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Review

Some Schemata for Applications of the Integral Transforms of Mathematical Physics

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Abstract: In this survey article, some schemata for applications of the integral transforms of mathematical physics are presented. First, integral transforms of mathematical physics are defined by using the notions of the inverse transforms and generating operators. The convolutions and generating operators of the integral transforms of mathematical physics are closely connected with the integral, differential, and integro-differential equations that can be solved by means of the corresponding integral transforms. Another important technique for applications of the integral transforms is the Mikusinski-type operational calculi that are also discussed in the article. The general schemata for applications of the integral transforms of mathematical physics are illustrated on an example of the Laplace integral transform. Finally, the Mellin integral transform and its basic properties and applications are briefly discussed.

Keywords: integral transforms; Laplace integral transform; transmutation operator; generating operator; integral equations; differential equations; operational calculus of Mikusinski type; Mellin integral transform

MSC: 45-02; 33C60; 44A10; 44A15; 44A20; 44A45; 45A05; 45E10; 45J05

1. Introduction

In this survey article, we discuss some schemata for applications of the integral transforms of mathematical physics to differential, integral, and integro-differential equations, and in the theory of special functions. The literature devoted to this subject is huge and includes many books and reams of papers. For more details regarding this topic we refer the readers to, say, [1–4]. Of course, in a short survey article it is not possible to mention all known integral transforms and their numerous applications. That is why we focus on just some selected integral transforms and their applications that are of general nature and valid for most of the integral transforms in one or another form.

We start with introducing the integral transforms of mathematical physics that possess the inverses in form of the linear integral transforms and can be interpreted as transmutation operators for their generating operators. The integral transforms of mathematical physics, their generating operators, and convolutions are closely related to each other. In particular, the integral transform technique can be employed for derivation of the closed form solutions to some integral equations of convolution type and to the integral, differential, or integro-differential equations with the generating operators.

Another powerful technique for applications of the integral transforms is the Mikusinski-type operational calculi. They can be developed for the left-inverse operators of the generating operators of the integral transforms. A basic element of this construction is the convolutions for the corresponding integral transforms that play the role of multiplication in some rings of functions. This ring is then extended to a field of convolution quotients following the standard procedure. One of the advantages of this extension is that the left-inverse operator $\mathcal{D}$ to the generating operator $\mathcal{L}$ of the given integral transform $\mathcal{T}$ can be then represented as multiplication with a certain field element. Thus, the differential or integro-differential equations with the operator $\mathcal{D}$ are reduced to some algebraic equations in the field of convolution quotients and can be solved in explicit form. The so obtained "generalized" solution can be sometimes represented as a conventional function from the initial ring of functions by using the so-called operational relations.

The general schemata for applications of the integral transforms of mathematical physics mentioned above are demonstrated on an example of the Laplace integral transform. The Laplace integral transform is a simple particular case of the general H-transform that is a Mellin convolution type integral transform with the Fox H-function in the kernel. The general schemata for applications of the integral transforms presented in this article are valid for the H-transform, too. For the theory of the generating operators, convolutions, and operational calculi of the Mikusinski type for the H-transforms we refer the interested readers to [4–8] (see also numerous references therein). In this article, we restrict ourselves to discussion of some fundamental properties of the Mellin integral transform that is a basis for the theory of the Mellin convolution type integral transforms in general and of the H-transform in particular.

The rest of the article is organized as follows: In the second section, general schemata for some applications of the integral transforms to analysis of the integral, differential, and integro-differential equations are presented. In particular, the main ideas behind an operational calculus of Mikusinski type are discussed. The third section illustrates these schemata on the example of the Laplace integral transform. The fourth section deals with the basic properties of the Mellin integral transform.

2. Integral Transforms of Mathematical Physics

The focus of this survey article is on properties of the integral transforms and their applications to different problems of analysis, differential and integral equations, and special functions. Thus, we do not discuss the integral transforms from the viewpoint of functional analysis by considering, say, their mapping properties in some spaces of functions. Instead, we try to illustrate the underlying ideas and procedures both for analysis of the integral transforms and for their applications.

2.1. Applications of the Integral Transforms

The integral transforms of mathematical physics are not arbitrary linear integral operators, but rather those with the known inverse operators and the known generating operators. For the sake of simplicity and clarity, in this article we restrict ourselves to the case of the one-dimensional integral transforms. However, a similar theory can be also developed for the multi-dimensional integral transforms. A one-dimensional integral transform (of mathematical physics) of a function $f : \mathbb{R} \to \mathbb{R}$ at the point $t \in \mathbb{R}$ is defined by the (convergent) integral

$$g(t) = \mathcal{T}\,\{f(x); t\} = \int_{-\infty}^{+\infty} K(t,x)\, f(x)\, dx\,. \tag{1}$$

Its inverse operator must be also a linear integral transform

$$f(x) = \mathcal{T}^{-1}\,\{g(t); x\} = \int_{-\infty}^{+\infty} \hat{K}(x,t)\, g(t)\, dt \tag{2}$$

with a known kernel function $\hat{K}$. The kernel functions K and $\hat{K}$ of the integral transforms (1) and (2) satisfy the relation

$$\int_{-\infty}^{+\infty} \hat{K}(x,t)K(t,y)\, dt \;=\; \delta(x-y) \tag{3}$$

with the Dirac δ-function.

Many applications of the integral transforms of mathematical physics are based on the operational relations of the following form:

$$\mathcal{T}\{(\mathcal{L}f)(x);t\} = L(t)\mathcal{T}\{f(x);t\}. \tag{4}$$

The integral transform T satisfying the relation (4) is called a transmutation that translates an operator $\mathcal{L}$ into multiplication by the function L. Following [4,8,9], we call the operator $\mathcal{L}$ the generating operator of the integral transform $\mathcal{T}$. For the general H-transform, one of the important classes of their generating operators can be represented in form of finite compositions of the fractional Erdelyi-Kober integrals and derivatives [4–8]. In the case of the Laplace integral transform, the generating operator is just the first derivative.

Let us now discuss a general schema for applications of the transmutation Formula (4) on an example of the equation

$$P(\mathcal{L})y(x) = f(x), \tag{5}$$

where P is a polynomial and f is a given function. Applying the integral transform (1) to Equation (5) and employing the transmutation Formula (4) lead to the algebraic (in fact, linear) equation

$$P(L(t))\mathcal{T}\{y(x);t\} = \mathcal{T}\{f(x);t\} \tag{6}$$

for the integral transform $\mathcal{T}$ of the unknown function y with a solution in form

$$\mathcal{T}\{y(x);t\} = \frac{\mathcal{T}\{f(x);t\}}{P(L(t))}. \tag{7}$$

In the system theory, the function $1/P(L(t))$ is often called the transfer function. The inversion Formula (2) allows then to represent the solution (7) as follows:

$$y(x) = \mathcal{T}^{-1}\left\{\frac{\mathcal{T}\{f(x);t\}}{P(L(t))};x\right\}. \tag{8}$$

In many applications of the integral transforms of mathematical physics, one deals with the linear differential operators of the form

$$\mathcal{L}(x,\frac{d}{dx})y = \sum_{k=0}^{n} l_k(x)\frac{d^k y}{dx^k}. \tag{9}$$

Let us suppose that $\mathcal{L}$ is a generating operator of the integral transform (1) with the inverse integral transform (2) such that the relation (4) holds true. By

$$\mathcal{L}^T(x,\frac{d}{dx})y = \sum_{k=0}^{n}(-1)^k\frac{d^k}{dx^k}(l_k(x)y) \tag{10}$$

we denote the operator conjugate to the operator $\mathcal{L}$.

