}^{(\alpha, \beta)}$, and $\mathcal{J}_{[-1,1]}^{(\alpha, \beta)} := \mathcal{J}^{(\alpha, \beta)}$. Furthermore, the shifted Laguerre functional on $[a, \infty)$ will be denoted by $\mathcal{L}_{[a,\infty)}^{(\alpha)}$, and $\mathcal{L}_{[0,\infty)}^{(\alpha)} := \mathcal{L}^{(\alpha)}$. In this way, the Jacobi functional $\mathcal{J}_{[a,b]}^{(\alpha, \beta)}$ satisfies:

$$D \left[(x - a)(x - b) \mathcal{J}_{[a,b]}^{(\alpha, \beta)} \right] = ((\alpha + \beta + 2)x + [a(\alpha + 1) + b(\beta + 1)]) \mathcal{J}_{[a,b]}^{(\alpha, \beta)},$$

and:

$$\left\langle \mathcal{J}_{[a,b]}^{(\alpha, \beta)}, p(x) \right\rangle = \int_a^b p(x) (b - x)^\alpha (x - a)^\beta dx, \quad p \in \mathbf{P}.$$

The Laguerre functional $\mathcal{L}_{[a,\infty)}^{(\alpha,\beta)}$ satisfies:

$$D \left((x - a) \mathcal{L}_{[a,\infty)}^\alpha \right) = (-x + \alpha + a + 1) \mathcal{L}_{[a,\infty)}^\alpha,$$

and:

$$\left\langle \mathcal{L}_{[a,\infty)}^\alpha, p(x) \right\rangle = \int_a^\infty p(x) e^{-x} (x - a)^\alpha dx, \quad p \in \mathbf{P}.$$

2.2. Symmetric Linear Functionals

A linear functional $U \in \mathbf{P}'$ is called symmetric if $u_{2n+1} = \langle U, x^{2n+1} \rangle = 0$, for every $n \in \mathbb{N}$ (see [3] for other characterizations of symmetric regular linear functionals). If $U \in \mathbf{P}'$ is symmetric and quasi-definite and $\{P_n\}_{n \geq 0}$ is its corresponding MOPS, then we can define $\tilde{u} \in \mathbf{P}'$ by:

$$\langle \tilde{u}, x^n \rangle = \langle U, x^{2n} \rangle, \quad n \in \mathbb{N}, \tag{23}$$

and the sequences of monic polynomials $\{A_n\}_{n \geq 0}$ and $\{\tilde{A}_n\}_{n \geq 0}$ by:

$$P_{2n}(x) = A_n(x^2) \quad \text{and} \quad P_{2n+1}(x) = x\tilde{A}_n(x^2). \tag{24}$$

Theorem 3 ([3]). *If $U \in \mathbf{P}'$ is a symmetric and quasi-definite linear functional and $\{P_n\}_{n \geq 0}$ is its corresponding MOPS, then \tilde{u} , defined by (23), is quasi-definite. Besides, $\{A_n\}_{n \geq 0}$ and $\{\tilde{A}_n\}_{n \geq 0}$ defined by (24) are the MOPS with respect to \tilde{u} and $x\tilde{u}$, respectively.*

Conversely, if $\tilde{u} \in \mathbf{P}'$ is quasi-definite, we can define the symmetric linear functional $U \in \mathbf{P}'$ given by:

$$\langle U, x^{2n} \rangle = \langle \tilde{u}, x^n \rangle \quad \text{and} \quad \langle U, x^{2n+1} \rangle = 0, \quad n \geq 0. \tag{25}$$

Theorem 4 ([3]). *If \tilde{u} and $x\tilde{u}$ are quasi-definite linear functionals on \mathbf{P}' and $\{A_n\}_{n \geq 0}$ and $\{\tilde{A}_n\}_{n \geq 0}$ are their corresponding MOPS, then the symmetric linear functional $U \in \mathbf{P}'$ defined by (25) is quasi-definite, and its MOPS $\{P_n\}_{n \geq 0}$ is given by (24).*

Remark 3. *Notice that $\{\tilde{A}_n\}_{n \geq 0}$ are the kernel polynomials with κ -parameter zero associated with \tilde{u} . Besides, U is called the symmetrized linear functional of \tilde{u} .*

Theorem 5 ([3]). *U is positive definite on $[-\sqrt{b}, \sqrt{b}]$ if and only if \tilde{u} and $x\tilde{u}$ are positive-definite on $[a, b]$ with $a \geq 0$.*

Now, we deduce some interesting consequences of (14).

Lemma 3. *Let U be the symmetrization of $\tilde{u} \in \mathbf{P}'$. Let σ be a polynomial with nonzero simple zeros. Then, for every polynomial q , we get:*

$$\left\langle \sigma^{-1}(x^2)U, q(x^2) \right\rangle = \left\langle \sigma^{-1}(x)\tilde{u}, q(x) \right\rangle. \tag{26}$$

Proof. If $\sigma(x) = \prod_{i=1}^k (x - x_i)$, let $\bar{\sigma}(x) = \sigma(x^2) = \prod_{i=1}^{2k} (x - y_i)$, where $y_{2j} = \sqrt{x_j}$ and $y_{2j-1} = -\sqrt{x_j}$ for $j = 1, \dots, k$. Then, from (14):

$$\left\langle \sigma^{-1}(x^2)U, q(x^2) \right\rangle = \left\langle \bar{\sigma}^{-1}(x)U, q(x^2) \right\rangle = \left\langle U, \frac{q(x^2) - L_{\bar{\sigma}}(x; q(x^2))}{\bar{\sigma}(x)} \right\rangle,$$

and from (15):

$$\begin{aligned} L_{\bar{\sigma}}(x; q(x^2)) &= \sum_{i=1}^{2k} q(y_i^2) \frac{\bar{\sigma}(x)}{(x - y_i)\bar{\sigma}'(y_i)} \\ &= \sum_{i=1}^k \frac{q(x_i)\sigma(x^2)}{\sigma'(x_i)2y_i} \left[\frac{1}{(x - \sqrt{x_i})} - \frac{1}{(x + \sqrt{x_i})} \right] \\ &= \sum_{i=1}^k \frac{q(x_i)\sigma(x^2)}{\sigma'(x_i)(x^2 - x_i)}, \end{aligned}$$

then:

$$\langle \sigma^{-1}(x^2)U, q(x^2) \rangle = \left\langle U, \frac{q(x^2) - \sum_{i=1}^k \frac{q(x_i)\sigma(x^2)}{\sigma'(x_i)(x^2 - x_i)}}{\sigma(x^2)} \right\rangle.$$

Besides, since U is the symmetrization of \tilde{u} , then for any polynomial p , $\langle U, p(x^2) \rangle = \langle \tilde{u}, p(x) \rangle$, and as a consequence,

$$\begin{aligned} \langle \sigma^{-1}(x^2)U, q(x^2) \rangle &= \left\langle \tilde{u}, \frac{q(x) - \sum_{i=1}^k \frac{q(x_i)\sigma(x)}{\sigma'(x_i)(x - x_i)}}{\sigma(x)} \right\rangle \\ &= \left\langle \tilde{u}, \frac{q(x) - L_{\sigma}(x; q(x))}{\sigma(x)} \right\rangle \\ &= \langle \sigma^{-1}(x)\tilde{u}, q(x) \rangle. \end{aligned}$$

□

Given a semiclassical quasi-definite linear functional \tilde{u} , the semiclassical character of the symmetrized linear functional of \tilde{u} , its class and the respective Pearson equation are described in the next theorem.

Theorem 6 ([38]). *Let $\tilde{u} \in \mathbf{P}'$ be semiclassical of class \tilde{s} satisfying the Pearson equation:*

$$D [\tilde{\phi}(x)\tilde{u}] + \tilde{\psi}(x)\tilde{u} = 0, \tag{27}$$

and $x\tilde{u}$ is a quasi-definite linear functional and $\hat{\Psi}(x) := \tilde{\phi}'(x) + 2\tilde{\psi}(x)$. Then, U , the symmetrization of \tilde{u} , is semiclassical of class s satisfying the Pearson equation:

$$D [\phi(x)U] + \psi(x)U = 0, \tag{28}$$

where the number s and the polynomials ϕ and ψ are defined according to the next cases:

(i) If

$$\tilde{\phi}(0) = 0 \quad \text{and} \quad \hat{\Psi}(0) = 0, \tag{29}$$

then:

$$\phi(x) = (\theta_0\tilde{\phi})(x^2), \quad \psi(x) = x [2(\theta_0\tilde{\psi})(x^2) + (\theta_0^2\tilde{\phi})(x^2)], \tag{30}$$

and $s = 2\tilde{s}$.

(ii) If

$$\tilde{\phi}(0) = 0 \quad \text{and} \quad \hat{\Psi}(0) \neq 0, \tag{31}$$

then:

$$\phi(x) = x(\theta_0\tilde{\phi})(x^2), \quad \psi(x) = 2\tilde{\psi}(x^2),$$

and $s = 2\tilde{s} + 1$.

(iii) If $\tilde{\phi}(0) \neq 0$, then:

$$\phi(x) = x\tilde{\phi}(x^2), \quad \psi(x) = 2 \left[x^2\tilde{\psi}(x^2) - \tilde{\phi}(x^2) \right],$$

and $s = 2\tilde{s} + 3$.

Corollary 1. *If s is odd, the polynomials ϕ and ψ in (28) are, respectively, odd and even functions. If s is even, the polynomials ϕ and ψ in (28) are, respectively, even and odd functions.*

In the cases $\mathcal{J}_{[0,1]}^{(\alpha,\beta)}$ and $\mathcal{L}^{(\alpha)}$, where the weight functions are $\omega(x) = (1-x)^\alpha x^\beta$ on $[0, 1]$ and $\omega(x) = x^\alpha e^{-x}$ on $[0, \infty)$, respectively, the new weight functions associated with the symmetrized linear functionals $\overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)}$ and $\overline{\mathcal{L}}^{(\alpha)}$ are $\omega(x) = (1-x^2)^\alpha |x|^{2\beta+1}$ on $[-1, 1]$ (the generalized Gegenbauer weight) and $\omega(x) = |x|^{2\alpha+1} e^{-x^2}$ on \mathbb{R} (the generalized Hermite weight), respectively. Notice that $\langle \overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)}, p(x^2) \rangle = \langle \mathcal{J}_{[0,1]}^{(\alpha,\beta)}, p(x) \rangle$ and $\langle \overline{\mathcal{L}}^{(\alpha)}, p(x^2) \rangle = \langle \mathcal{L}^{(\alpha)}, p(x) \rangle$ for any polynomial p .

If u is a positive-definite linear functional, with weight function ω on the interval I , and if p and q are polynomials, then the linear functional with weight $\frac{|p(x)|}{|q(x)|}\omega(x)$ (provided this is a weight function) will be represented by $\frac{|p(x)|}{|q(x)|}u$.

Remark 4. *In the symmetric framework, the quasi-definite semiclassical linear functionals of Class 1 were described in [38], and in [37], the symmetric quasi-definite linear functionals of Class 2 were given. Finally, examples of symmetric semiclassical linear functionals of Class 3 were studied in [42].*

2.3. (1, 1)-Coherent Pairs

In [9], the (1, 1)-coherence relation:

$$\frac{T'_{n+1}(x)}{n+1} + \tilde{a}_{n-1} \frac{T'_n(x)}{n} = Q_n(x) + \tilde{b}_{n-1} Q_{n-1}(x), \quad \tilde{a}_{n-1} \neq 0, \quad n \geq 1, \tag{32}$$

was studied, where $\{T_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ are orthogonal with respect to quasi-definite linear functionals u and v , respectively. In such a paper, the following result is proven.

