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Variational Inequalities Approaches to Minimization Problems with Constraints of Generalized Mixed Equilibria and Variational Inclusions

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Abstract: Multistep composite implicit and explicit extragradient-like schemes are presented for solving the minimization problem with the constraints of variational inclusions and generalized mixed equilibrium problems. Strong convergence results of introduced schemes are given under suitable control conditions.

Keywords: extragradient-like method; minimization problem; variational inclusion; generalized mixed equilibrium problem

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1. Introduction

Let H be a real Hilbert space and $\emptyset \neq C \subset H$ be a closed convex set. Let $\mathcal{A} : C \to H$ be a nonlinear operator. Let us consider the variational inequality problem (VIP) which aims to find $x^* \in C$ verifying

$$\langle \mathcal{A}x^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1}$$

We denote the solution set of VIP (1) by $\mathrm{VI}(C, \mathcal{A})$.

In [1], Korpelevich suggested an extragradient method for solving VIP (1). Korpelevich's method has been studied by many authors, see e.g., references [2–13] and references therein. In 2011, Ceng, Ansari and Yao [14] considered the following algorithm

$$u_{k+1} = P_C[\alpha_k \gamma V u_k + (I - \alpha_k \mu F) T u_k], \quad \forall k \geq 0. \tag{2}$$

They showed that $\{u_k\}$ strongly converges to $x^\dagger \in \mathrm{Fix}(T)$ which solves the following VIP

$$\langle (\mu F - \gamma V) x^\dagger, x^\dagger - \tilde{x} \rangle \leq 0, \quad \forall \tilde{x} \in \mathrm{Fix}(T). \tag{3}$$

Ceng, Guu and Yao [15] presented a composite implicit scheme

$$x_t = (I - \theta_t \mathcal{B})Tx_t + \theta_t[Tx_t - t(\mu F T x_t - \gamma V x_t)], \tag{4}$$

and another composite explicit scheme

$$\begin{cases} y_n = (I - \alpha_n \mu F)Tx_n + \alpha_n \gamma V x_n, \\ x_{n+1} = (I - \beta_n \mathcal{B})Tx_n + \beta_n y_n, \quad \forall n \geq 0. \end{cases} \tag{5}$$

Ceng, Guu and Yao proved that $\{x_t\}$ and $\{x_n\}$ converge strongly to the same point $x^\dagger \in \mathrm{Fix}(T)$, which solves the following VIP

$$\langle (\mathcal{B} - I)x^\dagger, x^\dagger - \tilde{x} \rangle \leq 0, \quad \forall \tilde{x} \in \mathrm{Fix}(T). \tag{6}$$

In [2], Peng and Yao considered the generalized mixed equilibrium problem (GMEP) which aims to find $x^\dagger \in C$ verifying

$$\varphi(y) - \varphi(x^\dagger) + \Theta(x^\dagger, y) + \langle \mathcal{A}x^\dagger, y - x^\dagger \rangle \geq 0, \quad \forall y \in C, \tag{7}$$

where $\varphi : C \to \mathbf{R}$ is a real-valued function and $\Theta : C \times C \to \mathbf{R}$ is a bifunction. The set of solutions of GMEP (7) is denoted by $\mathrm{GMEP}(\Theta, \varphi, \mathcal{A})$.

It is clear that GMEP (7) includes optimization problems, VIP and Nash equilibrium problems as special cases.

Special cases:

(i) Letting $\varphi = 0$, GMEP (7) reduces to find $x^\dagger \in C$ verifying

$$(GEP): \quad \Theta(x^\dagger, y) + \langle \mathcal{A}x^\dagger, y - x^\dagger \rangle \geq 0, \quad \forall y \in C,$$

which was studied in [13,16].

(ii) Letting $\mathcal{A} = 0$, GMEP (7) reduces to find $x^\dagger \in C$ verifying

$$(MEP): \quad \Theta(x^\dagger, y) + \varphi(y) - \varphi(x^\dagger) \geq 0, \quad \forall y \in C,$$

which was considered in [17].

(iii) Letting $\varphi = 0$ and $\mathcal{A} = 0$, GMEP (7) reduces to find $x^\dagger \in C$ verifying

$$(EP): \quad \Theta(x^\dagger, y) \geq 0, \quad \forall y \in C,$$

which was discussed in [18,19].

GMEP (7) has been discussed extensively in the literature; see e.g., [6,7,20–32].

Now, we consider the minimization problem (CMP):

$$\min_{x \in C} f(x), \tag{8}$$

where $f : C \to \mathbf{R}$ is a convex and continuously Fréchet differentiable functional. Use Ξ to denote the set of minimizers of CMP (8).

For solving (8), an efficient algorithm is the gradient-projection algorithm (GPA) which is defined by: for given the initial guess x_0,

$$z_{k+1} := P_C(z_k - \mu \nabla f(z_k)), \quad \forall k \geq 0, \tag{9}$$

or a general form,

$$z_{n+1} := P_C(z_k - \mu_k \nabla f(z_k)), \quad \forall k \geq 0. \tag{10}$$

It is known that $S := P_C(I - \lambda \nabla f)$ is a contractive operator if ∇f is α-strongly monotone and L-Lipschitz and $0 < \lambda < \frac{2\alpha}{L^2}$. In this case, GPA (9) has strong convergence.

Recall that the variational inclusion problem is to find a point $x \in C$ verifying

$$0 \in Bx + Rx. \tag{11}$$

where $B : C \to H$ is a single-valued mapping and R is a multivalued mapping with $D(R) = C$. Use $\mathrm{I}(B, R)$ to denote the solution set of (11).

Taking $B = 0$ in (11), it reduces to the problem introduced by Rockafellar [33]. Let $R : D(R) \subset H \to 2^H$ be a maximal monotone operator. The resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ is defined by $J_{R,\lambda} = (I + \lambda R)^{-1}$, $\forall x \in H$. In [34], Huang considered problem (11) under the assumptions that B is strongly monotone Lipschitz operator and R is maximal monotone. Zeng, Guu and Yao [35] discussed problem (11) under more general cases. Related work, please refer to [36,37] and the references therein.

In the present paper, we introduce two composite schemes for finding a solution of the CMP (8) with the constraints of finitely many GMEPs and finitely many variational inclusions for maximal monotone and inverse strongly monotone mappings in a real Hilbert space H. Strong convergence of the suggested algorithms are given. Our theorems complement, develop and extend the results obtained in [6,15,38], having as background [39–46].

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that the metric projection $P_C : H \to C$ is defined by $\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C)$. A mapping $A : H \to H$ is called strongly positive if $\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2$, $\forall x \in H$ for $\bar{\gamma} > 0$. A mapping $F : C \to H$ is called Lipschitz if $\|Fx - Fy\| \leq L\|x - y\|$, $\forall x, y \in C$ for some $L \geq 0$. F is called nonexpansive if $L = 1$ and it is called contractive provided $L \in [0, 1)$.