Then it is known that the kernel K of the integral transform (1) is an eigenfunction of the operator $\mathcal{L}^T$ and the kernel $\hat{K}$ of the inverse integral transform (2) is an eigenfunction of the operator $\mathcal{L}$ [9]:

$$\mathcal{L}^T(x,\frac{d}{dx})K(t,x) = L(t)K(t,x),$$

$$\mathcal{L}(x,\frac{d}{dx})\hat{K}(x,t) = L(t)\hat{K}(x,t).$$

Let us note that the Formulas (1) and (2) for the integral transform $\mathcal{T}$ of a function f and its inverse integral transform can be put together into the form

$$f(x) = \int_{-\infty}^{+\infty} \hat{K}(x,t)\, \mathcal{T}\,\{f(x);t\}\, dt$$

and then interpreted as an expansion of the function f by the eigenfunctions of the linear differential operator $\mathcal{L}$.

Thus, the integral transforms of mathematical physics are closely connected with the eigenvalues of some differential operators. However, the eigenvalues of the differential operators are known in explicit form only in a few cases and therefore the amount of the integral transforms of mathematical physics is very restricted.

As a rule, the generating operators of the integral transforms of mathematical physics are differential operators either of the first or of the second order. Examples of the integral transforms with the generating operators in form of the differential operators of the first order are:

(a) the Laplace integral transform with the kernel function $K(t,x) = e^{-xt}$ if $x > 0$ and $K(t,x) = 0$ if $x \leq 0$,
(b) the sine- and cosine Fourier integral transforms with the kernel functions $K(t,x) = \sqrt{2/\pi}\sin(xt)$ if $x > 0$ and $K(t,x) = 0$ if $x \leq 0$ and $K(t,x) = \sqrt{2/\pi}\cos(xt)$ if $x > 0$ and $K(t,x) = 0$ if $x \leq 0$, respectively,
(c) the Fourier integral transform with the kernel function $K(t,x) = e^{-ixt}$,
(d) the Mellin integral transform with the kernel function $K(t,x) = x^{t-1}$ if $x > 0$ and $K(t,x) = 0$ if $x \leq 0$.

Following integral transforms possess generating operators in form of the differential operators of the second order:

(a) the Hankel integral transform with the kernel function $K(t,x) = \sqrt{xt}J_\nu(xt)$ (J_ν stands for the Bessel function) if $x > 0$ and $K(t,x) = 0$ if $x \leq 0$,
(b) the Meijer integral transform with the kernel function $K(t,x) = \sqrt{xt}K_\nu(xt)$ (K_ν is the Macdonald function) if $x > 0$ and $K(t,x) = 0$ if $x \leq 0$,
(c) the Kontorovich- Lebedev integral transform with the kernel function $K(t,x) = K_{it}(x)$ if $x > 0$ and $K(t,x) = 0$ if $x \leq 0$,
(d) the Mehler-Fock integral transform with the kernel function $K(t,x) = P^k_{it-1/2}(x)$ (P^μ_ν denotes the Legendre function of the first kind) if $x > 1$ and $K(t,x) = 0$ if $x \leq 1$.

As already mentioned, the generating operators of the H-transform are certain compositions of the Erdelyi-Kober fractional integrals and derivatives. This connection allows to solve equations of type (5) with the operator $\mathcal{L}$ in form of a composition of the Erdelyi-Kober fractional integro-differential operators [4–8].

Another operation that plays a very important role in applications of the integral transforms of mathematical physics is a convolution of two functions associated with a certain integral transform.

In general, a convolution on a linear vector space of functions is defined as a bilinear, commutative, and associative operation defined on a direct product of a linear vector space by itself. Together with the usual addition of two elements of the vector space, the convolution thus equips the linear vector space with a structure of a commutative ring.

A convolution $\overset{\mathcal{T}}{*}$ associated with the integral transform $\mathcal{T}$ and defined on a linear functional vector space $\mathcal{X}$ satisfies the relation (convolution theorem)

$$\mathcal{T}\left\{(f \overset{\mathcal{T}}{*} g)(x);t\right\} = \mathcal{T}\,\{f(x);t\}\,\mathcal{T}\,\{g(x);t\}\,,\ \ \forall f,g \in \mathcal{X}. \tag{11}$$

The reader can find many examples of convolutions for different integral transforms of mathematical physics in [4,5].

One of the basic applications of the convolutions is for analysis of the integral equations of convolution type. The convolutions of the integral transforms of mathematical physics are often represented in form of some integrals. In these cases the convolution equations like, e.g.,

$$y(x) - \lambda(y \overset{\mathcal{T}}{*} K)(x) \;=\; f(x),\; \lambda \in \mathbb{R} \text{ or } \lambda \in \mathbb{C}, \tag{12}$$

where f and K are some known functions and the function y is unknown, are integral equations.

To solve the integral Equation (12), we apply the integral transform $\mathcal{T}$ to both parts of (12). Then we first get an algebraic (in fact, a linear) equation

$$\mathcal{T}\,\{y(x);t\} - \lambda\mathcal{T}\,\{y(x);t\}\,\mathcal{T}\,\{K(x);t\} \;=\; \mathcal{T}\,\{f(x);t\} \tag{13}$$

with the solution

$$\mathcal{T}\,\{y(x);t\} \;=\; \frac{\mathcal{T}\,\{f(x);t\}}{1-\lambda\mathcal{T}\,\{K(x);t\}}. \tag{14}$$

Applying the inverse transform $\mathcal{T}^{-1}$ to (14), the solution of the integral Equation (12) can be then represented in the form

$$y(x) \;=\; \mathcal{T}^{-1}\left\{\frac{\mathcal{T}\,\{f(x);t\}}{1-\lambda\mathcal{T}\,\{K(x);t\}};x\right\}. \tag{15}$$

In many cases, the right-hand side of (15) has a convolution form and thus the solution to (12) can be rewritten as follows:

$$y(x) \;=\; f(x) + \lambda(f \overset{\mathcal{T}}{*} M)(x), \tag{16}$$

where M is a known function.

2.2. Basic Ideas Behind an Operational Calculus of Mikusinski Type

Another useful technique employed for solution of both integral equations of convolution type (12) and differential or integro-differential equations of type (5) is an algebraic approach based on the operational calculi of Mikusinski type [4–6,10–14].

In an operational calculus of Mikusinski type, a close relation between an integral transform, its convolution and its generating operator plays a very essential role as investigated in detail in [15].