Theorem 7 ([9]). *If (u, v) is a (1, 1)-coherent pair given by (32), such that $\tilde{a}_0 \neq \tilde{b}_0$, (or, equivalently, $Q_n \neq \frac{T'_{n+1}}{n+1}$, for $n \geq 1$), then:*

(i) *Either u is a semiclassical linear functional of class at most 1, i.e., there exist polynomials $\tilde{\beta}$ and φ with $\deg(\tilde{\beta}) \leq 3$ and $\deg(\varphi) \leq 2$ such that:*

$$D[\tilde{\beta}u] = -\varphi u.$$

(ii) *Or v is a semiclassical linear functional of class at most 1, i.e., there exist polynomials $\tilde{\beta}$ and φ with $\deg(\tilde{\beta}) \leq 3$ and $\deg(\varphi) \leq 2$ such that:*

$$D[\tilde{\beta}v] = -\varphi v.$$

Furthermore, there exists a constant ζ such that the pair (u, v) satisfies:

$$(x - \zeta)v = \tilde{\beta}u. \tag{33}$$

Moreover, in [9], all (1, 1)-coherent pairs of linear functionals were determined. Besides, in each case (i) and (ii) of the above theorem, the pair (u, v) is called either *type I* or *type II*, respectively.

2.4. Symmetric (1,1)-Coherent Pairs

From now on, U and V will denote two symmetric quasi-definite linear functionals, and $\{P_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ will be their corresponding MOPS, respectively. For the above linear functionals, the normalization $\langle U, 1 \rangle = \langle V, 1 \rangle = 1$ is assumed, as well as the existence of sequences of nonzero real numbers $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$, with $a_n \neq 0$, such that:

$$\frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1} = R_{n+2}(x) + b_n R_n(x), \quad n \geq 0, \tag{34}$$

holds. In this case, (U, V) is said to be a *symmetric (1,1)-coherent pair*.

In [28], the relation (34) was studied. Indeed, when U is a classical linear functional, the authors obtained the coefficients of the three-term recurrence relation that the MOPS $\{R_n\}_{n \geq 0}$ satisfies. Besides, its companion linear functional is a rational modification of U .

Lemma 4 ([9]). *Let (U, V) be a symmetric (1,1)-coherent pair given by (34). $a_0 \neq b_0$ and $a_1 \neq b_1$ if and only if $R_n \neq \frac{P'_{n+1}}{n+1}$ for $n \geq 2$.*

Since U and V are symmetric quasi-definite linear functionals, from (24), we can define:

$$\begin{aligned} P_{2m}(x) &= A_m(x^2) & \text{and} & & P_{2m+1}(x) &= x\tilde{A}_m(x^2), \\ R_{2m}(x) &= B_m(x^2) & \text{and} & & R_{2m+1}(x) &= x\tilde{B}_m(x^2), \end{aligned}$$

where $\{A_n\}_{n \geq 0}$, $\{\tilde{A}_n\}_{n \geq 0}$, $\{B_n\}_{n \geq 0}$, and $\{\tilde{B}_n\}_{n \geq 0}$ are the MOPS with respect to \tilde{u} , $x\tilde{u}$, \tilde{v} , and $x\tilde{v}$, respectively.

Next, we will deduce some relevant results to be used in the sequel. For $n \geq 0$ and from (34), we obtain:

$$\frac{P'_{2n+3}(x)}{2n+3} = R_{2n+2}(x) + \sum_{j=0}^n (-1)^j \left(\prod_{k=0}^j \tilde{u}_{2n-2(k-1)} \right) (s_{2n-2j} - u_{2n-2j}) R_{2n-2j}(x),$$

where:

$$\tilde{u}_{2n-2(k-1)} = \begin{cases} u_{2n-2(k-1)}, & 1 \leq k \leq j, \\ 1, & k = 0, \end{cases}$$

and for $n \geq 1$:

$$\begin{aligned} & \frac{P'_{2n+2}(x)}{2n+2} \\ &= R_{2n+1}(x) + \sum_{j=0}^{n-1} (-1)^j \left(\prod_{k=0}^j \tilde{u}_{2n-2k+1} \right) (s_{2n-1-2j} - u_{2n-1-2j}) R_{2n-1-2j}(x), \end{aligned}$$

where:

$$\tilde{u}_{2n-2k+1} = \begin{cases} u_{2n-2k+1}, & 1 \leq k \leq j, \\ 1, & k = 0. \end{cases}$$

Let us define $r_{2n+1}(x) := R_{2n+1}(x) + A_{2n+1}x$, $n \geq 1$. Then:

$$\begin{aligned} 0 &= \left\langle (R_{2n+1} + A_{2n+1}x) v, \frac{P'_{2n+4}}{2n+4} \right\rangle \\ &= (s_{2n+1} - u_{2n+1}) \langle v, R_{2n+1}^2 \rangle + A_{2n+1} (-1)^n \left(\prod_{k=0}^{n-1} u_{2n+1-2k} \right) (s_1 - u_1) \langle v, R_1^2 \rangle, \end{aligned}$$

if and only if:

$$A_{2n+1} = \frac{(s_{2n+1} - u_{2n+1}) \langle v, R_{2n+1}^2 \rangle}{(-1)^{n+1} \left(\prod_{k=0}^{n-1} u_{2n+1-2k} \right) (s_1 - u_1) \langle v, R_1^2 \rangle}, \quad n \geq 1,$$

and inductively, we can prove that $\langle (R_{2n+1} + A_{2n+1}x)v, P'_{2n+2k} \rangle = 0$ for $k \geq 2$. On the other hand, for $n \geq 1$, we define $r_{2n}(x) := R_{2n}(x) + A_{2n}$. Then:

$$\begin{aligned} & \left\langle r_{2n}v, \frac{P'_{2n+3}}{2n+3} \right\rangle \\ &= (s_{2n} - u_{2n}) \langle v, R_{2n}^2 \rangle + A_{2n} \left\langle v, (-1)^n \left(\prod_{k=0}^n \tilde{u}_{2n-2(k-1)} \right) (s_0 - u_0) \right\rangle = 0 \end{aligned}$$

if and only if:

$$A_{2n} = \frac{(s_{2n} - u_{2n}) \langle v, R_{2n}^2 \rangle}{(-1)^{n+1} \left(\prod_{k=1}^n u_{2n-2(k-1)} \right) (s_0 - u_0)}.$$

Furthermore, $\langle r_{2n}v, \frac{P'_k}{k} \rangle = 0, k \geq 2n + 2$. On the other hand, let us consider the linear functional $r_{2n+1}v$ and its expansion in terms of the dual basis $\{\widehat{U}_n\}_{n \geq 0}$ associated with $\{\frac{P'_{n+1}}{n+1}\}_{n \geq 0}$. Namely,

$$r_{2n+1}v = \sum_{k=0}^{\infty} \tilde{\lambda}_{nk} \widehat{U}_k, \quad \tilde{\lambda}_{nk} = \left\langle r_{2n+1}v, \frac{P'_{k+1}}{k+1} \right\rangle = 0, \quad k \geq 2n + 2,$$

where $\tilde{\lambda}_{nk} = \left\langle r_{2n+1}v, \frac{P'_{k+1}}{k+1} \right\rangle = 0$, if $k + 1$ is odd. If $r_{2n+1}v = \sum_{k=0}^n \tilde{\lambda}_{n,2k+1} \widehat{U}_{2k+1}$, then we can apply the distributional derivative on both sides, and we obtain:

$$D[r_{2n+1}v] = - \sum_{k=0}^n \tilde{\lambda}_{n,2k+1} (2k + 2) U_{2k+2},$$

where $\{U_n\}_{n \geq 0}$ is the dual basis associated with $\{P_n\}_{n \geq 0}$. Since $U_m = \frac{P_m}{\langle u, P_m^2 \rangle} u$, then:

$$D[r_{2n+1}v] = - \left(\sum_{k=0}^n \tilde{\lambda}_{n,2k+1} (2k + 2) \frac{P_{2k+2}}{\langle u, P_{2k+1}^2 \rangle} \right) u.$$

In an analogous way, for $n \geq 1$, we consider $r_{2n}v$ and, as above,

$$r_{2n}v = \sum_{k=0}^{\infty} \lambda_{nk} \widehat{U}_k, \quad \lambda_{nk} = \left\langle r_{2n}v, \frac{P'_{k+1}}{k+1} \right\rangle = 0, \quad \text{if } k \geq 2n + 3,$$

and $\lambda_{nk} = \left\langle r_{2n}v, \frac{P'_{k+1}}{k+1} \right\rangle = 0$, if $k + 1$ is even. Then, $D[r_{2n}v] = - \sum_{k=0}^n \lambda_{n,2k} (2k + 1) U_{2k+1}$ and:

$$D[r_{2n}v] = - \left(\sum_{k=0}^n \lambda_{n,2k} (2k + 1) \frac{P_{2k+1}(x)}{\langle u, P_{2k+1}^2 \rangle} \right) u.$$

Next, we summarize the above results.

Proposition 4. For $m \geq 2$, there exist polynomials r_m and ϕ_{m+1} with $\deg r_m = m$, $\deg \phi_{m+1} \leq m + 1$ such that:

$$D[r_m v] = -\phi_{m+1} u, \quad m \geq 2, \tag{35}$$

with $r_m(x) := R_m(x) + A_m x^{\binom{1-(-1)^m}{2}}$,

$$A_m = \frac{(s_m - u_m) \langle v, R_m^2 \rangle}{(-1)^{\lfloor m/2 \rfloor + 1} \left(\prod_{k=1}^n u_{m+2-2k} \right) \left(s_{\frac{1-(-1)^m}{2}} - u_{\frac{1-(-1)^m}{2}} \right) \langle v, R_{\frac{1-(-1)^m}{2}}^2 \rangle}.$$

Moreover,

$$\phi_{2n+2}(x) = \sum_{k=0}^n \frac{\tilde{\lambda}_{n,2k+1}(2k+2)}{\langle u, P_{2k+2}^2 \rangle} P_{2k+2}(x), \quad n \geq 1, \tag{36}$$

and:

$$\phi_{2n+1}(x) = \sum_{k=0}^n \frac{\lambda_{n,2k}(2k+1)}{\langle u, P_{2k+1}^2 \rangle} P_{2k+1}(x), \quad n \geq 1. \tag{37}$$

3. Symmetric (1, 1)-Coherent Pairs and Symmetrization

The concept of symmetric (1, 1)-coherent pair was introduced in [9] where, among others, the relation between connection coefficients in the coherence relation and recurrence coefficients for the MOPS, and the particular case when u is classical, were deeply studied. The associated inverse problem was solved in [43]. Namely,

Theorem 8. Let (u, v) be a symmetric (1, 1)-coherent pair. There exist polynomials A, B , and C with $\deg(A) = 4$, $\deg(B) \leq 5$, and $\deg(C) \leq 6$, such that:

$$A(x)Dv = B(x)u, \tag{38}$$

$$B(x)v = C(x)Dv, \tag{39}$$

$$xC(x)u = xA(x)v, \tag{40}$$

where:

$$A(x) = \frac{r_4'(x)r_2(x) - r_4(x)r_2'(x)}{x}, \tag{41}$$

$$B(x) = \frac{r_2'(x)\phi_5(x) - r_4'(x)\phi_3(x)}{x}, \tag{42}$$

$$C(x) = \frac{r_4(x)\phi_3(x) - r_2(x)\phi_5(x)}{x}. \tag{43}$$

Depending on the nature of the zeros of A , it is possible to refine the rational relation (40). Besides, according to (41), A is an even function. In this way, either $A(x) = 2(x^2 - \zeta_1^2)(x^2 - \zeta_2^2)$, $\zeta_1^2 \neq \zeta_2^2$, or $A(x) = 2(x^2 - \zeta^2)^2$. In the sequel, we will assume that $\zeta_1^2, \zeta_2^2 \in \mathbb{R}$. Next, we study each case.