Proposition 1. *Let $x \in H$ and $z \in C$. Then, the following hold*

- $z = P_C x \;\Leftrightarrow\; \langle x - z, z - x^\dagger\rangle \geq 0, \quad \forall x^\dagger \in C;$
- $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 + \|x^\dagger - z\|^2 \leq \|x - x^\dagger\|^2, \quad \forall x^\dagger \in C;$
- $\langle P_C x - P_C x^\dagger, x - x^\dagger\rangle \geq \|P_C x - P_C x^\dagger\|^2, \; \forall x^\dagger \in H.$

Definition 1. *A mapping $F : C \to H$ is called*

- *monotone if $\langle Fx - Fx^\dagger, x - x^\dagger\rangle \geq 0, \quad \forall x, x^\dagger \in C;$*
- *η-strongly monotone if $\langle Fx - Fx^\dagger, x - x^\dagger\rangle \geq \eta\|x - x^\dagger\|^2$, $\forall x, y \in C$ for some $\eta > 0$;*
- *α-inverse-strongly monotone (ism) if $\langle Fx - Fx^\dagger, x - x^\dagger\rangle \geq \alpha\|Fx - Fx^\dagger\|^2$, $\forall x, x^\dagger \in C$ for some $\alpha > 0$. In this case, we have for all $u, u^\dagger \in C$,*

$$\|(I - \mu F)u - (I - \mu F)u^\dagger\|^2 \leq \|u - u^\dagger\|^2 + \mu(\mu - 2\alpha)\|Fu - Fu^\dagger\|^2, \tag{12}$$

where $\mu > 0$ is a constant.

Definition 2. *A mapping $T : H \to H$ is called firmly nonexpansive if $2T - I$ is nonexpansive. Thus, T is firmly nonexpansive iff $T = \frac{1}{2}(I + S)$, where $S : H \to H$ is nonexpansive.*

Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying the following assumptions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;
(A2) $\Theta(x, x^\dagger) + \Theta(x^\dagger, x) \leq 0$ for all $x, x^\dagger \in C$;
(A3) for $x, x^\dagger, y \in C$, $\limsup_{t \to 0^+} \Theta(ty + (1 - t)x, x^\dagger) \leq \Theta(x, x^\dagger)$;
(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

Let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function satisfying the following conditions:

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,
$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

Let $r > 0$. Define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ by the following form: for any $x \in H$,
$$T_r^{(\Theta,\varphi)}(x) := \{x^\dagger \in C : \Theta(x^\dagger, y) + \varphi(y) - \varphi(x^\dagger) + \frac{1}{r}\langle x^\dagger - x, y - x^\dagger\rangle \geq 0, \forall y \in C\}.$$

If $\varphi \equiv 0$, then $T_r^{\Theta}(x) := \{x^\dagger \in C : \Theta(x^\dagger, y) + \frac{1}{r}\langle x^\dagger - x, y - x^\dagger\rangle \geq 0, \forall y \in C\}$.

Proposition 2 ([17]). *Let a bifunction $\Theta : C \times C \to \mathbf{R}$ satisfying assumptions (A1)–(A4). Let a convex function $\varphi : C \to \mathbf{R}$ be a proper lower semi-continuous. Assume conditions (B1) or (B2) holds. Then, we have*

- $\forall x \in H$, $T_r^{(\Theta,\varphi)}(x) \neq \emptyset$ *is single-valued;*
- $T_r^{(\Theta,\varphi)}$ *is firmly nonexpansive;*
- $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$*;*
- $\mathrm{MEP}(\Theta, \varphi)$ *is convex and closed;*
- $\forall s, t > 0$, $\|T_s^{(\Theta,\varphi)} z^\dagger - T_t^{(\Theta,\varphi)} z^\dagger\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Theta,\varphi)} z^\dagger - T_t^{(\Theta,\varphi)} z^\dagger, T_s^{(\Theta,\varphi)} z^\dagger - z^\dagger\rangle$ *for* $z^\dagger \in H$.

Proposition 3 ([41]). *We have the following conclusions:*

- *A mapping T is nonexpansive iff the complement $I - T$ is $\frac{1}{2}$-ism.*
- *If a mapping T is α-ism, then γT is $\frac{\alpha}{\gamma}$-ism where $\gamma > 0$.*
- *A mapping T is averaged iff the complement $I - T$ is α-ism for some $\alpha > 1/2$.*

Lemma 1. *Let E be a real inner product space. Then,*
$$\|x + x^\dagger\|^2 \leq \|x\|^2 + 2\langle x^\dagger, x + x^\dagger\rangle, \ \forall x, x^\dagger \in E.$$

Lemma 2. *In a real Hilbert space H, we have the following results*

- $\|x - x^\dagger\|^2 = \|x\|^2 - 2\langle x - x^\dagger, x^\dagger\rangle - \|x^\dagger\|^2, \forall x, x^\dagger \in H;$
- $\|\lambda x + \gamma x^\dagger\|^2 = \lambda\|x\|^2 + \gamma\|x^\dagger\|^2 - \lambda\gamma\|x - x^\dagger\|^2, \forall x, x^\dagger \in H$ *and* $\lambda, \gamma \in [0, 1]$ *with* $\lambda + \gamma = 1$*;*
- $\{u_n\} \subset H$ *is a sequence satisfying* $u_n \rightharpoonup x$*. Then,*
$$\limsup_{n\to\infty} \|u_n - x^\dagger\|^2 = \limsup_{n\to\infty} \|u_n - x\|^2 + \|x - x^\dagger\|^2, \ \forall x^\dagger \in H.$$

Lemma 3 ([45]). *Let H be a real Hilbert space and $\emptyset \neq C \subset H$ be a closed convex set. Let $T : C \to C$ be a k-strictly pseudocontractive mapping. Then,*

- $\|Tx - Ty\| \leq \frac{1+k}{1-k}\|x - y\|, \forall x, y \in C.$
- *$I - T$ is demiclosed at* 0.
- $\mathrm{Fix}(T)$ *of T is closed and convex.*

Lemma 4 ([39]). *Let H be a real Hilbert space and $\emptyset \neq C \subset H$ be a closed convex set. Let $T : C \to C$ be a k-strictly pseudocontractive mapping. Then,*
$$\|\gamma(x - x^\dagger) + \mu(Tx - Tx^\dagger)\| \leq (\gamma + \mu)\|x - x^\dagger\|, \ \forall x, x^\dagger \in C,$$

where $\gamma \geq 0$ *and* $\mu \geq 0$ *with* $(\gamma + \mu)k \leq \gamma$.

Let $T : C \to C$ be a nonexpansive operator and the operator $F : C \to H$ be κ-Lipschitzian and η-strongly monotone. Define an operator $T^\lambda : C \to H$ by $T^\lambda x := Tx - \lambda\mu F(Tx),\ \forall x \in C$, where $\lambda \in (0,1]$ and $\mu > 0$ are two constants.