Following [15], we first introduce a convolution of a linear operator $\mathcal{L}$. Let $\mathcal{X}$ be a linear vector space and $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ a linear operator defined on the elements of $\mathcal{X}$. A bilinear, commutative, and associative operation $* : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is said to be a convolution of the linear operator $\mathcal{L}$ if and only if the relation

$$\mathcal{L}(f * g) = (\mathcal{L}f) * g \tag{17}$$

holds true for all $f, g \in \mathcal{X}$.

As shown in [4], if $\mathcal{L}$ is a generating operator of the integral transform $\mathcal{T}$ (the Formula (4) holds true) and if $\overset{\mathcal{T}}{*}$ is a convolution of $\mathcal{T}$ that satisfies the relation (11), then $\overset{\mathcal{T}}{*}$ is a convolution of the generating operator $\mathcal{L}$ in the sense of the relation (17).

Another important fact is that any of the convolution operators of the type

$$(\mathcal{L}f)(x) = (h \overset{\mathcal{T}}{*} f)(x), \tag{18}$$

where $\overset{\mathcal{T}}{*}$ is a convolution of the integral transform $\mathcal{T}$ and h is a fixed element of $\mathcal{X}$ can be interpreted as a generating operator of the integral transform $\mathcal{T}$, i.e., it satisfies the transmutation relation (4).

The generating operator $\mathcal{L}$ given by (18) is an integral operator that is defined on the functions from the convolution ring $\mathcal{R} = (\mathcal{X}, \overset{\mathcal{T}}{*}, +)$ as multiplication by a fixed element $h \in \mathcal{X}$. It is important to stress that the representations of this type are not possible for the generating operators of the differential type, e.g., for the left-inverse operators of the integral operator (18). However, similar representations of differential operators can be derived on an extension of the convolution ring $\mathcal{R} = (\mathcal{X}, \overset{\mathcal{T}}{*}, +)$ to the field of the convolution quotients. In fact, this extension is a basic element of any operational calculus of Mikusinski type. In the case when the ring $\mathcal{R} = (\mathcal{X}, \overset{\mathcal{T}}{*}, +)$ has no divisors of zero, the extension follows the pattern of the extension of the ring of integer numbers to the field of rational numbers.

If the ring $\mathcal{R} = (\mathcal{X}, \overset{\mathcal{T}}{*}, +)$ has some divisors of zero, the construction of the field of convolution quotients becomes more complicated (see, e.g., [15] for details). A divisor-free convolution ring is usually extended to a field $\mathcal{F}$ of convolution quotients by factorization of the set $\mathcal{X} \times (\mathcal{X} - \{0\})$ with respect to the equivalence relation

$$(f, g) \sim (f_1, g_1) \Leftrightarrow (f \overset{\mathcal{T}}{*} g_1)(x) = (g \overset{\mathcal{T}}{*} f_1)(x). \tag{19}$$

The elements of the field $\mathcal{F}$ are sets of all pairs (f, g), $f, g \in \mathcal{X}$ that are equivalent to each other with respect to the equivalence relation (19). They are often formally denoted as quotients f/g. The addition $+$ and multiplication $\cdot$ operations are defined on $\mathcal{F}$ in a standard way:

$$f/g + f_1/g_1 = (f \overset{\mathcal{T}}{*} g_1 + g \overset{\mathcal{T}}{*} f_1)/(g \overset{\mathcal{T}}{*} g_1), \tag{20}$$

$$f/g \cdot f_1/g_1 = (f \overset{\mathcal{T}}{*} f_1)/(g \overset{\mathcal{T}}{*} g_1). \tag{21}$$

It is an easy exercise in algebra to show that the results of the operations (20) and (21) do not depend on the representatives of the field elements f/g and f_1/g_1 and thus these operations are well defined. Equipped with the operations $+$ and $\cdot$, the set $\mathcal{F}$ becomes a commutative field that is denoted by $(\mathcal{F}, \cdot, +)$.

The ring $\mathcal{R} = (\mathcal{X}, \overset{\mathcal{T}}{*}, +)$ can be embedded into the field $(\mathcal{F}, \cdot, +)$, say, by the map

$$f \in \mathcal{R} \to (f \overset{\mathcal{T}}{*} h)/h \in \mathcal{F}, \tag{22}$$

where $h \in \mathcal{R}$ is any non-zero element of the ring $\mathcal{R}$. A natural choice for the element $h \in \mathcal{R}$ in the relation (22) is the function from the Formula (18) that defines the generating operator $\mathcal{L}$. In this case, the corresponding operational calculus is constructed for the differential operator $\mathcal{D}$ that is a left-inverse operator to the integral operator (18), i.e., for the operator $\mathcal{D}$ that satisfies the relation

$$\mathcal{D}(\mathcal{L}f) \;=\; f, \; \forall f \in \mathcal{X}. \tag{23}$$

On the ring $\mathcal{R} = (\mathcal{X}, \overset{\mathcal{T}}{*}, +)$, the operator $\mathcal{L}$ applied to a function $f \in \mathcal{X}$ is just multiplication of f with a fixed element $h \in \mathcal{X}$. On the other hand, the differential operator $\mathcal{D}$ (a left-inverse operator to the integral operator $\mathcal{L}$) can be represented on the field $(\mathcal{F}, \cdot, +)$ in the form

$$\mathcal{D}f \;=\; S \cdot f - S \cdot \mathcal{P}f, \tag{24}$$

where the operator $\mathcal{P} = \mathrm{Id} - \mathcal{L}\mathcal{D}$ is called a projector of the generating operator $\mathcal{L}$ and $S \in \mathcal{F}$ is the element reciprocal to $h \in \mathcal{R} \subset \mathcal{F}$ defined by the Formula (18), i.e.,

$$S = h^{-1} = I/h = h/(h \overset{\mathcal{T}}{*} h) = h/h^2 = \cdots = h^k/h^{k+1}, \; k = 0, 1, 2, \ldots. \tag{25}$$

The element $S \in \mathcal{F}$ is often called an algebraic inverse of the generating operator $\mathcal{L}$ in the field of convolution quotients.

The operational Formula (24) is very important in applications of the constructed operational calculus because it allows a reduction of the linear differential equations with the operator $\mathcal{D}$ to some algebraic equations in the field $(\mathcal{F}, \cdot, +)$ of convolution quotients. The obtained equations can be then often solved in explicit form that leads to the "generalized" solutions that belong to the convolution quotients field $(\mathcal{F}, \cdot, +)$. In some cases, by making use of the embedding (22) and of the so-called operational relations, these generalized solutions can be reduced to the conventional functions from the initial ring $\mathcal{R}$. In particular, the following operational relation plays a very important role in any operational calculus:

$$\begin{aligned}&(S-\rho)^{-1} = I/(S-\rho) = h/(I-\rho h) = h \cdot (I + \rho h + \rho^2 h^2 + \ldots) \\ &= h(x) + \rho h^2(x) + \rho^2 h^3(x) + \cdots = H(x) \in \mathcal{R},\ \rho \in \mathbb{C},\ h^k = \underbrace{h \overset{\mathcal{T}}{*} h \overset{\mathcal{T}}{*} \cdots \overset{\mathcal{T}}{*} h}_{k}.\end{aligned} \tag{26}$$

The general schema for construction of an operational calculus and for its applications that was presented above seems to be not especially complicated. However, in the case of a given generating operator a lot of serious problems can appear while developing the corresponding operational calculus. The main questions are how to construct an appropriate convolution, how to determine its divisors of zero in the corresponding ring of functions (or show that it is divisors-free), how to calculate the projector operator (the projector operator determines the form of the initial conditions for the ordinary or fractional differential equations that can be solved by employing the operational calculus), how to specify the operational relations such as the one given in the Formula (26), etc. For discussions regarding how to overcome all these difficulties for operational calculi for different operators of Fractional Calculus see, e.g., ref. [4] or [5–8].

