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Some Mann-Type Implicit Iteration Methods for Triple Hierarchical Variational Inequalities, Systems Variational Inequalities and Fixed Point Problems

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Abstract: This paper discusses a monotone variational inequality problem with a variational inequality constraint over the common solution set of a general system of variational inequalities (GSVI) and a common fixed point (CFP) of a countable family of nonexpansive mappings and an asymptotically nonexpansive mapping in Hilbert spaces, which is called the triple hierarchical constrained variational inequality (THCVI), and introduces some Mann-type implicit iteration methods for solving it. Norm convergence of the proposed methods of the iteration methods is guaranteed under some suitable assumptions.

Keywords: mann-type implicit iteration methods; triple hierarchical constrained variational inequality; general system of variational inequalities; strong convergence; hilbert spaces

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1. Introduction

Let C be a convex closed nonempty subset of a real Hilbert space H with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let P_C be the metric (or nearest point) projection from H onto C, that is, for all $x \in H$, $P_C x \in C$ and $\|x - P_C x\| = \inf_{y\in C}\|x-y\|$. Let $T : C \to C$ be a possible nonlinear mapping. Denote by $\mathrm{Fix}(T)$ the set of fixed points of T, i.e., $\mathrm{Fix}(T) = \{x \in C : x = Tx\}$. We use the notations $\mathbf{R}$, $\rightharpoonup$ and $\to$ to indicate the set of real numbers, weak convergence and strong convergence, respectively.

A mapping $T : C \to C$ is said to be asymptotically nonexpansive (see [1]), if there exists a sequence $\{\theta_n\} \subset [0,+\infty)$ with $\lim_{n\to\infty}\theta_n = 0$ such that

$$\|T^n x - T^n y\| \le (1+\theta_n)\|x-y\| \quad \forall n \ge 0,\ x,y \in C.$$

In particular, T is said to be nonexpansive if $\|Tx - Ty\| \le \|x-y\|$, $\forall x,y \in C$, that is, $\theta \equiv 0$. If C is also a bounded set, then the fixed-point set of T is nonempty, that is $\mathrm{Fix}(T) \ne \emptyset$. Via iterative techniques, fixed points of (asymptotically) nonexpansive mappings have been studied because of their applications in convex optimization problems; see [2–10] and the references therein.

Let $B_1, B_2 : C \to H$ be two nonlinear single-valued mappings. We consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x - x^*, \mu_1 B_1 y^* + x^* - y^*\rangle \ge 0, & \forall x \in C, \\ \langle x - y^*, \mu_2 B_2 x^* + y^* - x^*\rangle \ge 0, & \forall x \in C, \end{cases} \tag{1}$$

which is called a general system of variational inequalities (GSVI) with real number constants μ_1 and $\mu_2 > 0$, which covers as special subcases the problems arising, especially from nonlinear complementarity problems, quadratic mathematical programming and other variational problems. The reader is referred to [11–18] and the references therein. Particularly, if both B_1 and B_2 are equal to A and $x^* = y^*$, then problem (1) become the classical variational inequality (VI), that set of solutions is stated by $\mathrm{VI}(C, A)$. Note that, problem (1) can be transformed into a fixed-point problem in the following way.

Lemma 1 ([19]). *Let both x^* and y^* be points in C. (x^*, y^*) is a solution of GSVI (1) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, and $y^* = P_C(I - \mu_2 B_2)x^*$.*

A mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \qquad \forall x, y \in C.$$

It is called η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \qquad \forall x, y \in C.$$

Moreover, it is called α-inverse-strongly monotone (or α-cocoercive), if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \qquad \forall x, y \in C.$$

Obviously, each inverse-strongly monotone mapping is monotone and Lipschitzian, and each strongly monotone and Lipschitzian mapping is inverse-strongly monotone but the converse is not true.

Recently, Cai et al. [20] proposed a new implicit-rule for obtaining a common element of the solution set of GSVI (1) and the fixed point set of an asymptotically nonexpansive mapping T, and presented norm convergence of the sequence generated by the proposed rule to an element of $\mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T)$, which also solves certain VI.

On the other hand, Iiduka [21] considered a monotone variational inequality linked to a inequality constraint over the set of fixed points of a nonexpansive mapping. Iiduka's problem is a triple mathematical programming in contrast with bilevel mathematical programming problems or hierarchical constrained optimization problems or nonlinear hierarchical problem, it is referred as triple hierarchical constrained optimization problem (THCOP). Since the THCOP is a general variational inequality, we also call it a triple hierarchical variational inequality (THVI). This kind of problems play an important role in nonlinear minimizer problems and nonlinear operator equations; see [22–26] and the references therein.

To begin with, let us recall the variational inequality for a monotone mapping, $A_1 : H \to H$, over the fixed point set of a nonexpansive mapping, $T : H \to H$:

$$\begin{aligned}&\text{Find } \bar{x} \in \mathrm{VI}(\mathrm{Fix}(T), A_1)\\ &\quad := \{\bar{x} \in \mathrm{Fix}(T) : \langle A_1\bar{x}, y - \bar{x}\rangle \geq 0\ \forall y \in \mathrm{Fix}(T)\},\end{aligned}$$

where $\mathrm{Fix}(T) := \{x \in H : Tx = x\} \neq \emptyset$. Iiduka's THCOP and its algorithm (Algorithm 1) are stated below.

Problem 1. (see [21], Problem 3.1) *Assume that*

(C1) $T : H \to H$ *is a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$;
(C2) $A_2 : H \to H$ *is κ-Lipschitz continuous η-strongly monotone;*
(C3) $A_1 : H \to H$ *is ζ-inverse-strongly monotone;*
(C4) $\mathrm{VI}(\mathrm{Fix}(T), A_1) \neq \emptyset$.

Then the objective is to

$$\begin{aligned}&\text{Find } x^* \in \mathrm{VI}(\mathrm{VI}(\mathrm{Fix}(T), A_1), A_2)\\ &\qquad := \{x^* \in \mathrm{VI}(\mathrm{Fix}(T), A_1) : \langle v - x^*, A_2x^*\rangle \geq 0\ \forall v \in \mathrm{VI}(\mathrm{Fix}(T), A_1)\}.\end{aligned}$$

Algorithm 1. (see [21], Algorithm 4.1)
Step 0. Take $\{\alpha_n\}_{n=0}^{\infty}, \{\delta_n\}_{n=0}^{\infty} \subset (0,\infty)$, *and* $\mu > 0$, *choose* $x_0 \in H$ *arbitrarily, and let* $n := 0$.
Step 1. Given $x_n \in H$, *compute* $x_{n+1} \in H$ *as*

$$\begin{cases} y_n = T(x_n - \delta_n A_1 x_n), \\ x_{n+1} = y_n - \alpha_n \mu A_2 y_n. \end{cases}$$

Update $n := n + 1$ *and go to Step 1.*

The purpose of this paper is to introduce and analyze some Mann-type implicit iteration methods for treating a monotone variational inequality with a inequality constraint over the common solution set of the GSVI (1) for two inverse-strongly monotone mappings and a common fixed point problem (CFPP) of a countable family of nonexpansive mappings and an asymptotically nonexpansive mapping in Hilbert spaces, which is called the triple hierarchical constrained variational inequality (THCVI). Here the Mann-type implicit iteration methods are based on the Mann iteration method, viscosity approximation method, Korpelevich's extragradient method and hybrid steepest-descent method. Under some suitable assumptions, we prove strong convergence of the proposed methods to the unique solution of the THCVI.

2. Preliminaries

Now we recall some necessary concepts and facts. A mapping $F : C \to H$ is named to be κ-Lipschitzian if there is a real number $\kappa > 0$ with

$$\kappa\|x - y\| \geq \|F(x) - F(y)\|, \qquad \forall x, y \in C.$$

Particularly, if $\kappa \in (0, 1)$, then F is said to be contractive. If $\kappa = 1$, then F is said to be a nonexpansivity. A mapping $A : H \to H$ is named to be a strongly positive bounded linear operator if there is a real number $\gamma > 0$ with

$$\langle Ax, x\rangle \geq \gamma\|x\|^2, \qquad \forall x \in H.$$

For a fixed $x \in H$, we know that there is a unique point in C, presented by $P_C x$, with

$$\|x - y\| \geq \|x - P_C x\|, \qquad \forall y \in C.$$

P_C is called a metric projection of H onto C.

Lemma 2. *There hold the following important relations for metric projection* P_C*:*

(i) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \forall x, y \in H$;
(ii) $0 \geq \langle x - P_C x, y - P_C x\rangle, \forall x \in H, y \in C$;
(iii) $\|x - y\|^2 + 2\langle x - y, y\rangle = \|x\|^2 - \|y\|^2, \forall x, y \in H$;
(iv) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \forall x \in H, y \in C$.

Lemma 3 ([27])**.** *Let* $\{a_n\}$ *be a sequence of real numbers with the conditions:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\gamma_n, \qquad \forall n \geq 0,$$

where $\{\lambda_n\}$ *and* $\{\gamma_n\}$ *are sequences of real numbers such that (i)* $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$, *and (ii)* $\sum_{n=0}^{\infty} |\gamma_n \lambda_n| < \infty$ *or* $\limsup_{n\to\infty} \gamma_n \leq 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 4 ([27]). *Let* λ *be real number in* $(0,1]$. *Let* $T : C \to H$ *be a nonexpansive nonself mapping. Let* $T^{\lambda} : C \to H$ *be a nonself mapping defined by*

$$T^{\lambda}x := Tx - \lambda\mu F(Tx), \qquad \forall x \in C.$$

Here $F : H \to H$ *is* κ*-Lipschitzian and* η*-strongly monotone. So,* T^{λ} *is a contraction if* $0 < \mu < \frac{2\eta}{\kappa^2}$, *i.e.,*

$$\|T^{\lambda}x - T^{\lambda}y\| \leq (1 - \lambda\tau)\|x - y\|, \qquad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.