Definition 4. Given an even polynomial p of degree $2n$, the polynomial p^E , with $\deg(p^E) = n$, is defined as $p^E(x^2) := p(x)$.

Lemma 5. (i) Let u and v be the symmetrized functionals of \tilde{u} and \tilde{v} , respectively. If ϕ and ψ are even polynomials such that $\phi u = \psi v$, holds, then $\phi^E \tilde{u} = \psi^E \tilde{v}$. Besides, the converse also holds.

(ii) If u and v satisfy $D(x\phi u) = \psi v$, where ϕ and ψ are even polynomials, then $D(x\phi^E \tilde{u}) = \frac{1}{2}\psi^E \tilde{v}$.

3.1. Case $A(x) = 2(x^2 - \zeta^2)^2$

If $A(x) = 2(x^2 - \zeta^2)^2$, then (40) can be written as $x C(x)u = 2x(x^2 - \zeta^2)^2 v$, as well as $r_2(x) = x^2 - a^2$ and $r_4(x) = x^4 + fx^2 + g$. As a consequence, $A(x) = 2x^4 - 4x^2\zeta^2 + 2\zeta^4$, where $\zeta^2 = a^2$ and $fa^2 + g = \zeta^4$. Thus, $r_2(x) = x^2 - \zeta^2$. Since $(xA(x))' = r_4''(x)r_2(x) - r_4(x)r_2''(x)$, we deduce $r_4''(\zeta)r_2(\zeta) - 2r_4(\zeta) = 0$. From this expression, taking into account $r_2(\zeta) = 0$, we get $r_4(\zeta) = 0$. As a consequence, $r_4(x) = r_2(x)\rho_2(x)$, where ρ_2 is an even and monic polynomial with $\deg(\rho_2) = 2$. From (43),

$$C(x) = r_4(x) \frac{\phi_3(x)}{x} - r_2(x) \frac{\phi_5(x)}{x} = r_2(x) \left(\rho_2(x) \frac{\phi_3(x)}{x} - \frac{\phi_5(x)}{x} \right) = r_2(x)\sigma_4(x).$$

According to (42), $B(x) = 2\phi_5(x) - 2(\rho(x) + r_2(x))\phi_3(x)$, then from (38), we get:

$$r_2^2(x)Dv = (\phi_5(x) - (x^2 + r_2(x))\phi_3(x))u.$$

For $m = 2$, multiplying (35) by $r_2(x)$, we deduce $r_2^2(x)Dv = -r_2(x)\phi_3(x)u - 2xr_2(x)v$. On the other hand, from the above expressions, we get:

$$(\phi_5(x) - (\rho(x) + r_2(x))\phi_3(x))u + r_2(x)\phi_3(x)u + 2xr_2(x)v = 0,$$

i.e.,

$$(\phi_5(x) - \rho(x)\phi_3(x))u + 2xr_2(x)v = 0.$$

Thus, we get:

$$x\sigma_4(x)u = 2xr_2(x)v, \tag{44}$$

where:

$$\sigma_4(x) = \rho(x) \frac{\phi_3(x)}{x} - \frac{\phi_5(x)}{x}. \tag{45}$$

If a symmetric (1, 1)-coherent pair (u, v) satisfies (44), then:

$$x^2\sigma_4(x)u = 2x^2r_2(x)v. \tag{46}$$

Through the symmetrization process, we can find pairs (u, v) of symmetric linear functionals such that (46) holds. Among such pairs, we will identify all the symmetric (1, 1)-coherent ones later on.

Lemma 6. (i) For $m = 2n$, (35) implies:

$$xD \left(r_{2n}^E(x)\tilde{v} \right) = -\frac{1}{2}r_{2n}^E(x)\tilde{v} - \frac{1}{2}x\tilde{\phi}_{2n}^E(x)\tilde{u},$$

where $\phi_{2n+1}(x) := x\tilde{\phi}_{2n}(x)$.

(ii) For $m = 2n + 1$, (35) yields:

$$D \left(x\tilde{r}_{2n}^E(x)\tilde{v} \right) = -\frac{1}{2}x\phi_{2n+2}^E(x)\tilde{u},$$

where $r_{2n+1}(x) = x\tilde{r}_{2n}(x)$.

Proof. We will prove (ii). The proof of (i) is similar. The Pearson-type relation is equivalent to:

$$D(x\tilde{r}_{2n}(x)v) = -\phi_{2n+2}(x)u. \tag{47}$$

For every polynomial p ,

$$\begin{aligned} \langle D(x\tilde{r}_{2n}^E(x)\tilde{v}), p(x) \rangle &= -\langle \tilde{v}, x\tilde{r}_{2n}^E(x)p'(x) \rangle \\ &= -\langle v, x^2\tilde{r}_{2n}(x)p'(x^2) \rangle \\ &= \frac{1}{2} \langle D(x\tilde{r}_{2n}(x)v), p(x^2) \rangle, \end{aligned}$$

and from (47):

$$\begin{aligned} \langle D(x\tilde{r}_{2n}^E(x)\tilde{v}), p(x) \rangle &= -\frac{1}{2} \langle \phi_{2n+2}(x)u, p(x^2) \rangle \\ &= -\frac{1}{2} \langle \phi_{2n+2}^E(x)\tilde{u}, p(x) \rangle. \end{aligned}$$

Thus, our statement follows. \square

From the previous lemma, we get that $D(r_2(x)v) = -\phi_3(x)u$ implies:

$$xD(r_2^E(x)\tilde{v}) = -\frac{1}{2}r_2^E(x)\tilde{v} - \frac{1}{2}x\tilde{\phi}_2^E(x)\tilde{u}. \tag{48}$$

On the other hand, $D(r_3(x)v) = -\phi_4(x)u$ is equivalent to:

$$D(x\tilde{r}_2^E(x)\tilde{v}) = -\frac{1}{2}\phi_4^E(x)\tilde{u}, \tag{49}$$

and $D(r_4(x)v) = -\phi_5u = -x\tilde{\phi}_4(x)u$ yields:

$$xD(r_4^E(x)\tilde{v}) = -\frac{1}{2}r_4^E(x)\tilde{v} - \frac{1}{2}x\tilde{\phi}_4^E(x)\tilde{u}. \tag{50}$$

On the other hand, let u and v be the symmetrizations of \tilde{u} and \tilde{v} , respectively. Then,

Lemma 7. u and v satisfy (46) if and only if \tilde{u} and \tilde{v} satisfy:

$$x\sigma_4^E(x)\tilde{u} = 2xr_2^E(x)\tilde{v}. \tag{51}$$

Proof. We assume that $x^2\sigma_4(x)u = 2x^2r_2(x)v$. Let p be any polynomial. Then:

$$\langle x\sigma_4^E(x)\tilde{u}, p(x) \rangle = \langle u, x^2p(x^2\sigma_4^E(x^2)) \rangle = \langle 2xr_2^E(x)\tilde{v}, p(x) \rangle.$$

On the other hand, assume that $x\sigma_4^E(x)\tilde{u} = 2xr_2^E(x)\tilde{v}$. If $p(x) = \sum_{k=0}^n a_kx^k$, then $p^E(x^2) = \sum_{k=0}^{[n/2]} a_{2k}x^{2k}$.

As a consequence, $\langle x^2\sigma_4(x)u, p(x) \rangle = \langle 2x^2r_2(x)v, p(x) \rangle$. \square

Taking derivatives on both sides of (51) and by using (48), we get:

$$D(x\sigma_4^E(x)\tilde{u}) = 2D(xr_2^E\tilde{v}) = r_2^E(x)\tilde{v} - x\tilde{\phi}_2^E(x)\tilde{u}.$$

If we multiply by x , then from (51):

$$xD(x\sigma_4^E(x)\tilde{u}) = xr_2^E(x)\tilde{v} - x^2\tilde{\phi}_2^E(x)\tilde{u} = \left(\frac{1}{2}x\sigma_4^E(x) - x^2\tilde{\phi}_2^E(x)\right)\tilde{u}$$

and, equivalently,

$$D(x^2\sigma_4^E(x)\tilde{u}) = \left(\frac{1}{2}x\sigma_4^E(x) + x\sigma_4^E(x) - x^2\tilde{\phi}_2^E(x)\right)\tilde{u} = \left(\frac{3}{2}x\sigma_4^E(x) - x^2\tilde{\phi}_2^E(x)\right)\tilde{u}.$$

Next we summarize the above results.

Proposition 5. *If $A(x) = 2(x^2 - \zeta^2)^2$ and (u, v) is a symmetric $(1, 1)$ -coherent pair, then (\tilde{u}, \tilde{v}) satisfy (51) and:*

$$D(\tilde{\phi}\tilde{u}) + \tilde{\psi}\tilde{u} = 0, \tag{52}$$

where:

$$\begin{aligned} \tilde{\phi}(x) &= x^2\sigma_4^E(x), \\ \sigma_4^E(x) &= x\tilde{\phi}_2^E(x) - \tilde{\phi}_4^E(x), \end{aligned}$$

and:

$$\tilde{\psi}(x) = x^2\tilde{\phi}_2^E(x) - \frac{3}{2}x\sigma_4^E(x) = -\frac{1}{2}x^2\tilde{\phi}_2^E(x) + \frac{3}{2}x\tilde{\phi}_4^E(x).$$

Moreover, $\deg \tilde{\psi} \leq 3$ and $\deg \tilde{\phi} \leq 4$. As a consequence, \tilde{u} is a semiclassical linear functional of class at most 2.

In the sequel, given a linear functional \tilde{U} and its symmetrized U , $\{\tilde{\mu}_n^U\}_{n \geq 0}$ and $\{\mu_n^U\}_{n \geq 0}$ will denote the corresponding moment sequences. From (37), we get $\phi_3(x) = \frac{\lambda_{1,0}}{\langle u, P_1^2 \rangle}x + \frac{3\lambda_{1,2}}{\langle u, P_3^2 \rangle}P_3(x)$, with $\lambda_{1,0} = \langle v, r_2 \rangle = \mu_2^v - \zeta^2$. After some straightforward calculations, we get:

$$\lambda_{1,2} = \mu_4^v - \left(\zeta^2 + \frac{1}{3}\gamma_1^u + \frac{1}{3}\gamma_2^u\right)\mu_2^v + \frac{\zeta^2}{3}(\gamma_1^u + \gamma_2^u),$$

where $\{\gamma_n^u\}_{n \geq 1}$ are the coefficients of the three-term recurrence relation that the MOPS $\{P_n\}_{n \geq 0}$ satisfies. Then:

$$\tilde{\phi}_2^E(x) = \frac{3\lambda_{1,2}}{\langle u, P_3^2 \rangle}x + \left(\frac{\lambda_{1,0}}{\langle u, P_1^2 \rangle} - \frac{3\lambda_{1,2}}{\langle u, P_3^2 \rangle}(\gamma_1^u + \gamma_2^u)\right).$$

In particular,

$$\tilde{\phi}_2^E(0) = \frac{\mu_2^v - \zeta^2}{\langle u, P_1^2 \rangle} - \frac{3\mu_4^v - 3\left(\zeta^2 + \frac{1}{3}\gamma_1^u + \frac{1}{3}\gamma_2^u\right)\mu_2^v + \zeta^2(\gamma_1^u + \gamma_2^u)}{\langle u, P_3^2 \rangle}(\gamma_1^u + \gamma_2^u). \tag{53}$$

From (51) and taking into account that \tilde{u} is a linear functional of class $s \leq 2$, according to the above classification, we can find its companion \tilde{v} . As a consequence, we can deduce all the candidates (u, v) to be symmetric $(1, 1)$ -coherent pairs. From (52), we get:

$$x^2\sigma_4^E(x)D(\tilde{u}) = -\left(\tilde{\psi}(x) + \left(x^2\sigma_4^E(x)\right)'\right)\tilde{u}. \tag{54}$$

In the sequel, we consider $\tilde{s} \leq 1$. The case $\tilde{s} = 2$ will not be considered. From the classification of the semiclassical linear functionals of class $\tilde{s} \leq 1$, we will analyze the semiclassical character of \tilde{u} taking into account the algebraic structure of $\sigma_4^E(x)$.