3. The Laplace Integral Transform

The Laplace integral transform—along with the Fourier integral transform and the Mellin integral transform—is one of the most important classical integral transforms that is widely used in analysis, differential equations, theory of special functions and integral transforms, and for other problems of mathematical physics. For a function f, its Laplace transform at the point $p \in \mathbb{C}$ is defined by the following improper integral (in the case it is a convergent one):

$$\widetilde{f}(p) = \mathcal{L}\{f(t); p\} = \int_0^\infty \mathrm{e}^{-pt} f(t)\, dt\,, \quad \Re(p) > a_f. \tag{27}$$

A sufficient condition for existence of the Laplace integral at the right-hand side of (27) for a function $f \in L^c(0, +\infty)$ is the estimate of the type

$$|f(t)| \le M_f \mathrm{e}^{a_f t},\ t > T_f, \tag{28}$$

where M_f, a_f, and T_f are some constants depending on the function f. The space of functions $L^c(0, +\infty)$ consists of all real or complex-valued functions of a real variable that are continuous on the open interval $(0, +\infty)$ except, possibly, at a counted number of isolated points, where these functions can tend to infinity and for that the improper Riemann integral absolutely converges on $(0, +\infty)$. In this section, the set of all functions from $L^c(0, +\infty)$ that satisfy the estimate (28) with some constants depending on the functions will be denoted by $\mathcal{O}$. In the following discussions, we always assume that the functions we deal with belong to the space of functions $\mathcal{O}$. For the functions from $\mathcal{O}$, their Laplace transforms $\widetilde{f}(p)$ defined by the right-hand side of (27) are analytic function in the half complex plane $\Re(p) > a_f$. This feature makes the Laplace transform technique very powerful because all methods and ideas elaborated in the well-developed theory of analytical functions can be employed in the Laplace domain.

Let a function f be piecewise differentiable and its Laplace transform exist for $\Re(p) > a_f$. At all points where f is continuous, it can be represented via the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\left\{\widetilde{f}(p); t\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{e}^{pt}\, \widetilde{f}(p)\, dp\,, \quad \Re(p) = \gamma > a_f, \tag{29}$$

where the integral at the right-hand side is understood in the sense of the Cauchy principal value.

Let us mention here that in some cases the bilateral Laplace transform can be useful. It is defined by the formula

$$\mathcal{L}_{bl}\left\{f(t); p\right\} = \int_{-\infty}^{\infty} \mathrm{e}^{-pt}\, f(t)\, dt\,, \quad b_f > \Re(p) > a_f. \tag{30}$$

In our article, we assume that the model equations we deal with refer to the causal processes and thus we restrict ourselves to discussion of the Laplace transform.

The Laplace integral transform is treated in many textbooks (see, e.g., [11,16–21]). Here, we demonstrate on the example of the Laplace integral transform how the general schemata for applications of the integral transforms of mathematical physics work. It is worth mentioning that the same constructions can be applied for the general H-transform and its numerous particular cases ([4–8]).

Let the inclusion $f' \in \mathcal{O}$ be valid. The integration by parts formula applied to the Laplace integral leads to following transmutation relation for the Laplace integral transform:

$$\mathcal{L}\left\{\frac{d}{dt}f(t); p\right\} \;=\; p\,\mathcal{L}\left\{f(t); p\right\} \;-\; f(0). \tag{31}$$

This means that the Laplace integral transform is a transmutation operator for the first derivative that translates it into multiplication with the linear factor p.

The Formula (31) and its generalization (for $f^{(n)} \in \mathcal{O}$)

$$\mathcal{L}\left\{f^{(n)}(t); p\right\} \;=\; p^n\,\mathcal{L}\left\{f(t); p\right\} \;-\; p^{n-1}f(0) \;-\ldots-\; f^{(n-1)}(0) \tag{32}$$

are the basic formulas for application of the Laplace transform technique to solution of the linear differential equations.

As an example, let us consider the following initial value problem:

$$\begin{cases} x^{(n)}(t) + a_1 x^{(n-1)}(t) + \cdots + a_n x(t) \;=\; f(t),\; t > 0, \\ x(0) = x_0,\; \frac{dx}{dt}|_{t=0} = x_1, \ldots, \frac{d^{n-1}x}{dt^{n-1}}|_{t=0} = x_{n-1}. \end{cases} \tag{33}$$

Applying of the Laplace integral transform to Equation (33) to the initial value problem (33) we get an algebraic (in fact, a linear) equation for the Laplace transform of the unknown function x:

$$L(p)\mathcal{L}\left\{x(t); p\right\} \;=\; \mathcal{L}\left\{f(t); p\right\} \;+\; M(p), \tag{34}$$

where $L(p) = p^n + a_1 p^{n-1} + \cdots + a_n$ and $M(p) = p^{n-1}x_0 + \cdots + x_{n-1} + a_1(p^{n-2}x_0 + \cdots + x_{n-2}) + \cdots + a_{n-1}x_0$. The polynomial L is known as a characteristic polynomial of Equation (33).

For applicability of this technique, the function f from the left-hand side of Equation (33) must satisfy the condition (28).

The linear Equation (34) can be easily solved:

$$\mathcal{L}\left\{x(t); p\right\} \;=\; \frac{\mathcal{L}\left\{f(t); p\right\} \;+\; M(p)}{L(p)}. \tag{35}$$

Thus, the unique solution to (33) can be (formally) obtained by applying the inverse Laplace transform to the right-hand side of (35):

$$x(t) \ = \ \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\left\{f(t);p\right\} \ + \ M(p)}{L(p)};t\right\}. \tag{36}$$

Often, the inverse Laplace transform at the right-hand side of the Formula (36) can be evaluated by means of the Cauchy residue theorem or by employing the tables of the Laplace integral transforms [19,20].

The same procedure as above is applicable for the systems of linear ordinary differential equations with the constant coefficients. A similar method can be applied for the linear ordinary differential equations with the polynomial coefficients ([11]). In the case of the time-dependent partial linear differential equations with the constant coefficients, application of the Laplace integral transform with respect to the time variable transforms them to the stationary partial differential equations of elliptic type with some parameters dependent on the Laplace variable p.

A prominent role in several applications of the Laplace integral transform is played by its convolution that is defined by the well-known formula

$$(f \overset{\mathcal{L}}{*} g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{37}$$

The Borel convolution theorem states the main property of the Laplace convolution (37). Let the Laplace integral transforms of the functions $f,\ g \in \mathcal{O}$ be well defined for $\Re\,(p) > \gamma$. Then the Laplace convolution (37) also exists for $\Re\,(p) > \gamma$ and the convolution formula

$$\mathcal{L}\left\{(f \overset{\mathcal{L}}{*} g)(t);p\right\} = \mathcal{L}\left\{f(t);p\right\} \times \mathcal{L}\left\{g(t);p\right\} \tag{38}$$

holds true.