Lemma 5 ([17]). *Let the mapping* $A : C \to H$ *be* α*-inverse-strongly nonself monotone. Then, for a given* $\lambda \geq 0$, $\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2$. *In particular, if* $0 \leq \lambda \leq 2\alpha$, *then* $I - \lambda A$ *is nonexpansive.*

Lemma 6 ([17]). *Let the mappings* $B_1, B_2 : C \to H$ *be* α*-inverse-strongly monotone and* β*-inverse-strongly monotone, respectively. Let the mapping* $G : C \to C$ *be defined as* $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$. *If* $0 \leq \mu_1 \leq 2\alpha$ *and* $0 \leq \mu_2 \leq 2\beta$, *then* $G : C \to C$ *is nonexpansive.*

Lemma 7 ([28]). *Let* H *be a Hilbert space. We suppose that* C *is a convex closed nonempty set in* H, *and* $T : C \to C$ *is an asymptotically nonexpansive nonself mapping with a nonempty fixed point set, that is,* $\mathrm{Fix}(T) \neq \emptyset$. *Then* $I - T$ *is demiclosed at zero, i.e., if* $\{x_n\} \subset C$ *converges weakly to some* $x \in C$, *and* $\{(I - T)x_n\}$ *converges strongly to zero, then* $(I - T)x = 0$, *where* I *is the identity mapping on* H.

Lemma 8 ([29]). *Let* H *be a Hilbert space. We suppose that* $\{x_n\}$ *and* $\{w_n\}$ *are bounded vector sequences in* H *and* $\{\beta_n\}$ *is a real number sequence in* $(0,1)$ *such that* $\limsup_{n\to\infty} \beta_n \leq 1$ *and* $\liminf_{n\to\infty} \beta_n > 0$. *We also suppose that* $x_{n+1} = \beta_n x_n + (1 - \beta_n)w_n$, $\forall n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|w_n - x_n\| = 0$.

Let C be a convex closed nonempty set. Let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self mappings defined on C, and $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers in $[0,1]$. On C, we define a self mapping W_n:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = (1 - \lambda_n)I + \lambda_n S_n U_{n,n+1}, \\ U_{n,n-1} = (1 - \lambda_{n-1})I + \lambda_{n-1} S_{n-1} U_{n,n}, \\ \cdots, \\ U_{n,k} = (1 - \lambda_k)I + \lambda_k S_k U_{n,k+1}, \\ U_{n,k-1} = (1 - \lambda_{k-1})I + \lambda_{k-1} S_{k-1} U_{n,k}, \\ \cdots, \\ U_{n,1} = (1 - \lambda_1)I + \lambda_1 S_1 U_{n,2}, \\ W_n = U_{n,0} = (1 - \lambda_0)I + \lambda_0 S_0 U_{n,1}. \end{cases}$$

Such a W_n is named the W-mapping generated by $S_n, S_{n-1}, ..., S_0$ and $\lambda_n, \lambda_{n-1}, ..., \lambda_0$; see [30].

Lemma 9 ([30]). *Let C be a convex closed nonempty set in a Hilbert space H. Let $\{S_n\}_{n=0}^{\infty}$ be a mapping sequence of nonexpansivity on C with $\bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \neq \emptyset$. Let $\{\lambda_n\}_{n=0}^{\infty}$ be a number sequence in $(0, b]$ for some $b \in (0, 1)$. Then $\lim_{n\to\infty} U_{n,k}x$ exists for every $x \in C$ and $k \geq 0$.*

Using Lemma 9, $W : C \to C$ is defined by $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,0}x$, $\forall x \in C$. We call W is the W-mapping defined by $\{S_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$. Next, we assume that $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$.

Lemma 10 ([30]). *Let C be a convex closed nonempty set of a Hilbert space H. Let $\{S_n\}_{n=0}^{\infty}$ be a mapping sequence of nonexpansivity on C with $\bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \neq \emptyset$. Let $\{\lambda_n\}_{n=0}^{\infty}$ be a number sequence in $(0, b]$ for some $b \in (0, 1)$. Then $\bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) = \mathrm{Fix}(W)$.*

Lemma 11 ([30]). *Let C be a convex closed nonempty set of a Hilbert space H. Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive self-mappings on C with $\bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \neq \emptyset$, and $\{\lambda_n\}_{n=0}^{\infty}$ be a real sequence in $(0, b]$ for some $b \in (0, 1)$. If D is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in D} \|W_n x - Wx\| = 0$.*

Lemma 12 ([21]). *Let C be a convex closed nonempty set of a Hilbert space H. Let $A : C \to H$ be a hemicontinuous nonself monotone mapping. Then the following hold: (i) $\mathrm{VI}(C, A) = \{x^* \in C : \langle x^* - y, Ay\rangle \leq 0\ \forall y \in C\}$; (ii) $\mathrm{VI}(C, A) = \mathrm{Fix}(P_C(I - \lambda A))$ for all $\lambda > 0$; (iii) $\mathrm{VI}(C, A)$ consists of one point, if A is strongly monotone and Lipschitz continuous.*

3. Main Results

Let C be a convex closed nonempty set of a real Hilbert space H. Let the mappings $A_1, B_i : C \to H$ be monotone for $i = 1, 2$. Let $T : C \to C$ be an asymptotically nonexpansive self mapping and $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self mappings on C. We now consider the variational inequality for mapping A_1 over the common solution set Ω of the GSVI (1) and the CFPP of $\{S_n\}_{n=0}^{\infty}$ and T:

$$\begin{aligned} &\text{Find } \bar{x} \in \mathrm{VI}(\Omega, A_1) \\ &\qquad := \{\bar{x} \in \Omega : \langle A_1\bar{x}, y - \bar{x}\rangle \geq 0\ \forall y \in \Omega\}, \end{aligned}$$

where $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T) \neq \emptyset$. This section introduces the following general monotone variational inequality with the variational inequality constraint on the common solution set of the GSVI (1) and the CFPP of $\{S_n\}_{n=0}^{\infty}$ and T, which is named as the triple hierarchical constrained variational inequality (THCVI):

Problem 2. *Assume that*

- *(C1)* $T : C \to C$ *is an asymptotically nonexpansive self mapping with a sequence* $\{\theta_n\} \subset [0, +\infty)$;
- *(C2)* $\{S_n\}_{n=0}^{\infty}$ *is a countable family of nonexpansive self mappings on* C;
- *(C3)* $B_1, B_2 : C \to H$ *are* α*-inverse-strongly monotone and* β*-inverse-strongly monotone, respectively;*
- *(C4)* $\mathrm{GSVI}(C, B_1, B_2) := \mathrm{Fix}(G)$ *where* $G := P_C(P_C(I - \mu_2 B_2) - \mu_1 B_1 P_C(I - \mu_2 B_2))$ *for real numbers* $\mu_1, \mu_2 > 0$;
- *(C5)* $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T) \neq \emptyset$;
- *(C6)* W_n *is the* W*-mapping defined by* $S_n, S_{n-1}, ..., S_0$ *and* $\lambda_n, \lambda_{n-1}, ..., \lambda_0$, *where* $\{\lambda_n\}_{n=0}^{\infty} \subset (0, 1)$;
- *(C7)* $A_1 : C \to H$ *is* ζ*-inverse-strongly monotone;*
- *(C8)* $A_2 : C \to H$ *is* η*-strongly monotone and* κ*-Lipschitzian;*
- *(C9)* $f : C \to C$ *is a* δ*-contraction mapping with real coefficient* $\delta \in [0, 1)$;
- *(C10)* $\mathrm{VI}(\Omega, A_1) \neq \emptyset$.

Then the objective is to

$$\begin{aligned} &\text{find } x^* \in \mathrm{VI}(\mathrm{VI}(\Omega, A_1), \mu A_2 - f) \\ &\qquad := \{x^* \in \mathrm{VI}(\Omega, A_1) : \langle x^* - v, (\mu A_2 - f)x^*\rangle \leq 0\ \forall v \in \mathrm{VI}(\Omega, A_1)\}, \end{aligned}$$

for some $\mu > 0$.

Problem 3. *If we put* $f = 0$ *in Problem 2, then the objective is to*

$$\begin{aligned}&\text{find } x^* \in \mathrm{VI}(\mathrm{VI}(\Omega, A_1), A_2)\\ &\quad := \{x^* \in \mathrm{VI}(\Omega, A_1) : \langle A_2 x^*, v - x^*\rangle \geq 0\ \forall v \in \mathrm{VI}(\Omega, A_1)\}.\end{aligned}$$

Here we propose the following implicit Mann-type iteration algorithms (Algorithms 2 and 3) for solving Problems 2 and 3, respectively.

Algorithm 2.

Step 0. Take $\{\alpha_n\}_{n=0}^\infty, \{\beta_n\}_{n=0}^\infty, \{\gamma_n\}_{n=0}^\infty, \{\delta_n\}_{n=0}^\infty \subset (0, \infty)$, *and* $\mu > 0$, *choose* $x_0 \in C$ *arbitrarily, and let* $n := 0$.