3.1.1. \tilde{u} of Class $\tilde{s} = 0$

In order to arrive at a classical case, we start the discussion by considering the following situations:

(i) $\sigma_4^E(x) = x^2$, $\tilde{\psi}(x) = x^2 (\tilde{\phi}_2^E(x) - \frac{3}{2}x)$. From (52), we get $D(x^4\tilde{u}) = -x^2 (\tilde{\phi}_2^E(x) - \frac{3}{2}x) \tilde{u}$ or, equivalently,

$$D(x^2\tilde{u}) = - \left(\tilde{\phi}_2^E(x) + \frac{1}{2}x \right) \tilde{u} + N_1\delta(x) + N_2\delta'(x).$$

It is easy to see that $N_1 = \langle \tilde{u}, \tilde{\phi}_2^E(x) + \frac{1}{2}x \rangle$ and $N_2 = \frac{1}{2} \langle \tilde{u}, x^2 \rangle - \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle$. Thus, if $\langle \tilde{u}, \tilde{\phi}_2^E(x) \rangle + \frac{1}{2}\tilde{\mu}_1^u = \frac{1}{2}\tilde{\mu}_2^u - \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle = 0$, then \tilde{u} is the Bessel classical functional, since:

$$D(x^2\tilde{u}) = - \left(\tilde{\phi}_2^E(x) + \frac{1}{2}x \right) \tilde{u}.$$

(ii) $\sigma_4^E(x) = x(x-1)$, $\tilde{\psi}(x) = x^2 (\tilde{\phi}_2^E(x) - \frac{3}{2}(x-1))$. Then, $D(x^3(x-1)\tilde{u}) = -x^2 (\tilde{\phi}_2^E(x) - \frac{3}{2}(x-1)) \tilde{u}$. This yields:

$$D(x(x-1)\tilde{u}) = - \left(\tilde{\phi}_2^E(x) + \frac{1}{2}(x-1) \right) \tilde{u} + N_1\delta(x) + N_2\delta'(x),$$

where $N_1 = \langle \tilde{u}, \tilde{\phi}_2^E(x) + \frac{1}{2}(x-1) \rangle$ and $N_2 = \frac{1}{2} \langle \tilde{u}, x(x-1) \rangle - \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle$. If $\langle \tilde{u}, \tilde{\phi}_2^E(x) \rangle + \frac{1}{2}(\tilde{\mu}_1^u - 1) = \frac{1}{2}(\tilde{\mu}_2^u - \tilde{\mu}_1^u) - \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle = 0$, then $\tilde{u} = \mathcal{J}_{[0,1]}^{(\alpha,\beta)}$, i.e., the Jacobi classical functional on $[0, 1]$, such that:

$$D(x(x-1)\tilde{u}) = - \left(\tilde{\phi}_2^E(x) + \frac{1}{2}(x-1) \right) \tilde{u}.$$

(iii) $\sigma_4^E(x) = x$, $\tilde{\psi}(x) = x^2 (\tilde{\phi}_2^E(x) - \frac{3}{2})$. From (52), we get $x^3D(\tilde{u}) = - (x^2 (\tilde{\phi}_2^E(x) - \frac{3}{2}) + 3x^2) \tilde{u}$ and:

$$xD(\tilde{u}) = - \left(\tilde{\phi}_2^E(x) + \frac{3}{2} \right) \tilde{u} + N_1\delta(x) + N_2\delta'(x).$$

Then, $\langle xD(\tilde{u}), 1 \rangle = - \langle \tilde{u}, (\tilde{\phi}_2^E(x) + \frac{3}{2}) \rangle + N_1$ and $\langle xD(\tilde{u}), x \rangle = - \langle \tilde{u}, \tilde{\phi}_2^E(x) + \frac{3}{2}x \rangle - N_2$. If:

$$N_1 = \langle \tilde{u}, \tilde{\phi}_2^E(x) \rangle + \frac{1}{2} = 0,$$

and $N_2 = - \langle \tilde{u}, \tilde{\phi}_2^E(x) \rangle + \frac{1}{2}\tilde{\mu}_1^u = 0$, we get $D(x\tilde{u}) = - \left(\tilde{\phi}_2^E(x) + \frac{1}{2} \right) \tilde{u}$, i.e., \tilde{u} is the classical Laguerre linear functional.

Remark 5. We do not consider $\sigma_4^E(x) = 1$, since in such a case, \tilde{u} is the classical Hermite functional.

3.1.2. \tilde{u} of Class $\tilde{s} = 1$

In order to analyze the semiclassical case when $\tilde{s} = 1$, we will discuss two possible situations in order to reduce the degrees of the polynomials involved in the initial Pearson equation.

(a)

(i) $\sigma_4^E(x) = x^2$, $\tilde{\psi}(x) = x^2 (\tilde{\phi}_2^E(x) - \frac{3}{2}x)$. From (52):

$$x^3D(\tilde{u}) = - \left(\tilde{\psi}(x) + 4x^3 \right) \tilde{u} = -x \left(\left(\tilde{\phi}_2^E(x) - \frac{3}{2}x \right) + 4x \right) \tilde{u} + M\delta(x).$$

If:

$$M = \langle x^3D(\tilde{u}), 1 \rangle + \langle x \left(\tilde{\phi}_2^E(x) + \frac{5}{2}x \right) \tilde{u}, 1 \rangle = \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle - \frac{1}{2}\tilde{\mu}_2^u = 0,$$

then you can reduce the Pearson equation to:

$$D(x^3\tilde{u}) = -x \left(\tilde{\phi}_2^E(x) + \frac{5}{2}x \right) \tilde{u} + 3x^2\tilde{u} = \left(-x\tilde{\phi}_2^E(x) + \frac{1}{2}x^2 \right) \tilde{u},$$

and you have here $\tilde{\psi}(x) = x\tilde{\phi}_2^E(x) - \frac{1}{2}x^2$, $\tilde{\psi}(0) = 0$ and $\tilde{\psi}'(0) = \tilde{\phi}_2^E(0)$. If $\tilde{\phi}_2^E(0) \neq 0$, then \tilde{u} corresponds to the case $A_{3,2}$ of Belmehdi's classification in [13], and as a consequence, $\tilde{u} = x^{-1}\mathcal{B}^{(\alpha)} + M\delta(x)$.

(ii) $\sigma_4^E(x) = x(x-1)$, $\tilde{\psi}(x) = x^2 \left(\tilde{\phi}_2^E(x) - \frac{3}{2}(x-1) \right)$. In this case:

$$x^2(x-1)D(\tilde{u}) = - \left(x\tilde{\phi}_2^E(x) - \frac{3}{2}x^2 + \frac{3}{2}x + 4x^2 - 3x \right) \tilde{u} + M_1\delta(x).$$

Then:

$$\begin{aligned} M_1 &= -3 \langle \tilde{u}, x^2 \rangle + 2 \langle \tilde{u}, x \rangle + \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle + \frac{5}{2} \langle \tilde{u}, x^2 \rangle - \frac{3}{2} \langle \tilde{u}, x \rangle \\ &= \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle - \frac{1}{2}\tilde{\mu}_2^u + \frac{1}{2}\tilde{\mu}_1^u. \end{aligned}$$

If $M_1 = 0$, then $D(x^2(x-1)\tilde{u}) = - \left(x\tilde{\phi}_2^E(x) + \frac{1}{2}x - \frac{1}{2}x^2 \right) \tilde{u}$, and according to the case A_2 in [13], \tilde{u} has an integral representation in terms of the weight function $w(x) = (1-x)^\alpha x^\beta e^{-\frac{\gamma}{x}}$, on $[0, 1]$, with $\alpha\gamma \neq 0$, $\gamma > 0$, $\alpha > -1$.

(iii) $\sigma_4^E(x) = (x-1)(x-\zeta)$, $\zeta \neq 0, 1$, $\tilde{\psi}(x) = x \left(x\tilde{\phi}_2^E(x) - \frac{3}{2}(x-1)(x-\zeta) \right)$. Then,

$$D(x(x-1)(x-\zeta)\tilde{u}) = - \left(x\tilde{\phi}_2^E(x) - \frac{1}{2}(x-1)(x-\zeta) \right) \tilde{u} + M\delta(x).$$

If:

$$M = \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle - \frac{1}{2}\tilde{\mu}_2^u + \left(\frac{1}{2}\zeta + \frac{1}{2} \right) \tilde{\mu}_1^u - \frac{1}{2}\zeta = 0,$$

this corresponds to the case A_1 in [13] with $\tilde{\omega}(x) = (1-x)^\alpha x^\beta |x-\zeta|^\gamma$ on $[0, 1]$ with the conditions $\alpha\beta\gamma \neq 0$, $\alpha, \beta, \gamma > -1$, $\zeta \in (0, 1)$.

(b)

(i) $\sigma_4^E(x) = x$, $\tilde{\psi}(x) = x^2 \left(\tilde{\phi}_2^E(x) - \frac{3}{2} \right)$. As above, if $M = \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle - \frac{1}{2}\tilde{\mu}_1^u = 0$, then:

$$D(x^2\tilde{u}) = -x \left(\tilde{\phi}_2^E(x) - \frac{1}{2} \right) \tilde{u},$$

and according to the case B_2 in [13], we obtain an integral representation of \tilde{u} in terms of the weight function:

$$w(x) = x^\alpha (1+x)^{\beta+1} e^{-x+\frac{\beta}{x}},$$

on $[0, \infty)$, with $\beta < 0$, $\alpha, \beta > -1$.

(ii) $\sigma_4^E(x) = x-1$, $\tilde{\psi}(x) = x \left(x\tilde{\phi}_2^E(x) - \frac{3}{2}(x-1) \right)$. Then, $D(x(x-1)\tilde{u}) = - \left(x\tilde{\phi}_2^E(x) - \frac{1}{2}(x-1) \right) \tilde{u}$, when $M = \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle - \frac{1}{2}\tilde{\mu}_1^u + \frac{1}{2} = 0$. This is the case B_1 in [13] with $\omega(x) = (1-x)^{\alpha+1} x^{\beta+1} e^{-\lambda x}$ on $[0, 1]$ and the conditions $\alpha\beta \neq 0$, $\alpha, \beta > -1$.