In [4,5], a close relation of the Laplace convolution to the Euler Beta-function was established. It turned out that the Formula (37) for the Laplace convolution follows from the well-known formula

$$B(s,t) \ = \ \int_0^1 x^{s-1}(1-x)^{t-1}\,dx \ = \ \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}, \tag{39}$$

Γ being the Euler Gamma-function defined as

$$\Gamma(s) \ = \ \int_0^\infty e^{-x}x^{s-1}\,dx, \ \ \Re(s) > 0. \tag{40}$$

Applying the Formula (37) for the Laplace convolution and the Formula (39) for the B-function, convolutions of many other integral transforms including the general H-transform can be constructed [4,5].

The Borel Formula (38) can be employed for solving some integral equations of the Laplace convolution type. As an example, let us consider an integral equation of the second kind in the form

$$x(t) \ - \ \lambda\int_0^t k(t-\tau)x(\tau)\,d\tau \ = \ f(t). \tag{41}$$

Application of the Laplace integral transform to Equation (41) reduces it to the algebraic (in fact, linear) equation

$$\mathcal{L}\left\{x(t);p\right\} \ - \ \lambda\mathcal{L}\left\{x(t);p\right\}\mathcal{L}\left\{k(t);p\right\} \ = \ \mathcal{L}\left\{f(t);p\right\} \tag{42}$$

for the Laplace transform of the unknown function. Its solution is given by the formula

$$\mathcal{L}\{x(t);p\} = \frac{\mathcal{L}\{f(t);p\}}{1-\lambda\mathcal{L}\{k(t);p\}}. \tag{43}$$

To get a solution for the integral Equation (41), the inverse Laplace integral transform can be applied to Equation Lap-8. However, let us follow another approach and get a representation of the solution by using the Formula (38). To do this, let us take a closer look at the expression $\mathcal{L}\{x(t);p\} - \mathcal{L}\{f(t);p\}$. The Formula (43) leads to the representation

$$\begin{aligned} &\mathcal{L}\{x(t);p\} - \mathcal{L}\{f(t);p\} = \tfrac{\mathcal{L}\{f(t);p\}}{1-\lambda\mathcal{L}\{k(t);p\}} - \mathcal{L}\{f(t);p\} \\ &= \mathcal{L}\{f(t);p\}\left(\tfrac{1}{1-\lambda\mathcal{L}\{k(t);p\}} - 1\right) = \lambda\mathcal{L}\{f(t);p\}\,\tfrac{\mathcal{L}\{k(t);p\}}{1-\lambda\mathcal{L}\{k(t);p\}}. \end{aligned} \tag{44}$$

Because of the Formula (38), we can represent a solution for the integral Equation (41) in the form

$$x(t) = f(t) + \lambda\int_0^t h(t-\tau)f(\tau)\,d\tau. \tag{45}$$

In this formula, the function h is the inverse Laplace transform of the function $\mathcal{L}\{k(t);p\}/(1-\lambda\mathcal{L}\{k(t);p\})$.

The same method can be employed for the initial value problems for the ordinary differential equations in form (33). Let us revisit the Formula (35) and represent the rational functions $1/L$ and M/L as sums of partial fractions

$$\frac{1}{L(p)} = \frac{1}{(p-\lambda_1)^{m_1}\times\ldots\times(p-\lambda_k)^{m_k}} = \sum_{j=1}^{k}\sum_{r=1}^{m_j}\frac{c_{jr}}{(p-\lambda_j)^r},\ \lambda_j,\, c_{jr}\in\mathbb{C} \tag{46}$$

$$\frac{M(p)}{L(p)} = \sum_{j=1}^{m}\sum_{r=1}^{s_j}\frac{d_{jr}}{(p-\eta_j)^r},\ \eta_j,\, d_{jr}\in\mathbb{C}. \tag{47}$$

The operational formula

$$\mathcal{L}\left\{t^n\,e^{at};p\right\} = \int_0^\infty t^n\,e^{-t(p-a)}\,dt = \frac{\Gamma(n+1)}{(p-a)^{n+1}} = \frac{n!}{(p-a)^{n+1}},\ \Re(p-a)>0 \tag{48}$$

and the Borel convolution Formula (38) allow us to represent the solution of the initial value problem (33) in the form

$$x(t) = \int_0^t f(t-\tau)\left(\sum_{j=1}^{k}\sum_{r=1}^{m_j}\frac{c_{jr}}{(r-1)!}\tau^{r-1}\,e^{\lambda_j\tau}\right)d\tau + \sum_{j=1}^{m}\sum_{r=1}^{s_j}\frac{d_{jr}}{(r-1)!}t^{r-1}\,e^{\eta_j t}. \tag{49}$$

It is worth mentioning that the Formula (48) is valid not only for the power functions in the form $t^n, n\in\mathbb{N}$, but also for the arbitrary power functions with the exponents $\alpha>-1$:

$$\mathcal{L}\left\{t^\alpha\,e^{at};p\right\} = \int_0^\infty t^\alpha\,e^{-t(p-a)}\,dt = \frac{\Gamma(\alpha+1)}{(p-a)^{\alpha+1}},\ \alpha>-1,\ \Re(p-a)>0. \tag{50}$$

In order to solve the linear ordinary differential equations with the polynomial coefficients, a slightly different procedure compared to the method demonstrated above is often employed. Its main element consists in representation of an unknown solution to a differential equation in the form similar to the form of the inverse Laplace transform

$$x(t) = \int_C \phi(p)\,e^{pt}\,dp \tag{51}$$

with an unknown contour C. In the process of solution, the contour C must be appropriately chosen [22].

To illustrate this technique, let us consider another—a little bit exotic—application of the Laplace integral transform technique, namely, for solving a functional equation. The Euler Gamma-function (40) is a generalization of the factorial function. It satisfies the functional equation

$$F(s+1) = s\,F(s),\ \Re(s) > 0,\ F(1) = 1. \tag{52}$$

Let us show that the functional Equation (52) has a unique solution, namely, the Gamma-function in the functional space of smooth functions. To solve the functional Equation (52), the Laplace transform method mentioned above is employed. We look for the solutions F of (52) in form

$$F(s) = \int_{-\infty}^{\infty} \phi(p)\, e^{-ps}\, dp. \tag{53}$$

Then the relation

$$s\,F(s) = s\int_{-\infty}^{\infty} \phi(p)\, e^{-ps}\, dp = -\phi(p)\, e^{-ps}|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \phi'(p)\, e^{-ps}\, dp \tag{54}$$

holds valid if the unknown function ϕ is a smooth function on $\mathbb{R}$ and the limits $\lim_{p\to\pm\infty} \phi(p)\, e^{-ps}$ exist and are finite. Moreover, let the relations $\lim_{p\to\pm\infty} \phi(p)\, e^{-ps} = 0$ hold true. At this stage we just suppose that the unknown function ϕ satisfies the conditions above. However, after the functional Equation (52) will be solved, these conditions can be directly verified.