Lemma 5 ([42]). *Let* $0 < \mu < \frac{2\eta}{\kappa^2}$. *Then, for* $\forall x, y \in C$, *we have* $\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

Lemma 6 ([46]). *Let* $\{a_n\} \subset [0, +\infty)$, $\{\omega_n\} \subset [0,1]$, $\{\delta_n\}$ *and* $\{r_n\} \subset [0, +\infty)$ *be four sequences. If*

(i) $\sum_{n=0}^{\infty} \omega_n = \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$;
(ii) either $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \omega_n |\delta_n| < \infty$;
(iii) $a_{n+1} \leq (1 - \omega_n)a_n + \omega_n\delta_n + r_n,\ \forall n \geq 0$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 7 ([44]). *Let bounded linear operator* $A : H \to H$ *be* $\bar{\gamma}$*-strongly positive. Then,* $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$ *provided* $0 < \rho \leq \|A\|^{-1}$.

Let LIM be a Banach limit. Then, we have following properties:

- $a_n \leq c_n, n \geq 1 \Rightarrow \mathrm{LIM}_n a_n \leq \mathrm{LIM}_n c_n$;
- $\mathrm{LIM}_n a_{n+N} = \mathrm{LIM}_n a_n$, where N is a fixed positive integer;
- $\liminf_{n\to\infty} a_n \leq \mathrm{LIM}_n a_n \leq \limsup_{n\to\infty} a_n,\ \forall\{a_n\} \in l^\infty$.

Lemma 8 ([40]). *Assume that the sequence* $\{s_n\} \in l^\infty$ *satisfy* $\mathrm{LIM}_n s_n \leq M$, *where* M *is a constant. If* $\limsup_{n\to\infty}(s_{n+1} - s_n) \leq 0$, *then* $\limsup_{n\to\infty} s_n \leq M$.

Let the operator $R : D(R) \subset H \to 2^H$ be maximal monotone. Let $\lambda, \mu > 0$ be two constants.

Lemma 9 ([43]). *There holds the resolvent identity*

$$J_{R,\lambda}x = J_{R,\mu}\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_{R,\lambda}x\right),\ \forall x \in H.$$

Remark 1. *The resolvent has the following property*

$$\|J_{R,\lambda}x - J_{R,\mu}x^\dagger\| \leq \|x - x^\dagger\| + |\lambda - \mu|\left(\frac{1}{\lambda}\|J_{R,\lambda}x - x^\dagger\| + \frac{1}{\mu}\|x - J_{R,\mu}x^\dagger\|\right),\ \forall x, x^\dagger \in H. \quad (13)$$

Lemma 10 ([34,35]). $J_{R,\lambda}$ *satisfies*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x - y\rangle \geq \|J_{R,\lambda}x - J_{R,\lambda}y\|^2,\ \forall x, y \in H.$$

3. Main Results

Let H be a real Hilbert space and $\emptyset \neq C \subset H$ be a closed convex set. In what follows, we assume:

$f : C \to \mathbf{R}$ is a convex functional with gradient ∇f being L-Lipschitz, the operator $F : C \to H$ is η-strongly monotone and κ-Lipschitzian, $R_i : C \to 2^H$ is a maximal monotone mapping and $B_i : C \to H (i = 1, ..., N)$ is η_i-ism, $\Theta_j : C \times C \to \mathbf{R}$ satisfies (A1)–(A4), $\varphi_j : C \to \mathbf{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous satisfying (B1) or (B2), and $A_j : C \to H$ is μ_j-ism for each $j = 1, ..., M$;

$V : C \to H$ is a nonexpansive mapping and $A : H \to H$ is a $\bar{\gamma}$-strongly positive bounded linear operator such $1 < \bar{\gamma} < 2$, $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma < \tau$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;

$P_C(I-\lambda_t\nabla f)=s_tI+(1-s_t)T_t$ where T_t is nonexpansive, $s_t=\frac{2-\lambda_tL}{4}\in(0,\frac{1}{2})$ and $\lambda_t:(0,1)\to(0,\frac{2}{L})$ with $\lim_{t\to0}\lambda_t=\frac{2}{L}$;

The operator $\Lambda_t^N:C\to C$ is defined by $\Lambda_t^Nx=J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)\cdots J_{R_1,\lambda_{1,t}}(I-\lambda_{1,t}B_1)x, t\in(0,1)$, for $\{\lambda_{i,t}\}\subset[a_i,b_i]\subset(0,2\eta_i),\ i=1,...,N$;

The operator $\Lambda_n^N:C\to C$ is defined by $\Lambda_n^Nx=J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\cdots J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1)x$ with $\{\lambda_{i,n}\}\subset[a_i,b_i]\subset(0,2\eta_i)$ and $\lim_{n\to\infty}\lambda_{i,n}=\lambda_i$, for each $i=1,...,N$;

The operator $\Delta_t^M:C\to C$ is defined by $\Delta_t^Mx=T_{r_{M,t}}^{(\Theta_M,\varphi_M)}(I-r_{M,t}A_M)\cdots T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I-r_{1,t}A_1)x, t\in(0,1)$, for $\{r_{j,t}\}\subset[c_j,d_j]\subset(0,2\mu_j),\ j=1,...,M$;

The operator $\Delta_n^M:C\to C$ is defined by $\Delta_n^Mx=T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)\cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I-r_{1,n}A_1)x$ with $\{r_{j,n}\}\subset[c_j,d_j]\subset(0,2\mu_j)$ and $\lim_{n\to\infty}r_{j,n}=r_j$, for each $j=1,...,M$;

$\Omega:=\cap_{j=1}^M\mathrm{GMEP}(\Theta_j,\varphi_j,A_j)\cap\cap_{i=1}^N\mathrm{I}(B_i,R_i)\cap\Xi\neq\emptyset$;

$\{\alpha_n\}\subset[0,1],\ \{s_n\}\subset(0,\min\{\frac{1}{2},\|A\|^{-1}\})$ and $\{s_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})}\subset(0,\min\{\frac{1}{2},\|A\|^{-1}\})$.