Step 1. Given $x_n \in C$, *compute* $x_{n+1} \in C$ *as*

$$\begin{cases} u_n = (1-\gamma_n)W_n u_n + \gamma_n x_n, \\ y_n = P_C(I - \delta_n A_1)G u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)P_C[\alpha_n f(x_n) + (I - \alpha_n \mu A_2)T^n y_n]. \end{cases} \tag{2}$$

Update $n := n+1$ *and go to Step 1.*

Algorithm 3.

Step 0. Take $\{\alpha_n\}_{n=0}^\infty, \{\beta_n\}_{n=0}^\infty, \{\gamma_n\}_{n=0}^\infty, \{\delta_n\}_{n=0}^\infty \subset (0, \infty)$, *and* $\mu > 0$, *choose* $x_0 \in C$ *arbitrarily, and let* $n := 0$.

Step 1. Given $x_n \in C$, *compute* $x_{n+1} \in C$ *as*

$$\begin{cases} u_n = (1-\gamma_n)W_n z_n + \gamma_n x_n, \\ v_n = P_C(u_n - \mu_2 B_2 u_n), \\ z_n = P_C(v_n - \mu_1 B_1 v_n), \\ y_n = P_C(z_n - \delta_n A_1 z_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)P_C(I - \alpha_n \mu A_2)T^n y_n. \end{cases}$$

Update $n := n+1$ *and go to Step 1.*

We are now able to state and prove the main results of this paper: the following convergence analysis is presented for our Algorithms 2 and 3.

Theorem 1. *Assume that* μ_1 *is a real number in* $(0, 2\alpha)$, *and* μ_2 *is a real number in* $(0, 2\beta)$. *Let* $\delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$ *for* $\mu \in (0, \frac{2\eta}{\kappa^2})$. *We suppose* $\{\lambda_n\}_{n=0}^\infty$ *is a real sequence in* $(0, b]$ *for some real number* b *in* $(0, 1)$. *We also suppose that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ *and* $\{\delta_n\} \subset (0, 2\zeta]$ *such that*

(i) $\sum_{n=0}^\infty \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\delta_n \leq \alpha_n\ \forall n \geq 0$ *and* $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$ *and* $\limsup_{n\to\infty} \beta_n < 1$;
(iv) $\liminf_{n\to\infty} \gamma_n > 0$, $\limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;
(v) $\lim_{n\to\infty} \|T^{n+1} y_n - T^n y_n\| = 0$.

Then the sequence $\{x_n\}_{n=0}^\infty$ *generated by Algorithm 2 satisfies the following properties:*

(a) $\{x_n\}_{n=0}^\infty$ *is bounded;*
(b) $\lim_{n\to\infty} \|x_n - y_n\| = 0$, $\lim_{n\to\infty} \|x_n - Gx_n\| = 0$, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$;
(c) *if* $\lim_{n\to} \frac{\|x_n - y_n\|}{\delta_n} = 0$, *then* $x_n \to x^* \in \mathrm{VI}(\Omega, A_1)$.

Proof. First of all, for any $x, y \in C$, by Lemma 4, we have

$$\|P_{\mathrm{VI}(\Omega,A_1)}(f+I-\mu A_2)x - P_{\mathrm{VI}(\Omega,A_1)}(f+I-\mu A_2)y\| \leq \delta\|x-y\| + (1-\tau)\|x-y\| = [1-(\tau-\delta)]\|x-y\|,$$

which implies that $P_{\mathrm{VI}(\Omega,A_1)}(f+I-\mu A_2)$ is a contraction. Banach's Contraction Principle tells us that $P_{\mathrm{VI}(\Omega,A_1)}(f+I-\mu A_2)$ has a fixed point. Indeed, it is also unique, say $x^* \in C$, that is, $x^* = P_{\mathrm{VI}(\Omega,A_1)}(f+I-\mu A_2)x^*$. Utilizing Lemma 12, we get

$$\{x^*\} = \mathrm{Fix}(P_{\mathrm{VI}(\Omega,A_1)}(f+I-\mu A_2)) = \mathrm{VI}(\mathrm{VI}(\Omega, A_1), \mu A_2 - f).$$

That is, the Problem 2 has the unique solution. Since $\liminf_{n\to\infty} \gamma_n > 0$ and $\limsup_{n\to\infty} \gamma_n < 1$, we can suppose that $\{\gamma_n\} \subset [a_0, b_0]$ is subset of $(0,1)$ for some $a_0, b_0 \in (0,1)$. Since G : is defined from C to C as $G := P_C(P_C(I-\mu_2 B_2) - \mu_1 B_1 P_C(I-\mu_2 B_2))$. Here $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$, G is nonexpansive by Lemma 6. It is easy to see that for each $n \geq 0$ there exists a unique element $u_n \in C$ such that

$$u_n = \gamma_n x_n + (1-\gamma_n) W_n u_n. \tag{3}$$

As a matter of fact, we utilize $F_n x := \gamma_n x_n + (1-\gamma_n) W_n x \ \forall x \in C$. Since each $W_n : C \to C$ is a nonexpansive mapping, we get

$$\|F_n x - F_n y\| = (1-\gamma_n)\|W_n x - W_n y\| \leq (1-\gamma_n)\|x-y\|, \qquad \forall x, y \in C.$$

Also, from $\{\gamma_n\} \subset [a_0, b_0]$ and $[a_0, b_0] \subset (0,1)$ we have $0 < 1-\gamma_n < 1, \forall n \geq 0$. Thus, $F_n : C \to C$ is a contraction. Banach's Contraction Principle infers there exists a unique element u_n in set C satisfying (3).

Here, we are able to divide the rest of the proof into several steps.

Step 1. We claim that all the vector sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{v_n\}, \{T^n y_n\}$ and $\{A_2(T^n y_n)\}$ are bounded, where $v_n = P_C(u_n - \mu_2 B_2 u_n)$ and $z_n = P_C(v_n - \mu_1 B_1 v_n)$ for all $n \geq 0$. Indeed, it is clear that (2) can be rewritten as

$$\begin{cases} u_n = (1-\gamma_n) W_n u_n + \gamma_n x_n, \\ z_n = G u_n, \\ y_n = P_C(I - \delta_n A_1) z_n, \\ x_{n+1} = (1-\beta_n) P_C[(I - \alpha_n \mu A_2) T^n y_n + \alpha_n f(x_n)] + \beta_n x_n. \end{cases} \tag{4}$$

Take an arbitrary

$$p \in \Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T).$$

Then $p = W_n p$, $p = Tp$ and $p = Gp$. Since each $W_n : C \to C$ is nonexpansive, (4) infers to

$$\|u_n - p\| \leq (1-\gamma_n)\|u_n - p\| + \gamma_n \|x_n - p\|,$$

which hence yields

$$\|u_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 0. \tag{5}$$

It is easy to infer from (4) that

$$\|z_n - p\| = \|G u_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 0. \tag{6}$$

Since $\liminf_{n\to\infty}\beta_n > 0$ and $\limsup_{n\to\infty}\beta_n < 1$, we suppose that $\{\beta_n\} \subset [c,d]$. Since $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$, we can also suppose that

$$\theta_n \leq \frac{\alpha_n(\tau-\delta)(1-d)}{2} \leq \alpha_n(\tau-\delta).$$

Note that $\delta_n \leq \alpha_n, \forall n \geq 0$. ζ-inverse-strong monotonicity of A_1 and Lemma 5 yield

$$\|y_n - p\| \leq \|(I-\delta_n A_1)z_n - (I-\delta_n A_1)p - \delta_n A_1 p\| \leq \delta_n\|A_1 p\| + \|z_n - p\| \leq \|x_n - p\| + \delta_n\|A_1 p\|. \tag{7}$$

Utilizing Lemma 4 and (7), we obtain from (4) that

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\leq (1-\beta_n)\|\alpha_n(f(x_n) - \mu A_2 p) + (I - \alpha_n\mu A_2)T^n y_n - (I-\alpha_n\mu A_2)p\| + \beta_n\|x_n - p\| \\
&\leq (1-\beta_n)[\alpha_n\delta\|x_n - p\| + \alpha_n\|f(p) - \mu A_2 p\| + (1-\alpha_n\tau)(1+\theta_n)\|y_n - p\|] + \beta_n\|x_n - p\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\{\alpha_n\delta\|x_n - p\| + \alpha_n\|\mu A_2 p - f(p)\| \\
&\quad + (1-\alpha_n\tau)[\|x_n - p\| + \delta_n\|A_1 p\|] + \theta_n[\delta_n\|A_1 p\| + \|x_n - p\|]\} \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\{\alpha_n\delta\|x_n - p\| + \alpha_n\|\mu A_2 p - f(p)\| \\
&\quad + (1-\alpha_n\tau)[\|x_n - p\| + \delta_n\|A_1 p\|] + \theta_n\|x_n - p\| + (\tau-\delta)\alpha_n\delta_n\|A_1 p\|\} \\
&\leq [1 - \alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - p\| + \theta_n\|x_n - p\| + \alpha_n\|\mu A_2 p - f(p)\| + \alpha_n\|A_1 p\| \\
&\leq [1 - \tfrac{\alpha_n(1-d)(\tau-\delta)}{2}]\|x_n - p\| + \tfrac{\alpha_n(1-d)(\tau-\delta)}{2}\cdot\tfrac{2(\|A_1 p\| + \|\mu A_2 p - f(p)\|)}{(1-d)(\tau-\delta)} \\
&\leq \max\{\tfrac{2(\|A_1 p\| + \|\mu A_2 p - f(p)\|)}{(1-d)(\tau-\delta)}, \|x_n - p\|\}.
\end{aligned}$$