Using the representation (that directly follows from (53))

$$F(s+1) = \int_{-\infty}^{\infty} \phi(p)\, e^{-ps}\, e^{-p}\, dp,$$

and the Formula (54), the functional Equation (52) is reduced to a differential equation for the unknown function ϕ:

$$\phi(p)\, e^{-p} = \phi'(p).$$

The general solution formula of the above equation can be derived by separating the variables p and ϕ:

$$\phi(p) = -C\, e^{-e^{-p}},\ C \in \mathbb{R}. \tag{55}$$

Evidently, the function ϕ defined by the right-hand side of (55) satisfies the conditions $\lim_{p\to\pm\infty} \phi(p)\, e^{-ps} = 0$ for $\Re(s) > 0$. Substituting (55) into (53) and by the variables substitution $x = \exp(-p)$, we get the solution formula

$$F(s) = C \int_0^{\infty} e^{-x}\, x^{s-1}\, dx,\ \Re(s) > 0. \tag{56}$$

The initial condition from (52) leads to a unique value of the constant C:

$$1 = F(1) = C \int_0^{\infty} e^{-x}\, dx = C.$$

Thus, the solution Formula (56) coincides with the integral representation of the Gamma-function and the Gamma-function is the only smooth solution to the functional Equation (52) with the initial condition $F(1) = 1$.

In our last example of this section, we prove the Formula (39) for the Beta-function by employing the Laplace transform technique. We start with an observation that the Beta-function B is the Laplace convolution of two power functions evaluated at the point $x = 1$:

$$B(s,t) = (\tau^{s-1} \overset{\mathcal{L}}{*} \tau^{t-1})(x)|_{x=1} = \int_0^x \tau^{s-1}\, (x-\tau)^{t-1}\, d\tau|_{x=1}.$$

Let us define an auxiliary function

$$\beta(s,t,x) = (\tau^{s-1} \overset{\mathcal{L}}{*} \tau^{t-1})(x) = \int_0^x \tau^{s-1} (x-\tau)^{t-1} \, d\tau, \; x > 0.$$

Its Laplace integral transform is given by the formula

$$\mathcal{L}\{\beta(x); p\} = \frac{\Gamma(s)}{p^s} \frac{\Gamma(t)}{p^t} = \frac{\Gamma(s)\Gamma(t)}{p^{s+t}} \tag{57}$$

because of the operational relation (50) with $a = 0$ and the Borel convolution formula.

Again using the operational relation (50), we determine the function β with the Laplace transform given by (57) in the form:

$$\beta(s,t,x) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \, x^{s+t-1}. \tag{58}$$

Specifying the Formula (58) for $x = 1$, we receive the well-known representation of the Beta-function in terms of the Gamma-function:

$$B(s,t) = \int_0^x \tau^{s-1} (x-\tau)^{t-1} \, d\tau|_{x=1} = \beta(s,t,1) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

As to the Mikusinski-type operational calculus associated with the Laplace integral transform, we start with the Volterra integral operator that is one of the generating operators for the Laplace integral transform. Indeed, this operator can be represented as the Laplace convolution of the functions f and $\{1\}$ ($\{1\}$ is the function that is identically equal to 1):

$$(\mathcal{V} f)(t) = (f \overset{\mathcal{L}}{*} \{1\})(t) = \int_0^t f(\tau) \, d\tau. \tag{59}$$

Evidently, the Volterra integral operator is a linear operator on $C[0,\infty)$ with the Laplace convolution as its convolution in the sense of Formula (17).

As stated by the well-known Titchmarsh theorem [23], the space of functions $C[0,\infty)$ equipped with the operations $+$ and $\overset{\mathcal{L}}{*}$ is a commutative ring without divisors of zero. The basic idea behind the classical Mikusinski operational calculus is an extension of this ring to a field of convolution quotients according to the schema presented in the previous section. The so constructed Mikusinski operational calculus is closely related with the first derivative $\frac{d}{dt}$ that is a left-inverse operator to the Volterra integral operator (59). The projector of the Volterra integral operator is given by the expression

$$\mathcal{P} f = f - \mathcal{V} \frac{df}{dt} = f(0). \tag{60}$$

It is worth mentioning that the projector (60) determines the form of the initial conditions for the differential equations that can be solved by applying the Mikusinski operational calculus.

4. The Mellin Integral Transform

In this section, some basic definitions and formulas for the Mellin integral transform are presented. The Mellin integral transform is one of the main tools for a treatment of the integral transforms of the Mellin convolution type, their convolutions and generating operators. More details regarding the Mellin integral transform, its properties and particular cases can be found in [4,24–29].

The Mellin integral transform of a sufficiently well-behaved function f at the point $s \in \mathbb{C}$ is defined as

$$\mathcal{M}\{f(t); s\} = f^*(s) = \int_0^{+\infty} f(t) t^{s-1} dt, \tag{61}$$

and the inverse Mellin integral transform as

$$f(t) = \mathcal{M}^{-1}\{f^*(s);t\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)t^{-s}ds,\ t>0,\ \gamma = \Re(s), \tag{62}$$

where the integral is understood in the sense of the Cauchy principal value.

The Mellin integral transform can be interpreted as the Fourier integral transform changed by a variables substitution and by a rotation of the complex plane:

$$\mathcal{M}\{f(t);s\} \;=\; \int_0^{+\infty} f(t)t^{s-1}dt \;=\; \int_{-\infty}^{+\infty} f(e^t)e^{it(-is)}dt \;=\; \mathcal{F}\{f(e^t);-is\}.$$

The integral at the right-hand side of the Formula (61) is well defined, say, for the functions $f \in L^c(\epsilon, E)$, $0 < \epsilon < E < \infty$ that are continuous on the intervals $(0,\epsilon]$, $[E,+\infty)$ and satisfy the estimates $|f(t)| \le M t^{-\gamma_1}$ for $0 < t < \epsilon$, $|f(t)| \le M t^{-\gamma_2}$ for $t > E$, where M is a constant and $\gamma_1 < \gamma_2$. Under these conditions, the Mellin transform f^* exists and is an analytical function in the vertical strip $\gamma_1 < \Re(s) < \gamma_2$.

The Formula (62) for the inverse Mellin integral transform holds true at all points where the function f is continuous if f is piecewise differentiable, $f(t)\, t^{\gamma-1} \in L^c(0,+\infty)$, and its Mellin integral transform f^* is given by (61).

For the reader's convenience, we present in this section some important theorems concerning the Mellin integral transform (for the proofs see e.g., [29]).