Next, set

$$\Lambda_t^i=J_{R_i,\lambda_{i,t}}(I-\lambda_{i,t}B_i)J_{R_{i-1},\lambda_{i-1,t}}(I-\lambda_{i-1,t}B_{i-1})\cdots J_{R_1,\lambda_{1,t}}(I-\lambda_{1,t}B_1),\ \forall t\in(0,1),$$

$$\Lambda_n^i=J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)J_{R_{i-1},\lambda_{i-1,n}}(I-\lambda_{i-1,n}B_{i-1})\cdots J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1),\ \forall n\geq0,$$

$$\Delta_t^j=T_{r_{j,t}}^{(\Theta_j,\varphi_j)}(I-r_{j,t}A_j)T_{r_{j-1,t}}^{(\Theta_{j-1},\varphi_{j-1})}(I-r_{j-1,t}A_{j-1})\cdots T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I-r_{1,t}A_1),\ \forall t\in(0,1),$$

and

$$\Delta_n^j=T_{r_{j,n}}^{(\Theta_j,\varphi_j)}(I-r_{j,n}A_j)T_{r_{j-1,n}}^{(\Theta_{j-1},\varphi_{j-1})}(I-r_{j-1,n}A_{j-1})\cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I-r_{1,n}A_1),\ \forall n\geq0,$$

for $j=1,...,M$ and $i=1,...,N$, $\Lambda_t^0=\Lambda_n^0=I$ and $\Delta_t^0=\Delta_n^0=I$.

By Proposition 3, $\lambda\nabla f$ is $\frac{1}{\lambda L}$-ism for $\lambda>0$. In addition, hence, $I-\lambda\nabla f$ is $\frac{\lambda L}{2}$-averaged. It is clear that $P_C(I-\lambda_t\nabla f)$ is $\frac{2+\lambda_tL}{4}$-averaged for each $\lambda_t\in(0,\frac{2}{L})$. Thus, $P_C(I-\lambda_t\nabla f)=s_tI+(1-s_t)T_t$, where T_t is nonexpansive and $s_t:=s_t(\lambda_t)=\frac{2-\lambda_tL}{4}\in(0,\frac{1}{2})$ for each $\lambda_t\in(0,\frac{2}{L})$. Similarly, for each $n\geq0$, $P_C(I-\lambda_n\nabla f)$ is $\frac{2+\lambda_nL}{4}$-averaged for each $\lambda_n\in(0,\frac{2}{L})$ and $P_C(I-\lambda_n\nabla f)=s_nI+(1-s_n)T_n$. Please note that $\mathrm{Fix}(T_t)=\mathrm{Fix}(T_n)=\Xi$. Since $\{\lambda_{i,t}\}\subset[a_i,b_i]\subset(0,2\eta_i)$, by (12) and Lemma 10, we deduce

$$\|\Lambda_t^Nx-\Lambda_t^Ny\|=\|J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}x-J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}y\|\leq\|x-y\|.$$

On the other hand, since $\{r_{i,t}\}\subset[c_i,d_i]\subset(0,2\mu_i)$, according to (12) and Proposition 2, we get

$$\begin{aligned}\|\Delta_t^Mx-\Delta_t^My\|&=\|T_{r_{M,t}}^{(\Theta_M,\varphi_M)}(I-r_{M,t}A_M)\Delta_t^{M-1}x-T_{r_{M,t}}^{(\Theta_M,\varphi_M)}(I-r_{M,t}A_M)\Delta_t^{M-1}y\|\\&\leq\|(I-r_{M,t}A_M)\Delta_t^{M-1}x-(I-r_{M,t}A_M)\Delta_t^{M-1}y\|\\&\leq\|x-y\|.\end{aligned}$$

Next, we present the following net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})}$:

$$\begin{cases}u_t=T_{r_{M,t}}^{(\Theta_M,\varphi_M)}(I-r_{M,t}A_M)T_{r_{M-1,t}}^{(\Theta_{M-1},\varphi_{M-1})}(I-r_{M-1,t}A_{M-1})\cdots T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I-r_{1,t}A_1)x_t,\\v_t=J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)J_{R_{N-1},\lambda_{N-1,t}}(I-\lambda_{N-1,t}B_{N-1})\cdots J_{R_1,\lambda_{1,t}}(I-\lambda_{1,t}B_1)u_t,\\x_t=P_C[(I-s_tA)T_tv_t+s_t[Vx_t-t(\mu FVx_t-\gamma T_tv_t)].\end{cases}\tag{14}$$

We prove the strong convergence of $\{x_t\}$ as $t\to0$ to a point $\tilde{x}\in\Omega$ which solves

$$\langle(A-V)\tilde{x},p-\tilde{x}\rangle\geq0,\quad\forall p\in\Omega.\tag{15}$$

Let $x_0 \in C$, define a sequence $\{x_n\}$ as follows

$$\begin{cases} u_n = T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}A_M)T_{r_{M-1,n}}^{(\Theta_{M-1},\varphi_{M-1})}(I - r_{M-1,n}A_{M-1}) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}A_1)x_n, \\ v_n = J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)J_{R_{N-1},\lambda_{N-1,n}}(I - \lambda_{N-1,n}B_{N-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ y_n = \alpha_n\gamma T_n v_n + (I - \alpha_n\mu F)Vx_n, \\ x_{n+1} = P_C[(I - s_nA)T_nv_n + s_ny_n], \ \forall n \geq 0. \end{cases} \tag{16}$$

We will show the convergence of $\{x_n\}$ as $n \to \infty$ to $\tilde{x} \in \Omega$, which solves (15).

Now, for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma}\})$, and $s_t \in (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$, let $Q_t : C \to C$ be defined by $Q_tx = P_C[(I - s_tA)T_t\Lambda_t^N\Delta_t^Mx + s_t[Vx - t(\mu FVx - \gamma T_t\Lambda_t^N\Delta_t^Mx)], \ \forall x \in C$. By Lemmas 5 and 7, we have

$$\begin{aligned} \|Q_tx - Q_ty\| &\leq \|(I - s_tA)T_t\Lambda_t^N\Delta_t^Mx - (I - s_tA)T_t\Lambda_t^N\Delta_t^My\| \\ &\quad + s_t\|((I - t\mu F)Vx + t\gamma T_t\Lambda_t^N\Delta_t^Mx) - ((I - t\mu F)Vy + t\gamma T_t\Lambda_t^N\Delta_t^My)\| \\ &\leq (1 - s_t\bar{\gamma})\|x - y\| + s_t[(1 - t\tau)\|x - y\| + t\gamma\|x - y\|] \\ &= [1 - s_t(\bar{\gamma} - 1 + t(\tau - \gamma))]\|x - y\|. \end{aligned}$$

It is easy to check that $0 < 1 - s_t(\bar{\gamma} - 1 + t(\tau - \gamma)) < 1$. Therefore, $Q_t : C \to C$ is contraction and Q_t has a unique fixed point, denoted by x_t.