(iii) $\sigma_4^E(x) = 1$, $\tilde{\psi}(x) = x \left(x\tilde{\phi}_2^E(x) - \frac{3}{2} \right)$. If $M = \langle \tilde{u}, x\tilde{\phi}_2^E(x) \rangle - \frac{1}{2} = 0$, then $D(x\tilde{u}) = - \left(x\tilde{\phi}_2^E(x) - \frac{1}{2} \right) \tilde{u}$, and according to the case B_3 in [13], we get that \tilde{u} is represented in terms of the weight function $w(x) = x^{2\mu} e^{-x^2-\lambda x}$, on \mathbb{R}^+ , $\mu > -1/2$, $\lambda \in \mathbb{R}$.

As in the classical case, it is possible to reduce (51). Indeed, the general form of the Pearson equation is:

$$D(x\sigma_4^E(x)\tilde{u}) = -\left(x\tilde{\phi}_2^E(x) - \frac{1}{2}\sigma_4^E(x)\right)\tilde{u}.$$

Taking derivatives in (51) and using (48), we get $2D(xr_2^E(x)\tilde{v}) = -\left(x\tilde{\phi}_2^E(x) - \frac{1}{2}\sigma_4^E(x)\right)\tilde{u}$. In other words,

$$r_2^E(x)\tilde{v} - x\tilde{\phi}_2^E(x)\tilde{u} = -x\tilde{\phi}_2^E(x)\tilde{u} + \frac{1}{2}\sigma_4^E(x)\tilde{u},$$

and as a consequence, $2r_2^E(x)\tilde{v} = \sigma_4^E(x)\tilde{u}$.

Remark 6. Notice that according to Theorem 6, we get $\widehat{\Psi}(x) = x\left((\sigma_4^E)'(x) + 2\tilde{\phi}_2^E(x)\right)$, and as a consequence, the class of u is $s = 2$.

3.2. Case $A(x) = 2(x^2 - \xi_1^2)(x - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$

In this case, the following result is obtained in [43].

Theorem 9. Suppose that $A(x) = 2(x^2 - \xi_1^2)(x - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$. Then, there exist odd and even polynomials ψ and ϕ , respectively, with $\deg \psi \leq 3$ and $\deg \phi \leq 4$ such that:

$$D(\phi v) + \psi v = 0. \tag{55}$$

As a consequence, v is a semiclassical linear functional of class at most 2. Besides:

$$x\phi(x)u = x(x^2 - \xi^2)v, \tag{56}$$

holds, where $\xi \in \{\xi_1, \xi_2\}$. Furthermore, $(x^2 - \xi^2)Dv = -(\phi'(x) + \psi(x))u$.

Multiplying in (55) by x , if we define $\psi(x) := x\tilde{\psi}(x)$, where $\tilde{\psi}$ is an even polynomial of degree ≤ 2 , and using the symmetrization process, after straightforward calculations, we get:

$$D\left(x\phi^E(x)\tilde{v}\right) = -\frac{1}{2}\left(x\tilde{\psi}^E(x) - \phi^E(x)\right)\tilde{v}, \tag{57}$$

$$x\phi^E(x)D(\tilde{u}) = \frac{1}{2}\left(\rho^E(x) + 2x\right)\tilde{v} - \left(2x(\phi^E)'(x) + \frac{1}{2}x\tilde{\psi}^E(x) + \phi^E(x)\right)\tilde{u}, \tag{58}$$

$$x\rho^E(x)D(\tilde{v}) = -\frac{1}{2}\rho^E(x)\tilde{v} - \frac{1}{2}x\left(2(\phi^E)'(x) + \tilde{\psi}^E(x)\right)\tilde{u}, \tag{59}$$

and:

$$x\phi^E(x)\tilde{u} = x(x - \xi^2)\tilde{v}. \tag{60}$$

Notice that \tilde{v} is semiclassical of class $\tilde{s} \leq 1$. Next, the class of \tilde{v} will be analyzed according to the zeros of ϕ^E .

3.2.1. \tilde{v} Semiclassical of Class $\tilde{s} = 0$

(A1) $\phi^E(x) = x^2$. In this case, (57) can be written as $D(x^3\tilde{v}) = -\frac{1}{2}(x\tilde{\psi}^E(x) - x^2)\tilde{v}$. Since \tilde{v} is classical, we can reduce the degree of the polynomials involved in this relation in one degree, namely $D(x^2\tilde{v}) = -\frac{1}{2}(\tilde{\psi}^E(x) + x)\tilde{v} + N\delta(x)$. Since:

$$0 = \left\langle D\left(x^2\tilde{v}\right), 1 \right\rangle = -\frac{1}{2}\left\langle \left(\tilde{\psi}^E(x) + x\right)\tilde{v}, 1 \right\rangle + N,$$

if $N = 0$, equivalently, $\langle \tilde{v}, \tilde{\psi}^E(x) + x \rangle = 0$, then $D(x^2\tilde{v}) = -\frac{1}{2}(\tilde{\psi}^E(x) + x)\tilde{v}$. In such a way, it is well known that $\tilde{v} = \mathcal{B}^{(\alpha)}$.

(A2) $\phi^E(x) = x(x - 1)$. (57) reads $D(x(x - 1)\tilde{v}) = -\frac{1}{2}[\tilde{\psi}^E(x) + (x - 1)]\tilde{v} + N\delta(x)$. Since:

$$0 = \langle D(x(x - 1)\tilde{v}), 1 \rangle = -\frac{1}{2} \langle \tilde{v}, \tilde{\psi}^E(x) + x - 1 \rangle + N,$$

if $\langle \tilde{v}, \tilde{\psi}^E(x) + x - 1 \rangle = 0$, then $D(x(x - 1)\tilde{v}) = -\frac{1}{2}[\tilde{\psi}^E(x) + (x - 1)]\tilde{v}$. This means that $\tilde{v} = \mathcal{J}_{(0,1)}^{(\alpha,\beta)}$ and:

$$\langle \tilde{v}, p(x) \rangle = \int_0^1 p(x)(1 - x)^\alpha x^\beta dx.$$

(A3) $\phi^E(x) = x$. If $N = \langle \tilde{v}, \tilde{\psi}^E(x) + 1 \rangle = 0$, then $D(x\tilde{v}) = -\frac{1}{2}(\tilde{\psi}^E(x) + 1)\tilde{v}$. As a consequence, $\tilde{v} = \mathcal{L}^{(\alpha)}$.

On one hand, from the symmetrization process and since the class of \tilde{v} is 0, the class s of v is determined by the polynomial $\Psi(x) = (\phi^E)'(x) + \tilde{\psi}^E(x) - \frac{\phi^E(x)}{x}$. Indeed, if $\Psi(0) = 0$, then $s = 0$. If $\Psi(0) \neq 0$, then $s = 1$. In Table 3, we describe the conditions leading to $\Psi(0) = 0$.

Table 3. Conditions for v to be classical.

\tilde{v}	$\tilde{\psi}^E$	$\Psi(x)$	Conditions for $\Psi(0) = 0$
$\mathcal{B}^{(\alpha)}$	$-(2\alpha + 5)x - 4$	$(-2\alpha - 4)x - 4$	$\Psi(0) \neq 0$ always
$\mathcal{J}_{(0,1)}^{(\alpha,\beta)}$	$(2\alpha + 2\beta + 3)x - (2\beta + 1)$	$(2\alpha + 2\beta + 4)x - (2\beta + 1)$	$\beta = -1/2$
$\mathcal{L}^{(\alpha)}$	$2x - (2\alpha + 3)$	$2x - (2\alpha + 3)$	$\alpha = -3/2$

Next, we will prove that we can reduce (60) in order to obtain:

$$\phi^E(x)\tilde{u} = \rho^E(x)\tilde{v},$$

where $\rho^E(x) := x - \xi^2$. In general, the Pearson equation is:

$$D(\phi^E(x)\tilde{v}) = -\frac{1}{2} \left(\tilde{\psi}^E(x) + \frac{\phi^E(x)}{x} \right) \tilde{v}, \tag{61}$$

or, equivalently,

$$\phi^E(x)D\tilde{v} = -\frac{1}{2} \left(\tilde{\psi}^E(x) + \frac{\phi^E(x)}{x} + 2(\phi^E)'(x) \right) \tilde{v}, \tag{62}$$

under the condition $\langle \tilde{v}, \tilde{\psi}^E(x) + \frac{\phi^E(x)}{x} \rangle = 0$.

The case A1, where \tilde{v} is the classical Bessel functional, reads as:

$$D(x^2\tilde{v}) = ((\alpha + 2)x + 2)\tilde{v} = -\frac{1}{2}(\tilde{\psi}^E(x) + x)\tilde{v}.$$

Then, $\tilde{\psi}^E(x) = (-2\alpha - 5)x - 4$, and the above differential relation can be written as:

$$x^2D(\tilde{v}) = (\alpha x + 2)\tilde{v}, \tag{63}$$

with the condition $\langle \tilde{v}, (\alpha + 2)x + 2 \rangle = 0$. Furthermore, in this case, the linear functionals \tilde{u} and \tilde{v} are related by $x^3\tilde{u} = x\rho^E(x)\tilde{v}$, and as a consequence,

$$\tilde{u} = \frac{\rho^E(x)}{x^2}\tilde{v} + K_1\delta(x) + K_2\delta'(x) + K_3\delta''(x).$$

From (59) and (63), we get:

$$\left(\alpha x + 2 + \frac{1}{2}\rho^E(x)\right) \tilde{v} - \zeta^2 x D(\tilde{v}) = -\frac{1}{2}x((-2\alpha - 1)x - 4)\tilde{u}.$$

The action of the linear functionals of both sides on $p(x) = x$ yields:

$$\begin{aligned} \left\langle \tilde{v}, \alpha x^2 + 2x + \frac{1}{2}x\rho^E(x) + 2x\zeta^2 \right\rangle &= \frac{1}{2} \left\langle \tilde{u}, (2\alpha + 1)x^3 + 4x^2 \right\rangle \\ &= \frac{1}{2} \left\langle \rho^E(x)\tilde{v}, (2\alpha + 1)x + 4 \right\rangle + 8K_3. \end{aligned}$$

As a consequence,

$$\begin{aligned} 8K_3 &= \left\langle \tilde{v}, \alpha x^2 + 2x + \frac{1}{2}x\rho^E(x) + 2x\zeta^2 - \frac{1}{2}(2\alpha + 1)x\rho^E(x) - 2\rho^E(x) \right\rangle \\ &= \left\langle \tilde{v}, \alpha x^2 + 2x + 2x\zeta^2 + (-\alpha x - 2)\rho^E(x) \right\rangle \\ &= \zeta^2 \langle \tilde{v}, (2 + \alpha)x + 2 \rangle. \end{aligned}$$

Thus, $K_3 = 0$. In a similar way, in the case A2, we get:

$$\tilde{u} = \frac{\rho^E(x)}{x(x-1)}\tilde{v} + K_1\delta(x) + K_2\delta'(x) + K_3\delta(x-1).$$

The action of the linear functional of both sides on $p(x) = x - 1$ yields:

$$\begin{aligned} &-\frac{1}{2} \left\langle \left(\tilde{\psi}^E(x) + x - 1 + 4x - 2 - x + \zeta^2\right) \tilde{v}, (x - 1) \right\rangle - (1 - \zeta^2) \langle x D\tilde{v}, x - 1 \rangle \\ &= -\frac{1}{2} \left\langle \tilde{u}, x(x - 1) (4x - 2 + \tilde{\psi}^E(x)) \right\rangle \end{aligned}$$

or, equivalently,

$$\begin{aligned} &\left\langle \tilde{v}, -\frac{1}{2}(x - 1) \left(\tilde{\psi}^E(x) + 4x - 3 + \zeta^2\right) - (1 - \zeta^2)(2x - 1) \right\rangle \\ &= -\frac{1}{2} \left\langle \tilde{v}, \rho^E(x) (4x - 2 + \tilde{\psi}^E(x)) \right\rangle + \frac{1}{2}K_2 (-2 + \tilde{\psi}^E(0)). \end{aligned}$$

Then:

$$\frac{1}{2}K_2 (-2 + \tilde{\psi}^E(0)) = -\frac{1}{2}(\zeta - 1) \langle \tilde{v}, \tilde{\psi}^E(x) + x - 1 \rangle.$$

In this case, since $\tilde{v} = \mathcal{J}_{(0,1)}^{(\alpha,\beta)}$, it is well known that:

$$\frac{1}{2} \left(\tilde{\psi}^E(x) + (x - 1)\right) = (\alpha + \beta + 2)x - (\beta + 1).$$

In other words,

$$\tilde{\psi}^E(x) = (2\alpha + 2\beta + 3)x - (2\beta + 1).$$

If $\tilde{\psi}^E(0) = 2$, then $\beta = -3/2$. Notice that this up to $K_2 = 0$ for this value. In the same way, for the case A3, (60) becomes:

$$x\tilde{u} = (x - \zeta^2)\tilde{v},$$

when $2 + \tilde{\psi}^E(0) \neq 0$. This means that $\alpha \neq -\frac{1}{2}$.