By simple induction, we have

$$\|x_{n+1} - p\| \leq \max\{\frac{2(\|f(p) - \mu A_2 p\|) + \|A_1 p\|}{(1-d)(\tau-\delta)}, \|x_0 - p\|\}, \qquad \forall n \geq 0.$$

Therefore, $\{x_n\}$ is a bounded vector sequence, and so are all the other sequences $\{y_n\}, \{z_n\}, \{u_n\}, \{T^n y_n\}$ and $\{A_2(T^n y_n)\}$ (due to the Lipschitz continuity of T and A_2). Since each W_n enjoys the nonexpansivity on C, we get

$$\|W_n u_n\| \leq \|W_n u_n - p\| + \|p\| \leq \|u_n - p\| + \|p\|,$$

which yields that $\{W_n u_n\}$ is bounded too. In addition, from Lemma 2 and p is a element in $\Omega \subset \mathrm{GSVI}(C, B_1, B_2)$, it also follows that (p,q) is a solution of GSVI (1) where $q = P_C(I-\mu_2 B_2)p$. Note that $v_n = P_C(I-\mu_2 B_2)u_n$ for all $n \geq 0$. Then by Lemma 5, we get

$$\|v_n\| \leq \|P_C(I-\mu_2 B_2)u_n - P_C(I-\mu_2 B_2)p\| + \|q\| \leq \|u_n - p\| + \|q\|.$$

This yields vector sequence $\{v_n\}$ is bounded.

Step 2. We claim that $\|x_n - x_{n+1}\| \to 0$ and $\|y_n - y_{n+1}\| \to 0$ as $n \to \infty$. Indeed, we set $x_{n+1} = \beta_n x_n + (1-\beta_n)w_n, \forall n \geq 0$. Then $w_n = P_C[(I-\alpha_n\mu A_2)T^n y_n + \alpha_n f(x_n)]$. It follows from (4) that

$$\begin{aligned}
\|w_{n+1} - w_n\| &\leq \|\alpha_{n+1}f(x_{n+1}) + (I-\alpha_{n+1}\mu A_2)T^{n+1}y_{n+1} - \alpha_n f(x_n) - (I-\alpha_n\mu A_2)T^n y_n\| \\
&\leq \|T^{n+1}y_{n+1} - T^{n+1}y_n\| + \|T^{n+1}y_n - T^n y_n\| + \alpha_{n+1}\|\mu A_2(T^{n+1}y_{n+1})\| \\
&\quad + \alpha_n\|\mu A_2(T^n y_n)\| + \alpha_{n+1}\|f(x_{n+1})\| + \alpha_n\|f(x_n)\| \\
&\leq (1+\theta_{n+1})\|y_{n+1} - y_n\| + \|T^{n+1}y_n - T^n y_n\| + \alpha_{n+1}(\|f(x_{n+1})\| \\
&\quad + \|\mu A_2(T^{n+1}y_{n+1})\|) + \alpha_n(\|f(x_n)\| + \|\mu A_2(T^n y_n)\|).
\end{aligned} \tag{8}$$

Since vector sequence $\{\delta_n\}$ falls into $(0, 2\zeta]$ and A_1 is ζ-inverse-strongly monotone, by Lemma 5 we obtain

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \|(z_{n+1} - \delta_{n+1}A_1z_{n+1}) - (z_n - \delta_n A_1 z_n)\| \\ &\le \|(z_{n+1} - \delta_{n+1}A_1z_{n+1}) - (z_n - \delta_{n+1}A_1z_n)\| + |\delta_n - \delta_{n+1}|\|A_1z_n\| \\ &\le \|u_{n+1} - u_n\| + |\delta_n - \delta_{n+1}|\|A_1z_n\|.\end{aligned} \tag{9}$$

Since simple calculations show that

$$\begin{aligned}\|u_nu_{n+1}\| &\le \gamma_{n+1}\|x_n - x_{n+1}\| + (1-\gamma_{n+1})\|W_nu_n - W_{n+1}u_{n+1}\| \\ &\quad + |\gamma_n - \gamma_{n+1}|\|W_nu_n - x_n\| \\ &\le \gamma_{n+1}\|x_n - x_{n+1}\| + (1-\gamma_{n+1})[\|W_nu_{n+1} - W_{n+1}u_{n+1}\| \\ &\quad + \|W_nu_n - W_nu_{n+1}\|] + |\gamma_n - \gamma_{n+1}|\|W_nu_n - x_n\| \\ &\le (1-\gamma_{n+1})[\|W_nu_{n+1} - W_{n+1}u_{n+1}\| + \gamma_{n+1}\|x_n - x_{n+1}\| \\ &\quad + \|u_n - u_{n+1}\|] + |\gamma_n - \gamma_{n+1}|\|W_nu_n - x_n\|,\end{aligned}$$

it follows that

$$a_0\|u_n - u_{n+1}\| \le a_0\|x_{n+1} - x_n\| + \|W_{n+1}u_{n+1} - W_nu_{n+1}\| + a_0\|x_n - W_nu_n\||\gamma_{n+1} - \gamma_n|. \tag{10}$$

Since $D := \{u_n : n \ge 0\} \subset C$ is bounded subset, by the argument process in Lemma 11 we get $\sum_{n=0}^{\infty} \sup_{x\in D} \|W_{n+1}x - W_nx\| < \infty$. Thus we have

$$\sum_{n=0}^{\infty} \|W_{n+1}u_{n+1} - W_nu_{n+1}\| < \infty. \tag{11}$$

Therefore, from (8)–(10) we deduce that

$$\begin{aligned}&\|w_n - w_{n+1}\| \\ &\le |\delta_n - \delta_{n+1}|\|A_1z_n\| + \theta_{n+1}\|y_n - y_{n+1}\| + \|T^ny_n - T^{n+1}y_n\| + \|u_n - u_{n+1}\| \\ &\quad + \alpha_{n+1}(\|f(x_{n+1})\| + \|\mu A_2(T^{n+1}y_{n+1})\|) + \alpha_n(\|f(x_n)\| + \|\mu A_2(T^ny_n)\|) \\ &\le \frac{1}{a_0}\|W_nu_{n+1} - W_{n+1}u_{n+1}\| + \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\frac{\|W_nu_n - x_n\|}{a_0} + |\delta_n - \delta_{n+1}|\|A_1z_n\| \\ &\quad + \theta_{n+1}\|y_n - y_{n+1}\| + \|T^ny_n - T^{n+1}y_n\| + \alpha_{n+1}(\|f(x_{n+1})\| + \|\mu A_2(T^{n+1}y_{n+1})\| \\ &\quad + \alpha_n\|(\|f(x_n)\| + \mu A_2(T^ny_n)\|),\end{aligned}$$

which immediately attains

$$\begin{aligned}&\|w_n - w_{n+1}\| - \|x_n - x_{n+1}\| \\ &\le \frac{1}{a_0}\|W_nu_{n+1} - W_{n+1}u_{n+1}\| + |\gamma_n - \gamma_{n+1}|\frac{\|W_nu_n - x_n\|}{a_0} + |\delta_n - \delta_{n+1}|\|A_1z_n\| \\ &\quad + \theta_{n+1}\|y_n - y_{n+1}\| + \|T^ny_n - T^{n+1}y_n\| + \alpha_{n+1}(\|f(x_{n+1})\| + \|\mu A_2(T^{n+1}y_{n+1})\|) \\ &\quad + \alpha_n(\|f(x_n)\| + \|\mu A_2(T^ny_n)\|).\end{aligned} \tag{12}$$

Since

$$\lim_{n\to\infty} \|T^ny_n - T^{n+1}y_n\| = \lim_{n\to\infty} \theta_n = 0,$$

from (11) and conditions (i), (ii), (iv) we get $\limsup_{n\to\infty}(\|w_n - w_{n+1}\| - \|x_n - x_{n+1}\|) \le 0$. Hence, by condition (iii) and Lemma 8, we get $\lim_{n\to\infty} \|w_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty}(1-\beta_n)\|w_n - x_n\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{13}$$

Again from (9) and (10) we conclude that

$$\begin{aligned}
&a_0\|y_n - y_{n+1}\| \\
&\leq a_0\|x_n - x_{n+1}\| + \|W_n u_{n+1} - W_{n+1}u_{n+1}\| + a_0|\gamma_n - \gamma_{n+1}| \\
&\|W_n u_n - x_n\| + |\delta_n - \delta_{n+1}|a_0\|A_1 z_n\| \to 0
\end{aligned}$$

and $\|z_{n+1} - z_n\| = \|Gu_{n+1} - Gu_n\| \leq \|u_{n+1} - u_n\| \to 0$. Thus,

$$\lim_{n\to\infty}\|y_n - y_{n+1}\| = \lim_{n\to\infty}\|u_n - u_{n+1}\| = \lim_{n\to\infty}\|z_n - z_{n+1}\| = 0. \tag{14}$$

Step 3. We claim that $\lim_{n\to\infty}\|Gx_n - x_n\| = 0$ as $n \to \infty$. Indeed, noticing $w_n = P_C[(I - \alpha_n\mu A_2)T^n y_n + \alpha_n f(x_n)]$ $\forall n \geq 0$, we obtain from Lemma 2 that for each $p \in \Omega$,

$$\langle p - w_n, (I - \alpha_n\mu A_2)T^n y_n + \alpha_n f(x_n) - P_C[\alpha_n f(x_n) + (I - \alpha_n\mu A_2)T^n y_n]\rangle \leq 0. \tag{15}$$