Proposition 4. *Let $\{x_t\}$ be defined via (14). Then*

(i) $\{x_t\}$ *is bounded for* $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma}\})$;

(ii) $\lim_{t\to 0}\|x_t - T_tx_t\| = 0$, $\lim_{t\to 0}\|x_t - \Lambda_t^Nx_t\| = 0$ *and* $\lim_{t\to 0}\|x_t - \Delta_t^Mx_t\| = 0$ *provided* $\lim_{t\to 0}\lambda_t = \frac{2}{L}$;

(iii) $x_t, s_t, \lambda_{i,t}, r_{j,t}$ *are locally Lipschitzian.*

Proof. (i) Pick up $p \in \Omega$. Noting that $\text{Fix}(T_t) = \Xi$, $\Lambda_t^ip = p$ and $\Delta_t^jp = p$ for all $i \in \{1, ..., N\}$ and $j \in \{1, ..., M\}$, by the nonexpansivity of T_t, Λ_t^i and Δ_t^j and Lemmas 5 and 7 we get

$$\begin{aligned} \|x_t - p\| &\leq \|(I - s_tA)T_t\Lambda_t^N\Delta_t^Mx_t - (I - s_tA)T_t\Lambda_t^N\Delta_t^Mp\| + s_t\|(I - t\mu F)Vx_t + t\gamma T_t\Lambda_t^N\Delta_t^Mx_t - Ap\| \\ &\leq s_t\|(I - t\mu F)Vx_t - (I - t\mu F)Vp + t(\gamma T_t\Lambda_t^N\Delta_t^Mx_t - \mu FVp) + Vp - Ap\| \\ &\quad + (1 - s_t\bar{\gamma})\|T_t\Lambda_t^N\Delta_t^Mx_t - T_t\Lambda_t^N\Delta_t^Mp\| \\ &\leq [1 - s_t(\bar{\gamma} - 1 + t(\tau - \gamma))]\|x_t - p\| + s_t(t\|(\gamma I - \mu FV)p\| + \|(V - A)p\|). \end{aligned}$$

Thus,

$$\|x_t - p\| \leq \frac{\|(V - A)p\| + \|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1}.$$

Hence $\{x_t\}$ is bounded and so are $\{Vx_t\}, \{u_t\}, \{v_t\}, \{T_tv_t\}$ and $\{FVx_t\}$.

(ii) From (14), we obtain

$$\begin{aligned} \|x_t - T_tv_t\| &= \|P_C[(I - s_tA)T_tv_t + s_t((I - t\mu F)Vx_t + t\gamma T_tv_t)] - P_CT_tv_t\| \\ &\leq s_t\|Vx_t - AT_tv_t\| + t\|\gamma T_tv_t - \mu FVx_t\| \to 0 \text{ as } t \to 0. \end{aligned} \tag{17}$$

From (12), we have

$$\begin{aligned} \|v_t - p\|^2 &\leq \|\Lambda_t^iu_t - p\|^2 \\ &= \|J_{R_i,\lambda_{i,t}}(I - \lambda_{i,t}B_i)\Lambda_t^{i-1}u_t - J_{R_i,\lambda_{i,t}}(I - \lambda_{i,t}B_i)p\|^2 \\ &\leq \|x_t - p\|^2 + \lambda_{i,t}(\lambda_{i,t} - 2\eta_i)\|B_i\Lambda_t^{i-1}u_t - B_ip\|^2, \end{aligned} \tag{18}$$

and

$$\begin{aligned}\|u_t - p\|^2 &\le \|\Delta_t^j x_t - p\|^2 \\ &= \|T_{r_{j,t}}^{(\Theta_j,\varphi_j)}(I - r_{j,t}A_j)\Delta_t^{j-1}x_t - T_{r_{j,t}}^{(\Theta_j,\varphi_j)}(I - r_{j,t}A_j)p\|^2 \\ &\le \|x_t - p\|^2 + r_{j,t}(r_{j,t} - 2\mu_j)\|A_j\Delta_t^{j-1}x_t - A_jp\|^2.\end{aligned} \tag{19}$$

Simple calculations show that

$$\begin{aligned}x_t - p = x_t - w_t + (I - s_tA)(T_tv_t - T_tp) + s_t[(I - t\mu F)Vx_t - (I - t\mu F)Vp \\ + t\gamma(T_tv_t - T_tp) + t(\gamma I - \mu FV)p] + s_t(V - A)p,\end{aligned} \tag{20}$$

where $w_t = (I - s_tA)T_tv_t + s_t(t\gamma T_tv_t + (I - t\mu F)Vx_t)$. Then, by the nonexpansivity of T_t, Proposition 1, and Lemmas 5 and 7, from (18)–(20) we obtain that

$$\begin{aligned}\|x_t - p\|^2 &\le \langle (I - s_tA)(T_tv_t - T_tp), x_t - p\rangle + s_t[\langle (I - t\mu F)Vx_t - (I - t\mu F)Vp, x_t - p\rangle \\ &\quad + t\gamma\langle (T_tv_t - T_tp), x_t - p\rangle + t\langle (\gamma I - \mu FV)p, x_t - p\rangle] + s_t\langle (V - A)p, x_t - p\rangle \\ &\le (1 - s_t(\bar\gamma - t\gamma))\frac{1}{2}[\|x_t - p\|^2 + r_{j,t}(r_{j,t} - 2\mu_j)\|A_j\Delta_t^{j-1}x_t - A_jp\|^2 \\ &\quad + \lambda_{i,t}(\lambda_{i,t} - 2\eta_i)\|B_i\Lambda_t^{i-1}u_t - B_ip\|^2 + \|x_t - p\|^2] + s_t[(1 - t\tau)\|x_t - p\|^2 \\ &\quad + t\|(\gamma I - \mu FV)p\|\|x_t - p\|] + s_t\|(V - A)p\|\|x_t - p\| \\ &\le \|x_t - p\|^2 - \frac{1 - s_t(\bar\gamma - t\gamma)}{2}[r_{j,t}(2\mu_j - r_{j,t})\|A_j\Delta_t^{j-1}x_t - A_jp\|^2 + \lambda_{i,t}(2\eta_i - \lambda_{i,t}) \\ &\quad \times \|B_i\Lambda_t^{i-1}u_t - B_ip\|^2] + s_t(t\|(\gamma I - \mu FV)p\| + \|(V - A)p\|)\|x_t - p\|,\end{aligned} \tag{21}$$

which together with $\{\lambda_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i), i = 1, ..., N$ and $\{r_{j,t}\} \subset [c_j, d_j] \subset (0, 2\mu_j), j = 1, ..., M$, implies that

$$\begin{aligned}&\frac{1 - s_t(\bar\gamma - t\gamma)}{2}[c_j(2\mu_j - d_j)\|A_j\Delta_t^{j-1}x_t - A_jp\|^2 + a_i(2\eta_i - b_i)\|B_i\Lambda_t^{i-1}u_t - B_ip\|^2] \\ &\le \frac{1 - s_t(\bar\gamma - t\gamma)}{2}[r_{j,t}(2\mu_j - r_{j,t})\|A_j\Delta_t^{j-1}x_t - A_jp\|^2 + \lambda_{i,t}(2\eta_i - \lambda_{i,t})\|B_i\Lambda_t^{i-1}u_t - B_ip\|^2] \\ &\le s_t(t\|(\gamma I - \mu FV)p\| + \|(V - A)p\|)\|x_t - p\|.\end{aligned}$$