Theorem 1. *Let f be a function of bounded variation in the neighborhood of a point $t = x$, $t^{\gamma-1}f(t) \in L(0,\infty)$, and*

$$F(s) = \mathcal{M}\{f(t),s\} = \int_0^{+\infty} f(t)t^{s-1}dt,\ s = \gamma + i\tau. \tag{63}$$

Then

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} F(s)x^{-s}ds. \tag{64}$$

Theorem 2. *Let $F(s)$, $s = \gamma + i\tau$ be a function of bounded variation in the neighborhood of a point $\tau = x$, $F \in L(-\infty,+\infty)$, and*

$$f(t) = \mathcal{M}^{-1}\{F(s);t\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)t^{-s}ds. \tag{65}$$

Then

$$\frac{F(\gamma+i(x+0))+F(\gamma+i(x-0))}{2} = \lim_{\lambda\to\infty}\int_{1/\lambda}^{\lambda} f(t)t^{\gamma+ix-1}dt. \tag{66}$$

To formulate other theorems, some special spaces of functions are first introduced. By $L_p(\mu(t);\mathbb{R}_+)$, $p \ge 1$ the space of functions summable in the Lebesgue sense on the interval $(0,+\infty)$ to the power p and with the weight $\mu(t) > 0$, $t > 0$ is denoted. The norm of the space $L_p(\mu(t);\mathbb{R}_+)$ is defined by

$$\|f\|_{L_p(\mu(t);\mathbb{R}_+)} = \left\{\int_0^\infty \mu(t)|f(t)|^p dt\right\}^{1/p} < \infty. \tag{67}$$

In particular, when $\mu(t) \equiv 1$, $t > 0$, the space $L_p(\mu(t);\mathbb{R}_+)$ is reduced to the usual L_p-space. While estimating the integrals in L_p, the Hölder inequality

$$\int_0^\infty |f(t)g(t)|dt \le \|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+)}, \tag{68}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and the Minkowski inequality

$$\left\{ \int_0^\infty dx \left| \int_0^\infty f(x,y)dy \right|^p \right\}^{1/p} \le \int_0^\infty dy \left\{ \int_0^\infty |f(x,y)|^p dx \right\}^{1/p} \tag{69}$$

are often used.

Theorem 3. *Let $f \in L_2(t^{2\gamma-1}; \mathbb{R}_+)$. Then the function*

$$f^*(s,\lambda) = \int_{1/\lambda}^{\lambda} f(t) t^{\gamma+i\tau-1} dt, \; s = \gamma + i\tau \tag{70}$$

converges in the norm of $L_2(\gamma - i\infty, \gamma + i\infty)$ to a function f^* and the function

$$f(t,\lambda) = \frac{1}{2\pi i} \int_{\gamma-i\lambda}^{\gamma+i\lambda} f^*(s) t^{-s} ds \tag{71}$$

converges in the norm of $L_2(t^{2\gamma-1}; \mathbb{R}_+)$ to the function f, i.e.,

$$\lim_{\lambda\to\infty} \int_0^\infty |f(t) - f(t,\lambda)|^2 t^{2\gamma-1} dt = 0. \tag{72}$$

Moreover, the Parseval equality

$$\int_0^\infty |f(t)|^2 t^{2\gamma-1} dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f^*(\gamma + i\tau)|^2 d\tau \tag{73}$$

holds true.

Theorem 4. *Let $f \in L_2(t^{2\gamma-1}; \mathbb{R}_+)$, $g \in L_2(t^{1-2\gamma}; \mathbb{R}_+)$ and f^*, g^* be their Mellin integral transforms, respectively. Then the Mellin-Parseval equality*

$$\int_0^\infty f(t)g(t)dt = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) g^*(1-s) ds \tag{74}$$

holds true.

The Mellin convolution

$$(f \overset{\mathcal{M}}{*} g)(x) = \int_0^{+\infty} f(x/t) g(t) \frac{dt}{t} \tag{75}$$

is a very essential element of the integral transforms of the Mellin convolution type. Following [29], we formulate the following important theorem:

Theorem 5. *Let $f(t)\, t^{\gamma-1} \in L(0,\infty)$ and $g(t)\, t^{\gamma-1} \in L(0,\infty)$. Then the Mellin convolution $h = (f \overset{\mathcal{M}}{*} g)$ given by (75) is well defined, $h(x)\, x^{\gamma-1} \in L(0,\infty)$ and the convolution formula*

$$\mathcal{M}\left\{ (f \overset{\mathcal{M}}{*} g)(x); s \right\} = \mathcal{M}\left\{ f(t); s \right\} \times \mathcal{M}\left\{ g(t); s \right\} \tag{76}$$

holds true along with the Parseval equality

$$\int_0^{+\infty} f(x/t) g(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) g^*(s) x^{-s} ds. \tag{77}$$

In particular, the Parseval equality (77) can be employed while treating integrals of the Fourier *cos*- and *sin*-transforms type for $x > 0$:

$$I_c(x) = \frac{1}{\pi} \int_0^\infty f(t) \cos(t\,x)\, dt,$$

$$I_s(x) = \frac{1}{\pi} \int_0^\infty f(t) \sin(t\,x)\, dt.$$

Indeed, the integrals I_c and I_s can be interpreted as Mellin convolutions (75) of the function f and the functions

$$g_c(t) = \frac{1}{\pi\,x\,t} \cos\left(\frac{1}{t}\right),\; g_s(t) = \frac{1}{\pi\,x\,t} \sin\left(\frac{1}{t}\right),$$

respectively, evaluated at the point $1/x$.

The Mellin integral transforms of the functions g_c, g_s are well known ([26] or [27]):

$$g_c^*(s) = \mathcal{M}\{g_c(t);s\} = \frac{\Gamma(1-s)}{\pi\,x} \sin\left(\frac{\pi s}{2}\right),\; 0 < \Re(s) < 1,$$

$$g_s^*(s) = \mathcal{M}\{g_s(t);s\} = \frac{\Gamma(1-s)}{\pi\,x} \cos\left(\frac{\pi s}{2}\right),\; 0 < \Re(s) < 2\,.$$

Thus the integrals I_c and I_s can be represented by means of the Parseval equality (77) as follows:

$$I_c(x) = \frac{1}{\pi\,x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)\,\Gamma(1-s) \sin\left(\frac{\pi\,s}{2}\right) x^s\, ds\,,\; x > 0\,,\; 0 < \gamma < 1\,,$$

$$I_s(x) = \frac{1}{\pi\,x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)\,\Gamma(1-s) \cos\left(\frac{\pi\,s}{2}\right) x^s\, ds\,,\; x > 0\,,\; 0 < \gamma < 2\,.$$

In [4], the method presented above was employed to introduce the notion of the generalized H-transform.

In the applications, elementary properties of the Mellin integral transform are often employed. They are presented in the rest of this section.