From (15), we have

$$\begin{aligned}
\|w_n - p\|^2 &= \langle P_C[(I - \alpha_n\mu A_2)T^n y_n + \alpha_n f(x_n)] - \alpha_n f(x_n) - (I - \alpha_n\mu A_2)T^n y_n, w_n - p\rangle \\
&\quad + \langle (I - \alpha_n\mu A_2)T^n y_n + \alpha_n f(x_n) - p, w_n - p\rangle \\
&\leq \langle (I - \alpha_n\mu A_2)T^n y_n + \alpha_n f(x_n) - p, w_n - p\rangle \\
&= \langle w_n - p, (I - \alpha_n\mu A_2)T^n y_n - (I - \alpha_n\mu A_2)p\rangle + \alpha_n\langle f(x_n) - \mu A_2 p, w_n - p\rangle \\
&\leq [(1 - \alpha_n\tau)\|T^n y_n - p\| + \delta\alpha_n\|x_n - p\|]\|w_n - p\| + \alpha_n\langle w_n - p, f(p) - \mu A_2 p\rangle \\
&\leq \tfrac{[(1-\alpha_n\tau)\|T^n y_n - p\| + \alpha_n\delta\|x_n - p\|]^2}{2} + \tfrac{1}{2}\|w_n - p\|^2 + \alpha_n\langle w_n - p, f(p) - \mu A_2 p\rangle,
\end{aligned}$$

which leads to

$$\begin{aligned}
&\|w_n - p\|^2 \\
&\leq (1 - \alpha_n\tau)\|T^n y_n - p\|^2 + \delta\alpha_n\|x_n - p\|^2 - 2\alpha_n\langle w_n - p, \mu A_2 p - f(p)\rangle \\
&\leq (1 - \alpha_n\tau)(1 + \theta_n)^2\|y_n - p\|^2 + \alpha_n\delta\|x_n - p\|^2 - 2\alpha_n\langle w_n - p, \mu A_2 p - f(p)\rangle \\
&\leq (1 - \alpha_n\tau)\|y_n - p\|^2 + \alpha_n\delta\|x_n - p\|^2 + \theta_n(2 + \theta_n)\|y_n - p\|^2 - 2\alpha_n\langle w_n - p, \mu A_2 p - f(p)\rangle.
\end{aligned} \tag{16}$$

From (7) and (16), we get

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)\|y_n - p\|^2 + \theta_n(2 + \theta_n)\|y_n - p\|^2 \\
&\quad + 2\alpha_n\langle f(p) - \mu A_2 p, w_n - p\rangle] \\
&\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\{\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)(\|z_n - p\| + \delta_n\|A_1 p\|)^2 \\
&\quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n\langle f(p) - \mu A_2 p, w_n - p\rangle\} \\
&\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)\|z_n - p\|^2] \\
&\quad + \delta_n\|A_1 p\|(2\|z_n - p\| + \delta_n\|A_1 p\|) + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n\|f(p) - \mu A_2 p\|\|p - w_n\|.
\end{aligned} \tag{17}$$

We now note that $q = P_C(p - \mu_2 B_2 p)$, $v_n = P_C(u_n - \mu_2 B_2 u_n)$ and $z_n = P_C(v_n - \mu_1 B_1 v_n)$. Then $z_n = Gu_n$. By Lemma 5 we have

$$\|v_n - q\|^2 \leq \|u_n - p - \mu_2(B_2 u_n - B_2 p)\|^2 \leq \|u_n - p\|^2 - \mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 p\|^2 \tag{18}$$

and

$$\|z_n - p\|^2 \leq \|v_n - q - \mu_1(B_1 v_n - B_1 q)\|^2 \leq \|v_n - q\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 q\|^2. \tag{19}$$

Substituting (18) for (19), we obtain from (5) that

$$\|z_n - p\|^2 \le \mu_2(\mu_2 - 2\beta)\|B_2 u_n - B_2 p\|^2 + \mu_1(\mu_1 + 2\alpha)\|B_1 v_n - B_1 q\|^2 + \|x_n - p\|^2. \tag{20}$$

Combining (17) and (20), we get

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - p\|^2 + (1-\tau\alpha_n)[\|x_n - p\|^2 \\
&\quad - (2\beta - \mu_2)\mu_2\|B_2 p - B_2 u_n\|^2 - (2\alpha - \mu_1)\mu_1\|B_1 q - B_1 v_n\|^2]\} \\
&\quad + \delta_n\|A_1 p\|(2\|z_n - p\| + \delta_n\|A_1 p\|) + (2+\theta_n)\theta_n\|y_n - p\|^2 + 2\alpha_n\|\mu A_2 p - f(p)\|\|p - w_n\| \\
&= [1 - (\tau - \delta)\alpha_n(1-\beta_n)]\|x_n - p\|^2 - (1-\alpha_n\tau)(1-\beta_n)[\mu_2(2\beta - \mu_2)\|B_2 p - B_2 u_n\|^2 \\
&\quad + (2\alpha - \mu_1)\mu_1\|B_1 q - B_1 v_n\|^2] + \delta_n\|A_1 p\|(2\|z_n - p\| + \delta_n\|A_1 p\|) \\
&\quad + (2+\theta_n)\theta_n\|y_n - p\|^2 + 2\alpha_n\|\mu A_2 p - f(p)\|\|p - w_n\|,
\end{aligned}$$

which immediately yields

$$\begin{aligned}
&(1-\alpha_n\tau)(1-\beta_n)[\mu_2(2\beta - \mu_2)\|B_2 p - B_2 u_n\|^2 + (2\alpha - \mu_1)\mu_1\|B_1 q - B_1 v_n\|^2] \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \delta_n\|A_1 p\|(\delta_n\|A_1 p\| + 2\|z_n - p\|) \\
&\quad + (2+\theta_n)\theta_n\|y_n - p\|^2 + 2\alpha_n\|\mu A_2 p - f(p)\|\|p - w_n\| \\
&\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + \delta_n\|A_1 p\|(\delta_n\|A_1 p\| + 2\|z_n - p\|) \\
&\quad + (2+\theta_n)\theta_n\|y_n - p\|^2 + 2\alpha_n\|p - w_n\|\|f(p) - \mu A_2 p\|.
\end{aligned}$$

Due to condition (iii), $\liminf_{n\to\infty}(1-\beta_n) > 0$, $\mu_1 \in (0, 2\alpha)$, $\mu_2 \in (0, 2\beta)$, $\lim_{n\to\infty}\theta_n = 0$, $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\delta_n = 0$, we obtain from (13) that

$$\lim_{n\to\infty}\|B_2 u_n - B_2 p\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|B_1 v_n - B_1 q\| = 0. \tag{21}$$

On the other hand, from Lemma 2 we have

$$\begin{aligned}
\|v_n - q\|^2 &\le \langle v_n - q, u_n - (p - \mu_2 B_2 p) - \mu_2 B_2 u_n\rangle \\
&\le \tfrac{1}{2}[\|u_n - p\|^2 + \|v_n - q\|^2 - \|u_n - v_n - (p-q)\|^2] + \mu_2\|v_n - q\|\|B_2 u_n - B_2 p\|,
\end{aligned}$$

which implies that

$$\|v_n - q\|^2 \le \|u_n - p\|^2 - \|(p-q) - u_n + v_n\|^2 + 2\mu_2\|v_n - q\|\|B_2 u_n - B_2 p\|. \tag{22}$$

In the same way, we derive

$$\|z_n - p\|^2 \le \|v_n - q\|^2 - \|(p-q) - v_n + z_n\|^2 + 2\mu_1\|z_n - p\|\|B_1 v_n - B_1 q\|. \tag{23}$$

Substituting (22) for (23), we deduce from (5) that

$$\begin{aligned}
\|z_n - p\|^2 &\le \|x_n - p\|^2 - \|u_n - v_n - (p-q)\|^2 - \|v_n - z_n + (p-q)\|^2 \\
&\quad + 2\mu_2\|B_2 p - B_2 u_n\|\|v_n - q\| + 2\mu_1\|B_1 q - B_1 v_n\|\|z_n - p\|.
\end{aligned} \tag{24}$$