Since $\lim_{t\to 0} s_t = 0$ and $\{x_t\}$ is bounded, we have

$$\lim_{t\to 0}\|B_i\Lambda_t^{i-1}u_t - B_ip\| = 0 \text{ and } \lim_{t\to 0}\|A_j\Delta_t^{j-1}x_t - A_jp\| = 0 \ (i \in \{1, ..., N\}, j \in \{1, ..., M\}). \tag{22}$$

According to Proposition 2, we have

$$\begin{aligned}\|\Delta_t^jx_t - p\|^2 &\le \langle (I - r_{j,t}A_j)\Delta_t^{j-1}x_t - (I - r_{j,t}A_j)p, \Delta_t^jx_t - p\rangle \\ &\le \frac{1}{2}(\|\Delta_t^{j-1}x_t - p\|^2 + \|\Delta_t^jx_t - p\|^2 - \|\Delta_t^{j-1}x_t - \Delta_t^jx_t - r_{j,t}(A_j\Delta_t^{j-1}x_t - A_jp)\|^2),\end{aligned}$$

which implies that

$$\begin{aligned}\|\Delta_t^jx_t - p\|^2 &\le \|\Delta_t^{j-1}x_t - p\|^2 - \|\Delta_t^{j-1}x_t - \Delta_t^jx_t - r_{j,t}(A_j\Delta_t^{j-1}x_t - A_jp)\|^2 \\ &\le \|x_t - p\|^2 - \|\Delta_t^{j-1}x_t - \Delta_t^jx_t\|^2 + 2r_{j,t}\|\Delta_t^{j-1}x_t - \Delta_t^jx_t\|\|A_j\Delta_t^{j-1}x_t - A_jp\|.\end{aligned} \tag{23}$$

Also, by Lemma 10, we obtain that for each $i \in \{1, ..., N\}$

$$
\begin{aligned}
\|\Lambda_t^i u_t - p\|^2 &\le \langle (I-\lambda_{i,t}B_i)\Lambda_t^{i-1}u_t - (I-\lambda_{i,t}B_i)p, \Lambda_t^i u_t - p\rangle \\
&= \frac{1}{2}(\|(I-\lambda_{i,t}B_i)\Lambda_t^{i-1}u_t - (I-\lambda_{i,t}B_i)p\|^2 + \|\Lambda_t^i u_t - p\|^2 \\
&\quad - \|(I-\lambda_{i,t}B_i)\Lambda_t^{i-1}u_t - (I-\lambda_{i,t}B_i)p - (\Lambda_t^i u_t - p)\|^2) \\
&\le \frac{1}{2}(\|u_t - p\|^2 + \|\Lambda_t^i u_t - p\|^2 - \|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t - \lambda_{i,t}(B_i\Lambda_t^{i-1}u_t - B_i p)\|^2),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|\Lambda_t^i u_t - p\|^2 &\le \|u_t - p\|^2 - \|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t - \lambda_{i,t}(B_i\Lambda_t^{i-1}u_t - B_i p)\|^2 \\
&\le \|u_t - p\|^2 - \|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t\|^2 + 2\lambda_{i,t}\|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t\|\|B_i\Lambda_t^{i-1}u_t - B_i p\|.
\end{aligned} \tag{24}
$$

Thus, utilizing Lemma 1, from (21), (23) and (24) we have

$$
\begin{aligned}
\|x_t - p\|^2 &\le (1 - s_t(\bar{\gamma} - t\gamma))\frac{1}{2}(\|\Lambda_t^i u_t - p\|^2 + \|x_t - p\|^2) + s_t\|(V-A)p\|\|x_t - p\| \\
&\quad + s_t[(1-t\tau)\|x_t - p\|^2 + t\|(\gamma I - \mu FV)p\|\|x_t - p\|] \\
&\le (1 - s_t(\bar{\gamma} - t\gamma))\frac{1}{2}[\|x_t - p\|^2 - \|\Delta_t^{j-1}x_t - \Delta_t^j x_t\|^2 \\
&\quad + 2r_{j,t}\|\Delta_t^{j-1}x_t - \Delta_t^j x_t\|\|A_j\Delta_t^{j-1}x_t - A_j p\| - \|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t\|^2 \\
&\quad + 2\lambda_{i,t}\|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t\|\|B_i\Lambda_t^{i-1}u_t - B_i p\| + \|x_t - p\|^2] \\
&\quad + s_t[(1-t\tau)\|x_t - p\|^2 + t\|(\gamma I - \mu FV)p\|\|x_t - p\|] + s_t\|(V-A)p\|\|x_t - p\| \\
&\le \|x_t - p\|^2 - \frac{1 - s_t(\bar{\gamma} - t\gamma)}{2}(\|\Delta_t^{j-1}x_t - \Delta_t^j x_t\|^2 + \|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t\|^2) \\
&\quad + r_{j,t}\|\Delta_t^{j-1}x_t - \Delta_t^j x_t\|\|A_j\Delta_t^{j-1}x_t - A_j p\| + \lambda_{i,t}\|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t\|\|B_i\Lambda_t^{i-1}u_t - B_i p\| \\
&\quad + s_t(t\|(\gamma I - \mu FV)p\| + \|(V-A)p\|)\|x_t - p\|,
\end{aligned}
$$

which together with $\{\lambda_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{j,t}\} \subset [c_j, d_j] \subset (0, 2\mu_j)$, leads to

$$
\begin{aligned}
&\frac{1 - s_t(\bar{\gamma} - t\gamma)}{2}(\|\Delta_t^{j-1}x_t - \Delta_t^j x_t\|^2 + \|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t\|^2) \\
&\le d_j\|\Delta_t^{j-1}x_t - \Delta_t^j x_t\|\|A_j\Delta_t^{j-1}x_t - A_j p\| + b_i\|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t\|\|B_i\Lambda_t^{i-1}u_t - B_i p\| \\
&\quad + s_t(t\|(\gamma I - \mu FV)p\| + \|(V-A)p\|)\|x_t - p\|.
\end{aligned}
$$

Since $\lim_{t\to 0} s_t = 0,\ \lim_{t\to 0}\|A_j\Delta_t^{j-1}x_t - A_j p\| = 0$ and $\lim_{t\to 0}\|B_i\Lambda_t^{i-1}u_t - B_i p\| = 0$ (due to (22)), we deduce from the boundedness of $\{x_t\},\ \{\Lambda_t^i u_t\}$ and $\{\Delta_t^j x_t\}$ that

$$
\lim_{t\to 0}\|\Lambda_t^{i-1}u_t - \Lambda_t^i u_t\| = 0 \text{ and } \lim_{t\to 0}\|\Delta_t^{j-1}x_t - \Delta_t^j x_t\| = 0. \tag{25}
$$