Let us denote by $\stackrel{\mathcal{M}}{\leftrightarrow}$ the juxtaposition of a function f with its Mellin transform f^*. The basic Mellin transform rules are as follows:

$$f(at) \stackrel{\mathcal{M}}{\leftrightarrow} a^{-s} f^*(s),\; a > 0, \tag{78}$$

$$t^p f(t) \stackrel{\mathcal{M}}{\leftrightarrow} f^*(s+p), \tag{79}$$

$$f(t^p) \stackrel{\mathcal{M}}{\leftrightarrow} \frac{1}{|p|} f^*(s/p),\; p \neq 0, \tag{80}$$

$$f^{(n)}(t) \stackrel{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(n+1-s)}{\Gamma(1-s)} f^*(s-n) \quad \text{if } \lim_{t\to 0} t^{s-k-1} f^{(k)}(t) = 0,\; k = 0,1,\ldots,n-1, \tag{81}$$

$$\left(t\frac{d}{dt}\right)^n f(t) \stackrel{\mathcal{M}}{\leftrightarrow} (-s)^n f^*(s), \tag{82}$$

$$\left(\frac{d}{dt}t\right)^n f(t) \stackrel{\mathcal{M}}{\leftrightarrow} (1-s)^n f^*(s). \tag{83}$$

In [24,26,27], the Mellin transforms of the elementary and many of the special functions are given. Here we list just some basic Mellin transform formulas that are often used in applications.

$$e^{-t^p} \overset{\mathcal{M}}{\leftrightarrow} \frac{1}{|p|}\Gamma(\frac{s}{p}) \text{ if } \Re(\frac{s}{p}) > 0, \tag{84}$$

$$\frac{(1-t^p)_+^{\alpha-1}}{\Gamma(\alpha)} \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(\frac{s}{p})}{|p|\Gamma(\frac{s}{p}+\alpha)} \text{ if } \Re(\alpha) > 0,\ \Re(\frac{s}{p}) > 0, \tag{85}$$

$$\frac{(t^p-1)_+^{\alpha-1}}{\Gamma(\alpha)} \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(1-\alpha-\frac{s}{p})}{|p|\Gamma(1-\frac{s}{p})} \text{ if } 0 < \Re(\alpha) < 1-\Re(\frac{s}{p}), \tag{86}$$

$$\Gamma(\rho)(1+t)^{-\rho} \overset{\mathcal{M}}{\leftrightarrow} \Gamma(s)\Gamma(\rho-s) \text{ if } 0 < \Re(s) < \Re(\rho), \tag{87}$$

$$\frac{1}{\pi(1-t)} \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(s+1/2)\Gamma(1/2-s)} \text{ if } 0 < \Re(s) < 1, \tag{88}$$

$$\frac{\sin(2\sqrt{t})}{\sqrt{\pi}} \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(s+1/2)}{\Gamma(1-s)} \text{ if } |\Re(s)| < 1/2, \tag{89}$$

$$\sqrt{\pi}\,\mathrm{erf}\,(\sqrt{t}) \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(s+1/2)\Gamma(-s)}{\Gamma(1-s)} \text{ if } -1/2 < \Re(s) < 0, \tag{90}$$

$$J_\nu(2\sqrt{t}) \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(s+\frac{\nu}{2})}{\Gamma(1+\frac{\nu}{2}-s)} \text{ if } -\Re(\frac{\nu}{2}) < \Re(s) < \frac{3}{4}, \tag{91}$$

$$2K_\nu(2\sqrt{t}) \overset{\mathcal{M}}{\leftrightarrow} \Gamma(s+\frac{\nu}{2})\Gamma(s-\frac{\nu}{2}) \text{ if } \Re(s) > \frac{|\Re(\nu)|}{2}, \tag{92}$$

$$\frac{\Gamma(a)}{\Gamma(c)}{}_1F_1(a;c;-t) \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(c-s)} \text{ if } 0 < \Re(s) < \Re(a), \tag{93}$$

$$|1-t|^{\mu/2}P_\nu^\mu(\sqrt{t}) \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(s)\Gamma(s+\frac{1}{2})\Gamma(\frac{1+\nu-\mu}{2}-s)\Gamma(-\frac{\mu+\nu}{2}-s)}{\pi 2^{\mu+1}\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)} \tag{94}$$
$$\text{if } 0 < \Re(s) < \min\{\frac{1+\Re(\nu-\mu)}{2}, -\frac{\Re(\nu+\mu)}{2}\},$$

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}{}_2F_1(a,b;c;-t) \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)} \tag{95}$$
$$\text{if } 0 < \Re(s) < \min\{\Re(a), \Re(b)\},$$

$$\frac{(1-t)_+^{c-1}}{\Gamma(c)}{}_2F_1(a,b;c;1-t) \overset{\mathcal{M}}{\leftrightarrow} \frac{\Gamma(s)\Gamma(s+c-a-b)}{\Gamma(s+c-a)\Gamma(s+c-b)} \text{ if } 0 < \Re(s), \tag{96}$$
$$0 < \Re(c),\ 0 < \Re(s+c-a-b),$$

$${}_pF_q((a)_p;(b)_q;-t) \overset{\mathcal{M}}{\leftrightarrow} \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \frac{\prod_{j=1}^p \Gamma(a_j-s)\Gamma(s)}{\prod_{j=1}^q \Gamma(b_j-s)} \tag{97}$$
$$\text{if } 0 < \Re(s) < \min_{1\le j\le p} \Re(a_j),$$
$$b_k \neq 0, -1, \ldots,\ 1 \le k \le q$$
$$\text{and}$$
$$(1)\ q = p-1 \text{ or}$$
$$(2)\ q = p \text{ or}$$
$$(3)\ q = p+1 \text{ and}$$
$$\Re(s) < \frac{1}{4} - \frac{1}{2}\left(\Re\left(\sum_{j=1}^p a_j - \sum_{j=1}^q b_j\right)\right),$$

where J_ν is the Bessel function, K_ν is the Macdonald function, P_ν^μ denotes the Legendre function of the first kind, and ${}_pF_q((a)_p;(b)_q;z)$ stands for the generalized hypergeometric function.

As to applications of the Mellin integral transform, we mention here its applications in the theory of the integral transform of the Mellin convolution type [4], for evaluation of improper integrals [26,27], in the theory of special functions of the hypergeometric type [26], for construction of the operational calculi of Mikusinski type for the compositions of the fractional Erdelyi-Kober derivatives [5], for derivation of the fundamental solutions to the space-time fractional diffusion equation [30,31], for analysis of the multi-dimensional fractional diffusion-wave equations [32], for derivation of the subordination principles for the multi-dimensional space-time-fractional diffusion-wave equation [33], and for several other important problems in Fractional Calculus [34].

5. Conclusions

In this survey article, we considered some elements of theory and applications of the integral transforms of mathematical physics. These integral transforms are not arbitrary integral transforms but those possessing well defined inverse integral transforms and generating operators. The basic constructions for most of applications of these integral transforms are their convolutions and generating operators. They lead to some simple and efficient solution methods for the corresponding integral, differential, and integro-differential equations. Another important technique for applications of the integral transforms is the Mikusinski-type operational calculi that were also discussed in the article.

The general schemata for applications of the integral transforms of mathematical physics were illustrated in detail by considering the Laplace integral transform. Similar, but more complicated constructions and solution methods are valid for the general H-transform as a generalization of the Laplace integral transform. In this case, the "integral" generating operators are in form of the compositions of the fractional Erdelyi-Kober right- and left-hand sided fractional integrals and derivatives. Their left-inverse "differential" operators are in form of certain compositions of the fractional Erdelyi-Kober left- and right-hand sided fractional derivatives and integrals. The convolutions of the general H-transform can be constructed in explicit form as some multiple integrals.

In the article, some basic elements of the Mellin integral transform were discussed, too. The Mellin integral transform is a foundation for the theory of the Mellin convolution type integral transforms in general and of the H-transform in particular. For details we refer the interested readers to [4] or [5–8].

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