Combining (17) and (24), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)[\|x_n-p\|^2-\|p-q-u_n+v_n\|^2\\
&\quad -\|p-q+v_n-z_n\|^2+2\mu_1\|z_n-p\|\|B_1v_n-B_1q\|+2\mu_2\|v_n-q\|\|B_2u_n-B_2p\|]\}\\
&\quad +\delta_n\|A_1p\|(2\|z_n-p\|+\delta_n\|A_1p\|)+(2+\theta_n)\theta_n\|y_n-p\|^2+2\alpha_n\|\mu A_2p-f(p)\|\|w_n-p\|\\
&\le [1-(\tau-\delta)\alpha_n(1-\beta_n)]\|x_n-p\|^2-(1-\alpha_n\tau)(1-\beta_n)[\|p-q-u_n+v_n\|^2\\
&\quad +\|p-q+v_n-z_n\|^2]+2\mu_1\|B_1v_n-B_1q\|\|z_n-p\|+2\mu_2\|v_n-q\|\|B_2p-B_2u_n\|\\
&\quad +\delta_n\|A_1p\|(\delta_n\|A_1p\|+2\|z_n-p\|)+(2+\theta_n)\theta_n\|y_n-p\|^2+2\alpha_n\|w_n-p\|\|f(p)-\mu A_2p\|,
\end{aligned}$$

which hence yields

$$\begin{aligned}
&(1-\alpha_n\tau)(1-\beta_n)[\|p-q-u_n+v_n\|^2+\|p-q+v_n-z_n\|^2]\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_2\|v_n-q\|\|B_2p-B_2u_n\|\\
&\quad +2\mu_1\|z_n-p\|\|B_1q-B_1v_n\|+\delta_n\|A_1p\|(\delta_n\|A_1p\|+2\|z_n-p\|)\\
&\quad +(2+\theta_n)\theta_n\|y_n-p\|^2+2\alpha_n\|p-w_n\|\|f(p)-\mu A_2p\|\\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+2\mu_2\|z_n-q\|\|B_2p-B_2u_n\|\\
&\quad +2\mu_1\|y_n-p\|\|B_1q-B_1z_n\|+\delta_n\|A_1p\|(\delta_n\|A_1p\|+2\|z_n-p\|)\\
&\quad +(2+\theta_n)\theta_n\|y_n-p\|^2+2\alpha_n\|p-w_n\|\|f(p)-\mu A_2p\|.
\end{aligned}$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$, $\lim_{n\to\infty}\theta_n=0$, $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\delta_n=0$, we conclude from (13) and (21) that

$$\lim_{n\to\infty}\|u_n-v_n-(p-q)\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|v_n-z_n+(p-q)\|=0. \tag{25}$$

It follows that

$$\|u_n-Gu_n\|=\|u_n-z_n\|\le\|u_n-v_n-(p-q)\|+\|v_n-z_n+(p-q)\|\to 0\quad (n\to\infty). \tag{26}$$

Also, from (4) we have $\|u_n-p\|^2\le(1-\gamma_n)\|u_n-p\|^2+\gamma_n\langle u_n-p,x_n-p\rangle$, which together with Lemma 2, yields $\|u_n-p\|^2\le\langle u_n-p,x_n-p\rangle=\frac{1}{2}[\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2]$. Thus, we get

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|x_n-u_n\|^2,$$

which together with (17), yields

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)[(1-\alpha_n\tau)\|u_n-p\|^2+\delta\alpha_n\|x_n-p\|^2]\\
&\quad +\delta_n\|A_1p\|(\delta_n\|A_1p\|+2\|z_n-p\|)+(2+\theta_n)\theta_n\|y_n-p\|^2+2\alpha_n\|p-w_n\|\|f(p)-\mu A_2p\|\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)[\|x_n-p\|^2-\|x_n-u_n\|^2]\}\\
&\quad +\delta_n\|A_1p\|(\delta_n\|A_1p\|+2\|z_n-p\|)+(2+\theta_n)\theta_n\|y_n-p\|^2+2\alpha_n\|p-w_n\|\|f(p)-\mu A_2p\|\\
&= [1-\alpha_n(\tau-\delta)(1-\beta_n)]\|x_n-p\|^2-(1-\alpha_n\tau)(1-\beta_n)\|x_n-u_n\|^2\\
&\quad +\delta_n\|A_1p\|(\delta_n\|A_1p\|+2\|z_n-p\|)+(2+\theta_n)\theta_n\|y_n-p\|^2+2\alpha_n\|p-w_n\|\|f(p)-\mu A_2p\|.
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&(1-\alpha_n\tau)(1-\beta_n)\|x_n-u_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\delta_n\|A_1p\|(\delta_n\|A_1p\|+2\|z_n-p\|)\\
&\quad +(2+\theta_n)\theta_n\|y_n-p\|^2+2\alpha_n\|\mu A_2p-f(p)\|\|p-w_n\|\\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+\delta_n\|A_1p\|(\delta_n\|A_1p\|+2\|z_n-p\|)\\
&\quad +(2+\theta_n)\theta_n\|y_n-p\|^2+2\alpha_n\|f(p)-\mu A_2p\|\|p-w_n\|.
\end{aligned}
$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$, $\lim_{n\to\infty}\theta_n=0$, $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\delta_n=0$, we obtain from (13) that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{27}$$

Also, observe that $\|x_n-z_n\|\le\|x_n-u_n\|+\|Gu_n-u_n\|$, $\|x_n-Gx_n\|\le\|x_n-z_n\|+\|u_n-x_n\|$, and

$$\|x_n-y_n\|\le\|x_n-(z_n-\delta_nA_1z_n)\|\le\|x_n-z_n\|+\delta_n\|A_1z_n\|.$$

Then from (26) and (27) it follows that

$$\lim_{n\to\infty}\|x_n-z_n\|=0,\quad \lim_{n\to\infty}\|x_n-Gx_n\|=0\quad\text{and}\quad \lim_{n\to\infty}\|x_n-y_n\|=0. \tag{28}$$

Step 4. We claim that $\lim_{n\to\infty}\|Tx_n-x_n\|=0$ and $\lim_{n\to\infty}\|W_nx_n-x_n\|=0$. Indeed, combining (4) and (27), we obtain

$$\|W_nu_n-u_n\|=\frac{\gamma_n}{1-\gamma_n}\|x_n-u_n\|\le\frac{b_0}{1-b_0}\|x_n-u_n\|\to 0\quad(n\to\infty). \tag{29}$$

Since each W_n is nonexpansive on C, from (27) and (29) we get

$$
\begin{aligned}
\|W_nx_n-x_n\| &\le \|W_nu_n-u_n\|+\|u_n-x_n\|+\|W_nx_n-W_nu_n\|\\
&\le \|W_nu_n-u_n\|+2\|u_n-x_n\|\to 0\quad(n\to\infty).
\end{aligned} \tag{30}
$$

We note that $\{\beta_n\}\subset[c,d]$ and $[c,d]\subset(0,1)$ for some $c,d\in(0,1)$, and observe that

$$
\begin{aligned}
&\|x_n-T^ny_n\|\le\|x_n-x_{n+1}\|+\|T^ny_n-x_{n+1}\|\\
&\le\|x_n-x_{n+1}\|+\beta_n\|x_n-T^ny_n\|+(1-\beta_n)\|T^ny_n-P_C[(I-\alpha_n\mu A_2)T^ny_n+\alpha_nf(x_n)]\|\\
&\le\|x_n-x_{n+1}\|+\beta_n\|x_n-T^ny_n\|+(1-\beta_n)\alpha_n(\|\mu A_2(T^ny_n)\|+\|f(x_n)\|).
\end{aligned}
$$

Then we have

$$(1-d)\|x_n-T^ny_n\|\le\|x_n-x_{n+1}\|+(1-d)\alpha_n(\|f(x_n)\|+\|\mu A_2(T^ny_n)\|).$$

Hence we get

$$
\begin{aligned}
(1-d)\|y_n-T^ny_n\| &\le (1-d)\|y_n-x_n\|+(1-d)\|x_n-T^ny_n\|\\
&\le (1-d)\|y_n-x_n\|+\|x_n-x_{n+1}\|+\alpha_n(1-d)(\|f(x_n)\|+\|\mu A_2(T^ny_n)\|).
\end{aligned}
$$

Consequently, from (13), (28) and $\lim_{n\to\infty}\alpha_n=0$, it follows that

$$\lim_{n\to\infty}\|x_n-T^ny_n\|=0\quad\text{and}\quad\lim_{n\to\infty}\|y_n-T^ny_n\|=0. \tag{31}$$

We also note that

$$\begin{aligned}\|y_n - Ty_n\| &\leq \|y_n - T^n y_n\| + \|T^n y_n - T^{n+1} y_n\| + \|T^{n+1} y_n - Ty_n\| \\ &\leq (2+\theta_1)\|T^n y_n - y_n\| + \|T^{n+1} y_n - T^n y_n\|.\end{aligned}$$

From $\lim_{n\to\infty} \|T^n y_n - T^{n+1} y_n\| = 0$ and (31), we get

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0. \tag{32}$$

In addition, noticing that

$$\|x_n - Tx_n\| \leq \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \leq \|y_n - Ty_n\| + (2+\theta_1)\|x_n - y_n\|,$$

we deduce from (28) and (32) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{33}$$

Step 5. We claim that $W : C \to C$ is nonexpansive, $\mathrm{Fix}(W) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$ and $\lim_{n\to\infty} \|Wx_n - x_n\| = 0$ where $Wx := \lim_{n\to\infty} W_n x$ for all $x \in C$. Indeed, we observe that for all $x, y \in C$, $\lim_{n\to\infty} \|W_n x - Wx\| = 0$ and $\lim_{n\to\infty} \|W_n y - Wy\| = 0$. Since each W_n enjoys the nonexpansivity, we get

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|.$$

This means that W is nonexpansive. Also, noticing the boundedness of $\{x_n\}$ and putting $D := \{x_n : n \geq 0\}$, we obtain from Lemma 11 that $\lim_{n\to\infty} \sup_{x\in D} \|W_n x - Wx\| = 0$, which immediately sends to

$$\lim_{n\to\infty} \|W_n x_n - Wx_n\| = 0. \tag{34}$$

Thus, combining (30) with (34) we have

$$\|x_n - Wx_n\| \leq \|x_n - W_n x_n\| + \|W_n x_n - Wx_n\| \to 0 \quad (n \to \infty). \tag{35}$$