Hence we get

$$
\begin{aligned}
\|x_t - u_t\| &\le \|\Delta_t^0 x_t - \Delta_t^1 x_t\| + \|\Delta_t^1 x_t - \Delta_t^2 x_t\| + \cdots + \|\Delta_t^{M-1}x_t - \Delta_t^M x_t\| \\
&\to 0 \text{ as } t \to 0,
\end{aligned} \tag{26}
$$

and

$$\begin{aligned} \|u_t - v_t\| &\le \|\Lambda_t^0 u_t - \Lambda_t^1 u_t\| + \|\Lambda_t^1 u_t - \Lambda_t^2 u_t\| + \cdots + \|\Lambda_t^{N-1} u_t - \Lambda_t^N u_t\| \\ &\to 0 \text{ as } t \to 0. \end{aligned} \tag{27}$$

So, taking into account that $\|x_t - v_t\| \le \|x_t - u_t\| + \|u_t - v_t\|$, we have

$$\lim_{t\to 0} \|x_t - v_t\| = 0. \tag{28}$$

In the meantime, from the nonexpansivity of T_t and Λ_t^N, it is easy to see that

$$\begin{aligned} \|x_t - T_t x_t\| &\le \|x_t - T_t v_t\| + \|T_t v_t - T_t x_t\| \\ &\le \|x_t - T_t v_t\| + \|v_t - x_t\|, \end{aligned}$$

and

$$\begin{aligned} \|x_t - \Lambda_t^N x_t\| &\le \|x_t - \Lambda_t^N u_t\| + \|\Lambda_t^N u_t - \Lambda_t^N x_t\| \\ &\le \|x_t - v_t\| + \|u_t - x_t\|. \end{aligned}$$

Consequently, from (17), (26) and (28) we immediately deduce that

$$\lim_{t\to 0} \|x_t - T_t x_t\| = 0 \text{ and } \lim_{t\to 0} \|x_t - \Lambda_t^N x_t\| = 0. \tag{29}$$

(iii) Since ∇f is $\frac{1}{L}$-ism, $P_C(I - \lambda_t \nabla f)$ is nonexpansive for $\lambda_t \in (0, \frac{2}{L})$. Then,

$$\begin{aligned} \|P_C(I - \lambda_t \nabla f) v_{t_0}\| &\le \|P_C(I - \lambda_t \nabla f) v_{t_0} - P_C(I - \lambda_t \nabla f) p\| + \|p\| \\ &\le \|v_{t_0}\| + 2\|p\|. \end{aligned}$$

This implies that $\{P_C(I - \lambda_t \nabla f) v_{t_0}\}$ is bounded. Also, observe that

$$\begin{aligned} \|T_t v_{t_0} - T_{t_0} v_{t_0}\| &\le \frac{4L|\lambda_{t_0} - \lambda_t| \|P_C(I - \lambda_t \nabla f) v_{t_0}\|}{(2 + \lambda_t L)(2 + \lambda_{t_0} L)} + \frac{4L|\lambda_t - \lambda_{t_0}|}{(2 + \lambda_t L)(2 + \lambda_{t_0} L)} \|v_{t_0}\| \\ &\quad + \frac{4(2 + \lambda_t L) \|P_C(I - \lambda_t \nabla f) v_{t_0} - P_C(I - \lambda_{t_0} \nabla f) v_{t_0}\|}{(2 + \lambda_t L)(2 + \lambda_{t_0} L)} \\ &\le |\lambda_t - \lambda_{t_0}| [L\|P_C(I - \lambda_t \nabla f) v_{t_0}\| + 4\|\nabla f(v_{t_0})\| + L\|v_{t_0}\|] \\ &\le \widetilde{M} |\lambda_t - \lambda_{t_0}|, \end{aligned} \tag{30}$$

where $\sup_t \{L\|P_C(I - \lambda_t \nabla f) v_{t_0}\| + 4\|\nabla f(v_{t_0})\| + L\|v_{t_0}\|\} \le \widetilde{M}$ for some $\widetilde{M} > 0$. So, by (30), we have that

$$\begin{aligned} \|T_t v_t - T_{t_0} v_{t_0}\| &\le \|T_t v_t - T_t v_{t_0}\| + \|T_t v_{t_0} - T_{t_0} v_{t_0}\| \\ &\le \|v_t - v_{t_0}\| + \widetilde{M} |\lambda_t - \lambda_{t_0}| \\ &\le \|v_t - v_{t_0}\| + \frac{4\widetilde{M}}{L} |s_t - s_{t_0}|. \end{aligned} \tag{31}$$

Utilizing (12) and (13), we obtain that

$$
\begin{aligned}
\|v_t - v_{t_0}\| &\le \|J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}u_t - J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}u_t\| \\
&\quad + \|J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}u_t - J_{R_N,\lambda_{N,t_0}}(I-\lambda_{N,t_0}B_N)\Lambda_{t_0}^{N-1}u_{t_0}\| \\
&\le \|(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}u_t - (I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}u_t\| \\
&\quad + \|(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}u_t - (I-\lambda_{N,t_0}B_N)\Lambda_{t_0}^{N-1}u_{t_0}\| + |\lambda_{N,t}-\lambda_{N,t_0}| \\
&\quad \times (\frac{1}{\lambda_{N,t}}\|J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}u_t - (I-\lambda_{N,t_0}B_N)\Lambda_{t_0}^{N-1}u_{t_0}\| \\
&\quad + \frac{1}{\lambda_{N,t_0}}\|(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}u_t - J_{R_N,\lambda_{N,t_0}}(I-\lambda_{N,t_0}B_N)\Lambda_{t_0}^{N-1}u_{t_0}\|) \\
&\le |\lambda_{N,t}-\lambda_{N,t_0}|(\|B_N\Lambda_t^{N-1}u_t\| + \widehat{M}) + \|\Lambda_t^{N-1}u_t - \Lambda_{t_0}^{N-1}u_{t_0}\| \\
&\le \widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,t}-\lambda_{i,t_0}| + \|u_t - u_{t_0}\|,
\end{aligned} \tag{32}
$$

where

$$
\begin{aligned}
&\sup_t\{\frac{1}{\lambda_{i,t}}\|J_{R_i,\lambda_{i,t}}(I-\lambda_{i,t_0}B_i)\Lambda_t^{i-1}u_t - (I-\lambda_{i,t_0}B_i)\Lambda_{t_0}^{i-1}u_{t_0}\| \\
&\quad + \frac{1}{\lambda_{i,t_0}}\|(I-\lambda_{i,t_0}B_i)\Lambda_t^{i-1}u_t - J_{R_i,\lambda_{i,t_0}}(I-\lambda_{i,t_0}B_i)\Lambda_{t_0}^{i-1}u_{t_0}\|\} \le \widehat{M},
\end{aligned}
$$