3.2.2. \tilde{v} Semiclassical of Class $\tilde{s} = 1$

From (57), the following situations appear.

A $\deg(x\phi^E(x)) = 3, 1 \leq \deg(x\tilde{\psi}^E(x) - \phi^E(x)) \leq 2$.

(A1) $\phi^E(x) = x^2, \Psi(x) = \frac{1}{2}(x\tilde{\psi}^E(x) - x^2)$ and $D(x^3\tilde{v}) = -\frac{1}{2}(x\tilde{\psi}^E(x) - x^2)\tilde{v}$. This corresponds to the case A_{32} in [13], where:

$$D(x^3\tilde{v}) = x((\alpha + 2)x + 2)\tilde{v},$$

with the condition $\Psi'(x) \neq 0$. Since $\Psi'(x) = \frac{1}{2}(\tilde{\psi}^E(x) + x(\tilde{\psi}^E)'(x) - 2x)$, the above condition means $\Psi'(0) = \frac{1}{2}\tilde{\psi}^E(0)$ and $\tilde{\psi}^E(0) \neq 0$. In addition, $\tilde{v} = x^{-1}\mathcal{B}^{(\alpha)} + M\delta(x)$.

(A2) $\phi^E(x) = x(x - 1), \Psi(x) = \frac{1}{2}(x\tilde{\psi}^E(x) - x(x - 1))$ and $D(x^2(x - 1)\tilde{v}) = -\frac{1}{2}(x\tilde{\psi}^E(x) - x(x - 1))\tilde{v}$. This corresponds to the case A_2 in [13], where \tilde{v} satisfies $D(x^2(x - 1)\tilde{v}) = -x(-(\alpha + \beta + 3)x + \beta + 2)\tilde{v}$, and:

$$\tilde{v} = x^{-1}(\tau_{1/2} \circ h_{1/2})\mathcal{J}^{(\alpha, \beta + 1)} + s\delta(x), \quad s \neq 0,$$

taking into account that for every polynomial p and $\alpha, \beta + 1 > -1$,

$$\langle \mathcal{J}^{(\alpha, \beta + 1)}, p(x) \rangle = \int_{-1}^1 p(x)(1 - x)^\alpha(x + 1)^{\beta + 1} dx.$$

The affine transformation $2t = x + 1$ yields:

$$\begin{aligned} \langle (\tau_{1/2} \circ h_{1/2})\mathcal{J}^{(\alpha, \beta + 1)}, p(x) \rangle &= \int_{-1}^1 p\left(\frac{1}{2}x + \frac{1}{2}\right)(1 - x)^\alpha(x + 1)^{\beta + 1} dx, \\ &= \langle \mathcal{J}_{[0,1]}^{(\alpha, \beta + 1)}, p(x) \rangle. \end{aligned}$$

As a consequence, $\tilde{v} = \mathcal{J}_{[0,1]}^{(\alpha, \beta)} + s\delta(x), s \neq 0$.

(A3) $\phi^E(x) = (x - 1)(x - \zeta), \Psi(x) = \frac{1}{2}(x\tilde{\psi}^E(x) - x(x - 1))$. This corresponds to the case A_1 in [13], where \tilde{v} satisfies:

$$\begin{aligned} &D(x(x - 1)(x - \zeta)\tilde{v}) \\ &= -\left[-(\alpha + \beta + \gamma + 3)x^2 + ((\alpha + \beta + 2)\zeta + \alpha + \gamma + 2)x - \zeta(\alpha + 1)\right], \end{aligned}$$

and it has the integral representation:

$$\langle \tilde{v}, p(x) \rangle = \int_0^1 p(x)(1 - x)^\alpha x^\beta |x - \zeta|^\gamma dx,$$

with the conditions $\alpha\beta\gamma \neq 0, \alpha, \beta, \gamma > 0, \zeta \in (0, 1)$.

B $\deg(x\phi^E(x)) < 3, \deg(x\tilde{\psi}^E(x) - \phi^E(x)) = 2$.

(B1) $\phi^E(x) = x - 1, \Psi(x) = \frac{1}{2}(x\tilde{\psi}^E(x) - (x - 1))$. This corresponds to the case B_1 in [13], where \tilde{v} satisfies $D(x(x - 1)\tilde{v}) = -(2\lambda x^2 + (-\alpha - \beta - 2\lambda - 2)x + \beta + 1)\tilde{v}$ and has the integral representation:

$$\langle \tilde{v}, p(x) \rangle = \int_0^1 p(x)(1 - x)^{\alpha + 1} x^{\beta + 1} e^{-\lambda x} dx,$$

with the conditions $\alpha\beta \neq 0, \alpha, \beta > -1$, and $\deg \tilde{\psi}^E = 1$.

(B2) $\phi^E(x) = x, \Psi(x) = \frac{1}{2}(x\tilde{\psi}^E(x) - x)$. This is the case B_2 in [13], where \tilde{v} satisfies:

$$D(x^2\tilde{v}) = -x(x - \alpha - 2)\tilde{v}.$$

Besides, for $\alpha > -1$, we get:

$$\langle \tilde{v}, p(x) \rangle = \int_0^\infty p(x)x^\alpha e^{-x} dx + sp(0), \quad s \neq 0.$$

(B3) $\phi^E(x) = 1, \Psi(x) = \frac{1}{2} (x\tilde{\psi}^E(x) - 1)$. This corresponds to the case B_3 in [13], where \tilde{v} satisfies:

$$D(x\tilde{v}) = - (2x^2 - \lambda x - 2\mu - 1) \tilde{v},$$

and it has the integral representation:

$$\langle \tilde{v}, p(x) \rangle = \int_0^\infty p(x)x^{2\mu} e^{-x^2 - \lambda x} dx,$$

with the conditions $\mu > -1/2, \lambda \in \mathbb{R}$ and $\deg \tilde{\psi}^E = 1$.

Now, we will analyze the reduction of (60) in the positive-definite case in order to get integral representations of such linear functionals. Then, we assume that \tilde{v} has an integral representation in terms of a weight function $\omega_{\tilde{v}}$ on an interval $[a, b]$ with $a \geq 0$, that is:

$$\langle \tilde{v}, p(x) \rangle = \int_a^b p(x)\omega_{\tilde{v}} dx.$$

First, we analyze the A_2 and B_2 cases. We get the rational relation $x^2\sigma_1(x)\tilde{u} = x(x - \xi^2)\tilde{v}$ with $\sigma_1(x) = x - 1$ in A_2 and $\sigma_1(x) = 1$ in B_2 . Besides:

$$\langle \tilde{u}, p(x) \rangle = \int_a^b p(x) \frac{\rho(x)\omega_{\tilde{v}}}{\phi^E(x)} dx + M_1p(0) + M_2p'(0) + Np(1),$$

where $N = 0$ in B_2 . By using (59) and (60), we get:

$$\begin{aligned} \langle x\rho^E(x)D\tilde{v}, p(x) \rangle &= - \langle \tilde{v}, (x\rho^E p)' \rangle \\ &= \int_a^b p\rho^E \frac{x\phi^E(x)\omega_{\tilde{v}}'}{\phi^E(x)} dx \\ &= -\frac{1}{2} \int_a^b p(x) \left(x\tilde{\psi}^E(x) + 2x(\phi^E)' \right) \frac{\rho^E(x)\omega_{\tilde{v}}(x)}{\phi^E(x)} dx - \frac{1}{2} \int_a^b p(x)\omega_{\tilde{v}}(x) dx. \end{aligned}$$

Since:

$$\begin{aligned} &-\frac{1}{2} \int_a^b p(x) \left(x\tilde{\psi}^E(x) + 2x(\phi^E)' \right) \frac{\rho^E(x)\omega_{\tilde{v}}(x)}{\phi^E(x)} dx \\ &= -\frac{1}{2} \langle \tilde{u}, p(x) \left(x\tilde{\psi}^E(x) + 2x(\phi^E)' \right) \rangle \\ &\quad + \frac{1}{2} M_2 p(0) \left(\tilde{\psi}^E(0) + 2(\phi^E)'(0) \right) + \frac{1}{2} N p(1) \left(\tilde{\psi}^E(1) + 2(\phi^E)'(1) \right), \end{aligned}$$

we get:

$$M_2 p(0) \left(\tilde{\psi}^E(0) + 2(\phi^E)'(0) \right) + N p(1) \left(\tilde{\psi}^E(1) + 2(\phi^E)'(1) \right) = 0,$$

for every polynomial p . In particular, for $p(x) = x - 1$:

$$M_2 \left(\tilde{\psi}^E(0) + 2(\phi^E)'(0) \right) = 0.$$

Next, we deal with $\tilde{\psi}^E(0) + 2(\phi^E)'(0) \neq 0$. When $\phi^E(x) = x(x - 1)$, \tilde{v} is positive-definite if $\alpha, (\beta + 1) > -1$. Taking into account that in this case, $\tilde{\psi}^E(x) = -(2\alpha + 2\beta + 5)x + 2\beta + 3$, then:

$$\tilde{\psi}^E(0) + 2(\phi^E)'(0) = 2\beta + 1,$$

and $M_2 = 0$ if $\beta \neq -1/2$. In a similar way, we get $\tilde{\psi}^E(x) = 2x - 2\alpha - 3$, and \tilde{v} is positive definite if $\alpha > -1$. After straightforward calculations, we obtain:

$$\tilde{\psi}^E(0) + 2(\phi^E)'(0) = -2\alpha - 1.$$

Thus, $M_2 = 0$ if $\alpha \neq -1/2$.