In addition, utilizing Lemma 10 we get

$$\mathrm{Fix}(W) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n). \tag{36}$$

Step 6. We prove that

$$\limsup_{n\to\infty} \langle A_2 x^*, x^* - w_n \rangle \leq 0 \quad \text{and} \quad \limsup_{n\to\infty} \langle A_1 x^*, x^* - z_n \rangle \leq 0, \tag{37}$$

where $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega, A_1), \mu A_2 - f)$. Indeed, we choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty} \langle x^* - w_n, A_2 x^* \rangle = \lim_{i\to\infty} \langle x^* - w_{n_i}, A_2 x^* \rangle.$$

Utilizing the boundedness of $\{w_n\} \subset C$, we suppose that $w_{n_i} \rightharpoonup \bar{x} \in C$. Since $\lim_{n\to\infty} \|x_n - T^n y_n\| = 0$ (due to (31)) and $\lim_{n\to\infty} \alpha_n = 0$, it follows that

$$\begin{aligned}\|x_n - w_n\| &\leq \|x_n - T^n y_n\| + \|T^n y_n - \alpha_n f(x_n) - (I - \alpha_n \mu A_2) T^n y_n\| \\ &\leq \|T^n y_n x_n\| + \alpha_n (\|\mu A_2 (T^n y_n)\| + \|f(x_n)\|) \to 0 \quad (n \to \infty).\end{aligned} \tag{38}$$

Hence, from $w_{n_i} \rightharpoonup \bar{x}$, we get $x_{n_i} \rightharpoonup \bar{x}$.

Note that G and W are nonexpansive and T is asymptotical. Since $(I-G)x_n \to 0$, $(I-T)x_n \to 0$ and $(I-W)x_n \to 0$ (due to (28), (33) and (35)), by Lemma 7 we get $\bar{x} \in \mathrm{Fix}(G) = \mathrm{GSVI}(C, B_1, B_2)$, $\bar{x} \in \mathrm{Fix}(T)$ and $\bar{x} \in \mathrm{Fix}(W) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$. So,

$$\bar{x} \in \Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T).$$

We show $\bar{x} \in \mathrm{VI}(\Omega, A_1)$. Actually, let $y \in \Omega$ be fixed arbitrarily. From (4), (6) and ζ-inverse strong monotonicity of A_1, we get

$$\|y_n - y\|^2 \le \|(z_n - y) - \delta_n A_1 z_n\|^2 \le \|x_n - y\|^2 + 2\delta_n \langle y - z_n, A_1 y\rangle + \delta_n^2 \|A_1 z_n\|^2,$$

which implies that, for all $n \ge 0$,

$$\begin{aligned} 0 &\le \tfrac{1}{\delta_n}(\|x_n - y\|^2 - \|y_n - y\|^2) + 2\langle A_1 y, y - z_n\rangle + \delta_n \|A_1 z_n\|^2 \\ &\le (\|x_n - y\| + \|y_n - y\|)\tfrac{\|x_n - y_n\|}{\delta_n} + 2\langle A_1 y, y - z_n\rangle + \delta_n \|A_1 z_n\|^2. \end{aligned}$$

From (28) it is easy to see $x_{n_i} \rightharpoonup \bar{x}$ leads to $z_{n_i} \rightharpoonup \bar{x}$. Since $\lim_{n\to\infty} \delta_n = 0$ and $\|x_n - y_n\| = o(\delta_n)$, we have

$$\begin{aligned} 0 &\le \liminf_{n\to\infty} \{(\|x_n - y\| + \|y_n - y\|)\tfrac{\|x_n - y_n\|}{\delta_n} + 2\langle A_1 y, y - z_n\rangle + \delta_n \|A_1 z_n\|^2\} \\ &= \liminf_{n\to\infty} 2\langle y - z_n, A_1 y\rangle \le \lim_{i\to\infty} 2\langle y - z_{n_i}, A_1 y\rangle = 2\langle y - \bar{x}, A_1 y\rangle. \end{aligned}$$

It follows that $\langle A_1 y, y - \bar{x}\rangle \ge 0$, $\forall y \in \Omega$. So, Lemma 12 and the ζ-inverse-strong monotonicity of A_1 ensure that $\langle y - \bar{x}, A_1 \bar{x}\rangle \ge 0$, $\quad \forall y \in \Omega$, that is, $\bar{x} \in \mathrm{VI}(\Omega, A_1)$. Consequently, from $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega, A_1), \mu A_2 - f)$, we have

$$\limsup_{n\to\infty} \langle x^* - w_n, (\mu A_2 - f)x^*\rangle = \lim_{i\to\infty} \langle x^* - w_{n_i}, (\mu A_2 - f)x^*\rangle = \langle x^* - \bar{x}, (\mu A_2 - f)x^*\rangle \le 0.$$

Also, we pick a subsequence $\{z_{n_k}\} \subset \{z_n\}$ such that

$$\limsup_{n\to\infty} \langle x^* - z_n, A_1 x^*\rangle = \lim_{k\to\infty} \langle x^* - z_{n_k}, A_1 x^*\rangle.$$

Since vector sequence $\{z_n\}$ is bounded in C, we suppose that $z_{n_k} \rightharpoonup \hat{x} \in C$. From (28) it is clear that $z_{n_k} \rightharpoonup \hat{x}$ yields $x_{n_k} \rightharpoonup \hat{x}$. By the same arguments as in the proof of $\bar{x} \in \Omega$, we have $\hat{x} \in \Omega$. From $x^* \in \mathrm{VI}(\Omega, A_1)$, we get

$$\limsup_{n\to\infty} \langle x^* - z_n, A_1 x^*\rangle = \lim_{k\to\infty} \langle x^* - z_{n_k}, A_1 x^*\rangle = \langle x^* - \hat{x}, A_1 x^*\rangle \le 0.$$

Therefore, the inequalities in (37) hold.

Step 7. We propose $x_n \to x^*$ as $n \to \infty$. Indeed, putting $p = x^*$ in (6) and (16) we obtain that $\|z_n - x^*\| \le \|x_n - x^*\|$ and

$$\|w_n - x^*\|^2 \le \alpha_n \delta \|x_n - x^*\|^2 + (1 - \alpha_n \tau)\|y_n - x^*\|^2 + \theta_n(2 + \theta_n)\|y_n - x^*\|^2 + 2\alpha_n \langle (\mu A_2 - f)x^*, x^* - w_n\rangle. \quad (39)$$

From (4) and the ζ-inverse-strong monotonicity of A_1 it follows that

$$\|y_n - x^*\|^2 \le \|(z_n - x^*) - \delta_n A_1 z_n\|^2 \le \|x_n - x^*\|^2 + 2\delta_n \langle A_1 x^*, x^* - z_n\rangle + \delta_n^2 \|A_1 z_n\|^2. \quad (40)$$

Thus, in terms of (4), (39) and (40), we get

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \le (1-\beta_n)\|w_n-x^*\|^2+\beta_n\|x_n-x^*\|^2\\
&\le (1-\beta_n)[\alpha_n\delta\|x_n-x^*\|^2+(1-\alpha_n\tau)\|y_n-x^*\|^2+\theta_n(2+\theta_n)\|y_n-x^*\|^2+\beta_n\|x_n-x^*\|^2\\
&\quad +2\alpha_n\langle(\mu A_2-f)x^*,x^*-w_n\rangle]\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2+(1-\alpha_n\tau)[2\delta_n\langle A_1x^*,x^*-z_n\rangle+\|x_n-x^*\|^2\\
&\quad +\delta_n^2\|A_1z_n\|^2]+\theta_n(2+\theta_n)\|y_n-x^*\|^2+2\alpha_n\langle(\mu A_2-f)x^*,x^*-w_n\rangle\}\\
&\le [1-\alpha_n(\tau-\delta)(1-\beta_n)]\|x_n-x^*\|^2+\alpha_n(\tau-\delta)(1-\beta_n)\{\tfrac{(1-\alpha_n\tau)2\delta_n}{(\tau-\delta)\alpha_n}\langle x^*-z_n,A_1x^*\rangle\\
&\quad +\alpha_n\tfrac{\|A_1z_n\|^2}{\tau-\delta}+\tfrac{(2+\theta_n)\theta_n\|y_n-x^*\|^2}{(\tau-\delta)\alpha_n}+\tfrac{2}{\tau-\delta}\langle(\mu A_2-f)x^*,x^*-w_n\rangle\}.
\end{aligned}\tag{41}$$

Obviously, (37) yields

$$\limsup_{n\to\infty}\frac{(1-\alpha_n\tau)2\delta_n}{(\tau-\delta)\alpha_n}\langle x^*-z_n,A_1x^*\rangle\le 0$$

and

$$\limsup_{n\to\infty}\frac{2}{\tau-\delta}\cdot\langle x^*-w_n,(\mu A_2-f)x^*\rangle\le 0.$$

Actually, from $\limsup_{n\to\infty}\langle A_1x^*,x^*-z_n\rangle\le 0$ it follows that for any given $\varepsilon>0$ there exists an integer $n_0\ge 1$ such that $\langle A_1x^*,x^*-z_n\rangle\le\varepsilon$, $\forall n\ge n_0$. Then from $\delta_n\le\alpha_n$ we get

$$\frac{2\delta_n(1-\alpha_n\tau)}{(\tau-\delta)\alpha_n}\langle A_1x^*,x^*-z_n\rangle\le\frac{2\delta_n(1-\alpha_n\tau)}{(\tau-\delta)\alpha_n}\varepsilon\le\frac{2}{\tau-\delta}\varepsilon,\quad\forall n\ge n_0,$$

which hence yields

$$\limsup_{n\to\infty}\frac{2\delta_n(1-\alpha_n\tau)\langle A_1x^*,x^*-z_n\rangle}{(\tau-\delta)\alpha_n}\le\frac{2}{\tau-\delta}\varepsilon.$$

Letting $\varepsilon\to 0$, we get

$$\limsup_{n\to\infty}\frac{2\delta_n(1-\alpha_n\tau)\langle x^*-z_n,A_1x^*\rangle}{(\tau-\delta)\alpha_n}\le 0.$$

Since $\sum_{n=0}^\infty\alpha_n=\infty$, $\liminf_{n\to\infty}(1-\beta_n)>0$ and $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}=0$, we deduce that

$$\sum_{n=0}^\infty\alpha_n(\tau-\delta)(1-\beta_n)=\infty$$

and

$$\begin{aligned}\limsup_{n\to\infty}\{&\tfrac{(1-\alpha_n\tau)2\delta_n}{(\tau-\delta)\alpha_n}\langle A_1x^*,x^*-z_n\rangle+\alpha_n\tfrac{\|A_1z_n\|^2}{\tau-\delta}\\
&+\tfrac{\theta_n(2+\theta_n)\|y_n-x^*\|^2}{(\tau-\delta)\alpha_n}+\tfrac{2}{\tau-\delta}\langle x^*-w_n,(\mu A_2-f)x^*\rangle\}\le 0.\end{aligned}$$

We can infer Lemma 3 to the relation (41) and conclude that $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

From Theorem 1, we have the following sub-result.

Corollary 1. *Assume that μ_1 is a real number in $(0,2\alpha)$, and μ_2 is a real number in $(0,2\beta)$. Let $\delta<\tau:=1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}\in(0,1]$ for $\mu\in(0,\frac{2\eta}{\kappa^2})$. We suppose $\{\lambda_n\}_{n=0}^\infty$ is a real sequence in $(0,b]$ for some real number b in $(0,1)$. We also suppose that $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset(0,1]$ and $\{\delta_n\}\subset(0,2\zeta]$ such that*

(i) $\sum_{n=0}^\infty\alpha_n=\infty$ *and* $\lim_{n\to\infty}\alpha_n=0$;
(ii) $\delta_n\le\alpha_n\ \forall n\ge 0$ *and* $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}=0$;
(iii) $\liminf_{n\to\infty}\beta_n>0$ *and* $\limsup_{n\to\infty}\beta_n<1$;

(iv) $\liminf_{n\to\infty} \gamma_n > 0$, $\limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;
(v) $\lim_{n\to\infty} \|T^{n+1}y_n - T^n y_n\| = 0$.

Let $\{x_n\}_{n=0}^{\infty}$ *be a sequence defined by*

$$\begin{cases} y_n = (1-\gamma_n)W_n y_n + \gamma_n x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)P_C[\alpha_n f(x_n) + (I - \alpha_n \mu A_2)T^n y_n]. \end{cases}$$

Then we have

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*
(b) $\lim_{n\to\infty} \|x_n - y_n\| = 0$, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$;
(c) *if* $\lim_{n\to} \frac{\|x_n - y_n\|}{\delta_n} = 0$, *then* $\{x_n\}$ *converges to a common fixed point of the asymptotically nonexpansive and nonexpansive mappings.*

Theorem 2. *Assume that* μ_1 *is a real number in* $(0, 2\alpha)$, *and* μ_2 *is a real number in* $(0, 2\beta)$. *Let* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$ *for* μ *in* $(0, \frac{2\eta}{\kappa^2})$, *and let* $\{\lambda_n\}_{n=0}^{\infty}$ *be a real sequence in* $(0, b]$ *for some* b *in* $(0, 1)$. *Suppose that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ *and* $\{\delta_n\} \subset (0, 2\zeta]$ *such that*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\delta_n \le \alpha_n \ \forall n \ge 0$ *and* $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$ *and* $\limsup_{n\to\infty} \beta_n < 1$;
(iv) $\liminf_{n\to\infty} \gamma_n > 0$, $\limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;
(v) $\lim_{n\to\infty} \|T^{n+1}y_n - T^n y_n\| = 0$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ *generated by Algorithm 3 satisfies the following properties:*

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*
(b) $\lim_{n\to\infty} \|x_n - y_n\| = 0$, $\lim_{n\to\infty} \|x_n - Gx_n\| = 0$, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$;
(c) *If* $\frac{\|x_n - y_n\|}{\delta_n} = 0$, $x_n \to x^* \in \mathrm{VI}(\Omega, A_1)$.

Proof. Since $A_2 : C \to H$ is κ-Lipschitzian and η-strongly monotone, by Lemma 12 we know that the Problem 2 has the unique solution. We let $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega, A_1), A_2)$. For each $n \ge 0$, we consider the mapping $F_n x := G(\gamma_n x_n + (1-\gamma_n)W_n x)$, $\forall x \in C$. Utilizing the same argument as in the proof of Theorem 1, we can deduce from Banach's contraction principle that for each $n \ge 0$ there exists a unique element $z_n \in C$ such that $z_n = G(\gamma_n x_n + (1-\gamma_n)W_n z_n)$. Thus, the iterative scheme in Algorithm 3 can be rewritten as

$$\begin{cases} u_n = \gamma_n x_n + (1-\gamma_n)W_n z_n, \\ z_n = Gu_n, \\ y_n = P_C(z_n - \delta_n A_1 z_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)P_C(I - \mu\alpha_n A_2)T^n y_n. \end{cases}$$

Here, we divide the rest of the proof into several steps.

Step 1. We prove $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{v_n\}, \{T^n y_n\}$ and $\{A_2(T^n y_n)\}$ are bounded vector sequences, where $v_n = P_C(u_n - \mu_2 B_2 u_n)$ and $z_n = P_C(v_n - \mu_1 B_1 v_n)$ for all $n \ge 0$. Indeed, utilizing the similar argument to that of Step 1 in the proof of Theorem 1, we obtain the desired assertion.

Step 2. We prove $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$. Indeed, utilizing the similar argument to that of Step 2 in the proof of Theorem 1, we obtain the desired assertion.

Step 3. We prove $\|x_n - Gx_n\| \to 0$ as $n \to \infty$. Indeed, utilizing the similar argument to that of Step 3 in the proof of Theorem 1, we obtain the desired assertion.

Step 4. We prove $\|Tx_n - x_n\| \to 0$ and $\|W_n x_n - x_n\| \to 0$ as $n \to \infty$. Indeed, utilizing the similar argument to that of Step 4 in the proof of Theorem 1, we obtain the desired assertion.

Step 5. We prove $W : C \to C$ enjoys the nonexpansivity, $\mathrm{Fix}(W) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$ and $\lim_{n\to\infty} \|Wx_n - x_n\| = 0$ where $Wx := \lim_{n\to\infty} W_n x$ for all $x \in C$. Indeed, utilizing the similar argument to that of Step 5 in the proof of Theorem 1, we obtain the desired assertion.

Step 6. We prove $\limsup_{n\to\infty}\langle A_2x^*, x^* - w_n\rangle \le 0$ and $\limsup_{n\to\infty}\langle A_1x^*, x^* - z_n\rangle \le 0$, where $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega, A_1), A_2)$. Indeed, utilizing the similar argument to that of Step 6 in the proof of Theorem 1, we obtain the desired assertion.

Step 7. We prove $x_n \to x^*$ as $n \to \infty$. Indeed, utilizing the similar argument to that of Step 7 in the proof of Theorem 1, we obtain the desired assertion.

This completes the entire proof. □

Corollary 2. *Assume that μ_1 is a real number in $(0, 2\alpha)$, and μ_2 is a real number in $(0, 2\beta)$. Let $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$ for μ in $(0, \frac{2\eta}{\kappa^2})$, and let $\{\lambda_n\}_{n=0}^{\infty}$ be a real sequence in $(0, b]$ for some b in $(0, 1)$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ and $\{\delta_n\} \subset (0, 2\zeta]$ such that*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\delta_n \le \alpha_n\ \forall n \ge 0$ *and* $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$ *and* $\limsup_{n\to\infty} \beta_n < 1$;
(iv) $\liminf_{n\to\infty} \gamma_n > 0$, $\limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;
(v) $\lim_{n\to\infty} \|T^{n+1}y_n - T^n y_n\| = 0$.

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by

$$\begin{cases} u_n = (1 - \gamma_n)W_n u_n + \gamma_n x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)P_C(I - \alpha_n\mu A_2)T^n u_n. \end{cases}$$

Then we have

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*
(b) $\lim_{n\to\infty} \|x_n - u_n\| = 0$, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$;
(c) *If* $\frac{\|x_n - u_n\|}{\delta_n} = 0$, $\{x_n\}$

converges to a common fixed point of the asymptotically nonexpansive and nonexpansive mappings.

4. Concluding Remark

This paper discussed a monotone variational inequality problem with a variational inequality constraint over the common solution set of a general system of variational inequalities and a common fixed point of a countable family of nonexpansive mappings and an asymptotically nonexpansive mapping in Hilbert spaces, which is called the triple hierarchical constrained variational inequality, and introduced some Mann-type implicit iteration methods for solving it. Norm convergence of the proposed methods of the iteration methods is guaranteed under some suitable assumptions.

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