In A3 and B1, we get:

$$\langle \tilde{u}, p(x) \rangle = \int_a^b p(x) \frac{\rho(x)\omega_{\tilde{v}}}{\phi^E(x)} dx + M_1p(0) + M_2p(1) + M_3p(\zeta),$$

where $M_3 = 0$ in B1. An iteration of the above procedure yields:

$$\begin{aligned} & -\frac{1}{2} \int_a^b p(x) \left(x\tilde{\psi}^E(x) + 2x(\phi^E)' \right) \frac{\rho^E(x)\omega_{\tilde{v}}(x)}{\phi^E(x)} dx \\ = & -\frac{1}{2} \left\langle \tilde{u}, p(x) \left(x\tilde{\psi}^E(x) + 2x(\phi^E)' \right) \right\rangle \\ & + \frac{1}{2} M_2 p(1) \left(\tilde{\psi}^E(1) + 2(\phi^E)'(1) \right) + \frac{1}{2} M_3 p(\zeta) \left(\zeta\tilde{\psi}^E(\zeta) + 2\zeta(\phi^E)'(\zeta) \right). \end{aligned}$$

Then:

$$\frac{1}{2} M_2 p(1) \left(\tilde{\psi}^E(1) + 2(\phi^E)'(1) \right) + \frac{1}{2} M_3 p(\zeta) \left(\zeta\tilde{\psi}^E(\zeta) + 2\zeta(\phi^E)'(\zeta) \right) = 0. \tag{64}$$

On the one hand, in A3:

$$\begin{aligned} & x\tilde{\psi}^E(x) \\ = & -(2\alpha + 2\beta + 2\gamma + 5)x^2 \\ & + 2 \left((\alpha + \beta + 2)\zeta + \alpha + \gamma + 2 - \frac{1}{2}(1 + \zeta) \right) x - 2\zeta(\alpha + 1) + \zeta. \end{aligned}$$

Then, $\alpha = -1/2$. In this way, the case A3 will not be considered. On the other hand, in the case B1:

$$x\tilde{\psi}^E(x) = 4\lambda x^2 + (-2\alpha - 2\beta - 4\lambda - 3)x + (2\beta + 1),$$

and thus, $\beta = -1/2$ and $\tilde{\psi}^E(x) = 4\lambda x - 2\alpha - 4\lambda - 2$. Then, $\tilde{\psi}^E(1) = -2\alpha - 2$ and:

$$\tilde{\psi}^E(1) + 2(\phi^E)'(1) = -2\alpha.$$

Therefore, $M_2 = 0$ if $\alpha \neq 0$.

In the case B3, we cannot simplify the factor x . However, we get:

$$x\tilde{\psi}^E(x) = 4x^2 - 2\lambda x - 4\mu - 1,$$

and, as a consequence, $\mu = -1/4$. Then, $\tilde{\psi}^E(x) = 4x - 2\lambda$, and \tilde{v} satisfies $D(x\tilde{v}) = -1/2(4x^2 - 2\lambda x - 1)\tilde{v}$, as well as:

$$\langle \tilde{v}, p(x) \rangle = \int_0^\infty p(x)x^{-1/2}e^{-x^2-\lambda x} dx.$$

4. Positive-Definite Symmetric (1, 1)-Coherent Pairs (u, v)

According to the functionals \tilde{u} and \tilde{v} obtained in the previous section when $A(x) = 2(x^2 - \zeta_1^2)(x - \zeta_2^2)$, $\zeta_1^2 \neq \zeta_2^2$, or $A(x) = 2(x^2 - \zeta^2)^2$, respectively, the symmetrization process allows us to recover the original symmetric functionals u and v , and as a consequence, we get a classification of symmetric (1, 1)-coherent pairs. Of course, if we recover one pair (u, v) , we must also prove that it is a symmetric (1, 1)-coherent one. For this purpose, we state the next results.

Theorem 10 ([43]). *Let u be a symmetric, semiclassical, and quasi-definite linear functional of odd class s satisfying:*

$$D(\phi u) + \psi u = 0,$$

where $\deg \phi \leq s + 2$ and $\deg \psi \leq s + 1$. Notice that, ϕ and ψ are even and odd polynomials, respectively. $\{P_n\}_{n \geq 0}$ will denote the corresponding MOPS. We assume that the linear functional $w = x\phi(x)u$ is quasi-definite, with $\{W_n\}_{n \geq 0}$ as the corresponding MOPS. Then:

$$\frac{P'_{n+1}(x)}{n+1} = W_n(x) + \sum_{k=1}^{(s+1)/2} \eta_{n,n-2k} W_{n-2k}(x), \quad n \geq s+1,$$

with $\eta_{n,n-(s+1)} \neq 0$.

Theorem 11. *Let u be a symmetric, semiclassical, and quasi-definite linear functional of even class s satisfying:*

$$D(\phi u) + \psi u = 0,$$

where $\deg \phi \leq s + 2$ and $\deg \psi \leq s + 1$. Notice that ϕ and ψ are odd and even polynomials, respectively. $\{P_n\}_{n \geq 0}$ will denote the corresponding MOPS. We assume that the linear functional $w = \phi(x)u$ is quasi-definite, and $\{W_n\}_{n \geq 0}$ is the corresponding MOPS. Then:

$$\frac{P'_{n+1}(x)}{n+1} = W_n(x) + \sum_{k=1}^{s/2} \eta_{n,n-2k} W_{n-2k}(x), \quad n \geq s,$$

with $\eta_{n,n-s} \neq 0$.

Proof. It is enough to expand the sequence $\left\{ \frac{P'_{n+1}}{n+1} \right\}_{n \geq 0}$ in terms of the basis $\{W_n\}_{n \geq 0}$ and to consider its quasi-orthogonal character described in Theorem 2, B). \square

As a consequence of the above theorems, we get the next result.

Corollary 2. *Let u be as above with class s either 1 or 2. Let v denote a symmetric and quasi-definite linear functional such that there exist even polynomials p and q , with $0 \leq \deg p \leq 4$ and $\deg q = 2$, such that:*

$$p(x)u = q(x)v$$

holds. In addition, let $\{Q_n\}_{n \geq 0}$ be the MOPS associated with v . Then, (u, v) is a symmetric (1, 1)-coherent pair.

Proof. We consider the above theorems with $s = 1$ and $s = 2$, respectively. In both cases, we get:

$$Q_n(x) = W_n(x) + \beta_n W_{n-2}(x),$$

and:

$$\frac{P'_{n+1}(x)}{n+1} = W_n(x) + \lambda_n W_{n-2}(x),$$

where $\beta_n \lambda_n \neq 0$. From the above equations, we obtain:

$$\frac{P'_{n+1}(x)}{n+1} + \beta_{n-2} \frac{(\lambda_n - \beta_n)}{(\lambda_{n-2} - \beta_{n-2})} \frac{P'_{n-1}(x)}{n-1} = Q_n(x) + \lambda_{n-2} \frac{(\lambda_n - \beta_n)}{(\lambda_{n-2} - \beta_{n-2})} Q_{n-2}(x),$$

where $\beta_n \neq \lambda_n$ for every n . \square

4.1. Case $A(x) = 2(x^2 - \zeta^2)^2$

According to Theorem 6, if the class of \tilde{u} is $\tilde{s} = 0$, then the class of u is either 0 or 1. The classical cases (Gegenbauer, Hermite) have been analyzed in [9]. We suppose that $s = 1$, i.e.,

$$\tilde{\Psi}(0) = (\sigma_4^E)'(0) + \lim_{x \rightarrow 0} \frac{\sigma_4^E(x)}{x} + 2\tilde{\phi}_2^E(0) \neq 0.$$

(i) If $\sigma_4^E(x) = x^2$, assuming that $\tilde{\phi}_2^E(0) \neq 0$, then $u = \overline{\mathcal{B}}^{(\alpha)}$ satisfies:

$$D(x^3u) = -2(\tilde{\phi}_2(x) + x^2)u.$$

(ii) If $\sigma_4^E(x) = x(x - 1)$, assuming that $\tilde{\phi}_2^E(0) \neq 1$, then:

$$D(x(x^2 - 1)u) = -\left(2\tilde{\phi}_2(x) + \frac{1}{2}(x^2 - 1)\right)u.$$

Notice that $u = \overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)}$.

(iii) If $\sigma_4^E(x) = x$, assuming $\tilde{\phi}_2^E(0) \neq -1$, then $u = \overline{\mathcal{L}}^{(\alpha)}$ and:

$$D(xu) = -\left(2\tilde{\phi}_2(x) + \frac{1}{2}\right)u.$$

On the other hand, if \tilde{u} is of class $\tilde{s} = 1$, then from the symmetrization theorem, we deduce that the class of u is $s = 2$. Next, we will describe u according to σ_4^E .

(i) If $\sigma_4^E(x) = x^2$ and $\tilde{\phi}_2^E(0) \neq 0$, then $D(x^4u) = -2x\tilde{\phi}_2(x)u$. Thus, $u = x^{-2}\overline{\mathcal{B}}^{(\alpha)} + M\delta(x)$.

(ii) If $\sigma_4^E(x) = x(x - 1)$, then u satisfies $D(x^2(x^2 - 1)u) = -2x\tilde{\phi}_2(x)u$, and it has the integral representation:

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x) (1 - x^2)^\alpha |x|^{2\beta+1} e^{-\frac{\gamma}{x^2}} dx,$$

with the conditions $\alpha\gamma \neq 0, \gamma > 0, \alpha > -1$.

(iii) If $\sigma_4^E(x) = (x - 1)(x - \zeta)$, with $\zeta \in (0, 1)$, then u satisfies $D((x^2 - 1)(x^2 - \zeta)u) = -2x\tilde{\phi}_2(x)u$. Moreover,

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x) (1 - x^2)^\alpha |x|^{2\beta+1} |x^2 - \zeta|^\gamma dx,$$

with the conditions $\alpha\beta\gamma \neq 0, \alpha, \beta, \gamma > -1$.

(iv) If $\sigma_4^E(x) = x$, then u satisfies $D(x^2u) = -2x\tilde{\phi}_2(x)u$, as well as:

$$\langle u, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} (1 + x^2)^{\beta+1} e^{-x^2 + \frac{\beta}{x^2}} dx,$$

with $\beta < 0, \alpha, \beta > -1$.

(v) If $\sigma_4^E(x) = x - 1$, then u satisfies $D((x^2 - 1)u) = -2x\tilde{\phi}_2(x)u$. Moreover,

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x) (1 - x^2)^{\alpha+1} |x|^{2\beta+3} e^{-\lambda x^2} dx,$$

with the conditions $\alpha\beta \neq 0, \alpha, \beta > -1$.

(vi) If $\sigma_4^E(x) = 1$, then u satisfies $Du = -2x\tilde{\phi}_2(x)u$, and it has the integral representation:

$$\langle u, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{4\mu+1} e^{-x^4 - \lambda x^2} dx,$$

under the conditions $\mu > -1/2, \lambda \in \mathbb{R}$.

Since in the previous cases, u and v are related by:

$$\sigma_4(x)u = 2r_2(x)v,$$

then according to Corollary 2, in each case, the pair (u, v) is a symmetric $(1, 1)$ -coherent pair. Next, the corresponding symmetric $(1, 1)$ -coherent pairs are described in the positive-definite framework.

Theorem 12. Let (u, v) be a symmetric $(1, 1)$ -coherent pair satisfying:

$$\sigma_4(x)u = 2(x^2 - \xi^2)v,$$

such that σ_4 is an even polynomial with $\deg \sigma_4 \leq 4$ and u is a semiclassical linear functional of class at most 2. In addition, u and v are positive-definite, and $A(x) = (x^2 - \xi^2)^2$ in (40).

(A) u of class $s = 1$.

$S_{1,1}$. If $\sigma_4(x) = x^2(x^2 - 1)$ and either $\xi^2 = 0$ or $\xi^2 = 1$, then:

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x)(1 - x^2)^\alpha |x|^{2\beta+1} dx,$$

and:

$$\begin{aligned} \langle v, p(x) \rangle &= \int_{-1}^1 p(x) \frac{(1 - x^2)^{\alpha+1} |x|^{2\beta+3}}{(x^2 - \xi^2)} dx, \\ &+ \frac{M}{2} (\delta(x + |\xi|) + \delta(x - |\xi|)). \end{aligned}$$

$S_{1,2}$. If $\sigma_4(x) = x^2$ and $\xi^2 = 0$, then $u = \overline{\mathcal{L}}^{(\alpha)}$ and $v = \overline{\mathcal{L}}^{(\alpha)} + M\delta(x)$.

(B) u of class $s = 2$.

$S_{1,3}$. If $\sigma_4(x) = x^2(x^2 - 1), \alpha\gamma \neq 0, \gamma > 0, \alpha > -1$, and either $\xi^2 = 0$ or $\xi^2 = 1$, then:

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x) (1 - x^2)^\alpha |x|^{2\beta+1} e^{-\frac{\gamma}{x^2}} dx,$$

and:

$$\langle v, p(x) \rangle = \int_{-1}^1 p(x) \frac{(1 - x^2)^{\alpha+1}}{(x^2 - \xi^2)} |x|^{2\beta+3} e^{-\frac{\gamma}{x^2}} dx + \frac{M}{2} (p(|\xi|) + p(-|\xi|)).$$

S_{1.4}. If $\sigma_4(x) = (x^2 - 1)(x^2 - \zeta)$, with $\zeta \in (0, 1)$, $\alpha\beta\gamma \neq 0$, $\alpha, \beta, \gamma > 0$, and either $\zeta^2 = 0$ or $\zeta^2 = 1$, then:

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x) (1 - x^2)^\alpha |x|^{2\beta+1} |x^2 - \zeta|^\gamma dx,$$

and:

$$\langle v, p(x) \rangle = \int_{-1}^1 p(x) \frac{(1 - x^2)^{\alpha+1}}{(x^2 - \zeta^2)} |x|^{2\beta+1} |x^2 - \zeta|^{\gamma+1} dx + \frac{M}{2} (p(|\zeta|) + p(-|\zeta|)).$$

S_{1.5}. If $\sigma_4(x) = x^2$, $\beta \in (-1, 0)$, $\alpha > -1$ and $\zeta^2 = 0$, then:

$$\langle u, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} (1 + x^2)^{\beta+1} e^{-x^2 + \frac{\beta}{x^2}} dx,$$

and:

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} (1 + x^2)^{\beta+1} e^{-x^2 + \frac{\beta}{x^2}} dx + Mp(0).$$

S_{1.6}. If $\sigma_4(x) = 1$, $\mu > 0$, $\lambda \in \mathbb{R}$, and $\zeta^2 = 0$, then:

$$\langle u, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{4\mu+1} e^{-x^4 - \lambda x^2} dx,$$

and:

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{4\mu-1} e^{-x^4 - \lambda x^2} dx + Mp(0).$$

S_{1.7}. If $\sigma_4(x) = x^2 - 1$, $\alpha\beta \neq 0$, $\alpha, \beta > -1$, and either $\zeta^2 = 0$ or $\zeta^2 = 1$, then:

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x) (1 - x^2)^{\alpha+1} |x|^{2\beta+3} e^{-\lambda x^2} dx,$$

and:

$$\langle v, p(x) \rangle = \int_{-1}^1 p(x) \frac{(1 - x^2)^{\alpha+2}}{(x^2 - \zeta^2)} |x|^{2\beta+3} e^{-\lambda x^2} dx + \frac{M}{2} (p(|\zeta|) + p(-|\zeta|)).$$

4.2. Case $A(x) = 2(x^2 - \zeta_1^2)(x - \zeta_2^2)$, $\zeta_1^2 \neq \zeta_2^2$

When \tilde{v} is a semiclassical linear functional of class $\tilde{s} = 0$, the class of v is either 0 or 1. When the class of v is $s = 0$, we get $\mathcal{J}_{(0,1)}^{(\alpha,-1/2)}$ and $\mathcal{L}^{(-3/2)}$, which are nonpositive-definite linear functionals. Next, we describe the cases when the class of v is $s = 1$ according to the expression of ϕ^E .

(i) If $\phi^E(x) = x^2$, then $v = \overline{\mathcal{B}}^{(\alpha)}$, and it satisfies $D(x^3v) = -(\tilde{\psi}(x) + x^2)v$. Notice that this is not a positive-definite case.

(ii) If $\phi^E(x) = x(x - 1)$ and $\beta \neq -1/2$, then $v = \overline{\mathcal{J}}_{(0,1)}^{(\alpha,\beta)}$; moreover, $D(x(x^2 - 1)v) = -(\tilde{\psi}(x) + (x^2 - 1))v$.

Notice that:

$$\langle v, p(x) \rangle = \int_{-1}^1 p(x)(1 - x^2)^\alpha |x|^{2\beta+1} dx.$$

(iii) If $\phi^E(x) = x$, then v satisfies $D(xv) = -(\tilde{\psi}(x) + 1)v$, and as a consequence, $v = \overline{\mathcal{L}}^{(\alpha)}$. Thus,

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} e^{-x^2} dx.$$

If \tilde{v} is a semiclassical linear functional of class $\tilde{s} = 1$, notice that, according to Theorem 6, v must be semiclassical of class $s = 2$. Next, we describe the possible choices for v .

(iv) If $\phi^E(x) = x^2$, then v satisfies $D(x^4v) = -x\tilde{\psi}(x)v$, i.e., $v = x^{-2}\overline{\mathcal{B}}^{(\alpha)} + M\delta(x)$.

(v) If $\phi^E(x) = x(x - 1)$, then $v = \overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)} + s\delta(x)$, $s \neq 0$, and v satisfies:

$$D(x^2(x^2 - 1)v) = -x\tilde{\psi}(x)v = -x(- (2\alpha + 2\beta + 5)x + 2\beta + 3)v,$$

with $\beta \neq -1/2$.

(vi) If $\phi^E(x) = x - 1$, $\alpha \neq 0$, $\alpha > -1$, $\lambda \neq 0$, then v satisfies:

$$D((x^2 - 1)v) = -2x(2\lambda x^2 - \alpha - 2\lambda - 1)v,$$

i.e.,

$$\langle v, p(x) \rangle = \int_{-1}^1 p(x) (1 - x^2)^{\alpha+1} x^2 e^{-\lambda x^2} dx.$$

(vii) If $\phi^E(x) = x$, $\alpha > -1$, $\alpha \neq -1/2$ and $s \neq 0$, then:

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} e^{-x^2} dx + sp(0),$$

and $D(x^2v) = -x(2x^2 - 2\alpha - 3)v$.

(viii) If $\phi^E(x) = 1$, then v satisfies $Dv = -x(4x^2 - 2\lambda)v$ and:

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) e^{-x^4 - \lambda x^2} dx.$$

Moreover,

$$\langle u, p(x) \rangle = \int_{-\infty}^{\infty} p(x) (x^2 - \zeta^2) e^{-x^4 - \lambda x^2} dx + Mp(0).$$

In Cases (i)–(v) and (vii), we will assume that $\zeta^2 = 0$. From (58), we get:

$$x\phi^E(x)D(\tilde{u}) = \frac{3}{2}x\tilde{v} - \left(2x(\phi^E)'(x) + \frac{1}{2}x\tilde{\psi}^E(x) + \phi^E(x)\right)\tilde{u}.$$

Taking into account $\phi^E(x)\tilde{u} = x\tilde{v}$, then:

$$D(x\phi^E(x)\tilde{u}) = \left(\frac{5}{2}\phi^E(x) - (x\phi^E(x))' - \frac{1}{2}x\tilde{\psi}^E(x)\right)\tilde{u}.$$

As a consequence, \tilde{u} is semiclassical of class at most 1. According to Theorem 6 and Corollary 2, since $\phi^E(0) = 0$, then the class of u must be at most 2, and the pairs (u, v) are symmetric $(1, 1)$ -coherent. For Cases (vi) and (viii), we get $x^2u = x^2(x^2 - \zeta^2)v$ and $x^2u = x^2v$, respectively. Then, it is enough to apply the arguments of the above lemma, but by using the fact that v is of class $s \leq 2$.

For the positive-definite case, the previous analysis is summarized next.

Theorem 13. Let (u, v) be a symmetric $(1, 1)$ -coherent pair satisfying:

$$x\phi(x)u = x(x^2 - \zeta^2)v$$

such that ϕ is an even polynomial with $\deg \phi(x) \leq 4$ and v is semiclassical of class at most 2. In addition, let us assume that u and v are positive-definite, as well as in (40) $A(x) = (x^2 - \zeta_1^2)(x^2 - \zeta_2^2)$, $\zeta_1^2 \neq \zeta_2^2$.

(A) v classical.

S_{2,1}. If $\phi(x) = x^2(x^2 - 1)$, then $v = \overline{\mathcal{J}}_{(0,1)}^{(\alpha,-1/2)} = \mathcal{G}^{(\lambda)}$, $\lambda > -1$, $\lambda \neq 0$, i.e., the classical Gegenbauer functional. Thus:

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x)(1 - x^2)^{\lambda-1/2} dx + M_1 p(0) + \frac{M_2}{2} (p(1) + p(-1)).$$

(B) *v* of Class 1.

S_{2,2}. If $\phi(x) = x^2(x^2 - 1)$, $\beta \neq -1/2$, then:

$$\langle v, p(x) \rangle = \int_{-1}^1 p(x)(1 - x^2)^\alpha |x|^{2\beta+1} dx,$$

and:

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x)(1 - x^2)^{\alpha-1} |x|^{2\beta+1} dx + M_1 p(0) + \frac{M_2}{2} (p(1) + p(-1)).$$

S_{2,3}. If $\phi(x) = x^2$ then:

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} e^{-x^2} dx,$$

and:

$$\langle u, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} e^{-x^2} dx + Mp(0).$$

(C) *v* of Class 2.

S_{2,4}. If $\phi(x) = x^2(x^2 - 1)$, $\beta \neq -1/2$, then:

$$\langle v, p(x) \rangle = \int_{-1}^1 p(x)(1 - x^2)^\alpha |x|^{2\beta+1} dx + Mp(0),$$

and:

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x)(1 - x^2)^{\alpha-1} |x|^{2\beta+1} dx + Mp(0) + \frac{N}{2} (p(1) + p(-1)).$$

S_{2,5}. If $\phi(x) = x^2$, $\alpha > -1$, $\alpha \neq -1/2$, and $M \neq 0$, then:

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} e^{-x^2} dx + Mp(0),$$

and:

$$\langle u, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x|^{2\alpha+1} e^{-x^2} dx + Mp(0).$$

S_{2,6}. If $\phi(x) = x^2 - 1$, $\zeta^2 = 1$, $\alpha \neq 0$, $\alpha > -1$, $\lambda \neq 0$, then:

$$\langle v, p(x) \rangle = \int_{-1}^1 p(x) (1 - x^2)^{\alpha+1} x^2 e^{-\lambda x^2} dx,$$

and:

$$\langle u, p(x) \rangle = \int_{-1}^1 p(x) (1 - x^2)^\alpha e^{-\lambda x^2} dx + Mp(0).$$

S_{2,9}. If $\phi(x) = 1$, then:

$$\langle v, p(x) \rangle = \int_{-\infty}^{\infty} p(x) e^{-x^4 - \lambda x^2} dx,$$

as well as:

$$\langle u, p(x) \rangle = \int_{-\infty}^{\infty} p(x) |x^2 - \xi^2| e^{-x^4 - \lambda x^2} dx + Mp(0).$$

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