for some $\widehat{M} > 0$ and $\sup_t\{\sum_{i=1}^N \|B_i\Lambda_t^{i-1}u_t\| + \widehat{M}\} \le \widetilde{M}_0$ for some $\widetilde{M}_0 > 0$. Also, utilizing Proposition 2, we deduce that

$$
\begin{aligned}
\|u_t - u_{t_0}\| &\le \|T_{r_{M,t}}^{(\Theta_M,\varphi_M)}(I-r_{M,t}B_M)\Delta_t^{M-1}x_t - T_{r_{M,t_0}}^{(\Theta_M,\varphi_M)}(I-r_{M,t}B_M)\Delta_t^{M-1}x_t\| \\
&\quad + \|T_{r_{M,t_0}}^{(\Theta_M,\varphi_M)}(I-r_{M,t}B_M)\Delta_t^{M-1}x_t - T_{r_{M,t_0}}^{(\Theta_M,\varphi_M)}(I-r_{M,t_0}B_M)\Delta_t^{M-1}x_t\| \\
&\quad + \|(I-r_{M,t_0}B_M)\Delta_t^{M-1}x_t - (I-r_{M,t_0}B_M)\Delta_{t_0}^{M-1}x_{t_0}\| \\
&\le |r_{M,t}-r_{M,t_0}|[\|B_M\Delta_t^{M-1}x_t\| + \frac{1}{r_{M,t}}\|T_{r_{M,t}}^{(\Theta_M,\varphi_M)}(I-r_{M,t}B_M)\Delta_t^{M-1}x_t \\
&\quad - (I-r_{M,t}B_M)\Delta_t^{M-1}x_t\|] + \cdots + \|\Delta_t^0 x_t - \Delta_{t_0}^0 x_{t_0}\| \\
&\quad + |r_{1,t}-r_{1,t_0}|[\|B_1\Delta_t^0 x_t\| + \frac{1}{r_{1,t}}\|T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I-r_{1,t}B_1)\Delta_t^0 x_t - (I-r_{1,t}B_1)\Delta_t^0 x_t\|] \\
&\le \widetilde{M}_1\sum_{j=1}^{M}|r_{j,t}-r_{j,t_0}| + \|x_t - x_{t_0}\|,
\end{aligned} \tag{33}
$$

where $\widetilde{M}_1 > 0$ is a constant and

$$
\sum_{j=1}^{M}[\|B_j\Delta_t^{j-1}x_t\| + \frac{1}{r_{j,t}}\|T_{r_{j,t}}^{(\Theta_j,\varphi_j)}(I-r_{j,t}B_j)\Delta_t^{j-1}x_t - (I-r_{j,t}B_j)\Delta_t^{j-1}x_t\|] \le \widetilde{M}_1.
$$

In terms of (31)–(33), we calculate

$$
\|T_t v_t - T_{t_0}v_{t_0}\| \le \|x_t - x_{t_0}\| + \widetilde{M}_1\sum_{j=1}^{M}|r_{j,t}-r_{j,t_0}| + \widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,t}-\lambda_{i,t_0}| + \frac{4\widetilde{M}}{L}|s_t - s_{t_0}|,
$$

and hence

$$\begin{aligned}\|x_t - x_{t_0}\| &\leq \|(I - s_t A)T_t v_t + s_t((I - t\mu F)Vx_t + t\gamma T_t v_t) - (I - s_{t_0}A)T_{t_0}v_{t_0} \\ &\quad - s_{t_0}((I - t_0\mu F)Vx_{t_0} + t_0\gamma T_{t_0}v_{t_0})\| \\ &\leq |s_t - s_{t_0}|\|A\|\|T_t v_t\| + (1 - s_{t_0}\bar{\gamma})\|T_t v_t - T_{t_0}v_{t_0}\| + t_0\gamma\|T_t v_t - T_{t_0}v_{t_0}\| \\ &\quad + |s_t - s_{t_0}|[\|Vx_t\| + t(\gamma\|T_t v_t\| + \mu\|FVx_t\|)] + s_{t_0}[(\gamma\|T_t v_t\| + \mu\|FVx_t\|)|t - t_0| \\ &\quad + (1 - t_0\tau)\|x_t - x_{t_0}\|] \\ &\leq (1 - s_{t_0}(\bar{\gamma} - 1 + t_0(\tau - \gamma)))(\|x_t - x_{t_0}\| + \widetilde{M}_1\sum_{j=1}^{M}|r_{j,t} - r_{j,t_0}| + \widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,t} - \lambda_{i,t_0}| \\ &\quad + \frac{4\widetilde{M}}{L}|s_t - s_{t_0}|) + |s_t - s_{t_0}|[\|A\|\|T_t v_t\| + \|Vx_t\| + t(\gamma\|T_t v_t\| + \mu\|FVx_t\|)] \\ &\quad + s_{t_0}(\gamma\|T_t v_t\| + \mu\|FVx_t\|)|t - t_0|.\end{aligned}$$

This immediately implies that

$$\begin{aligned}\|x_t - x_{t_0}\| &\leq \frac{\|A\|\|T_t v_t\| + \|Vx_t\| + t(\gamma\|T_t v_t\| + \mu\|FVx_t\|) + \frac{4\widetilde{M}}{L}}{s_{t_0}(\bar{\gamma} - 1 + t_0(\tau - \gamma))}|s_t - s_{t_0}| \\ &\quad + \frac{\gamma\|T_t v_t\| + \mu\|FVx_t\|}{\bar{\gamma} - 1 + t_0(\tau - \gamma)}|t - t_0| + \frac{\widetilde{M}_1}{s_{t_0}(\bar{\gamma} - 1 + t_0(\tau - \gamma l))}\sum_{j=1}^{M}|r_{j,t} - r_{j,t_0}| \\ &\quad + \frac{\widetilde{M}_0}{s_{t_0}(\bar{\gamma} - 1 + t_0(\tau - \gamma l))}\sum_{i=1}^{N}|\lambda_{i,t} - \lambda_{i,t_0}|.\end{aligned}$$

Since $s_t, \lambda_{i,t}, r_{j,t}$ are locally Lipschitzian, we conclude that x_t is locally Lipschitzian. □

Theorem 1. *Assume that* $\lim_{t\to 0} s_t = 0$. *Then,* x_t *defined by* (14) *converges strongly to* $\tilde{x} \in \Omega$ *as* $t \to 0$, *which solves* (15).

Proof. Let $\tilde{x}$ be the unique solution of (15). Let $p \in \Omega$. Then, we have

$$\begin{aligned}x_t - p &= x_t - w_t + s_t[(I - t\mu F)Vx_t - (I - t\mu F)Vp + t\gamma(T_t v_t - T_t p) + t(\gamma I - \mu FV)p] \\ &\quad + (I - s_t A)(T_t v_t - T_t p) + s_t(V - A)p,\end{aligned}$$

where $w_t = (I - s_t A)T_t v_t + s_t((I - t\mu F)Vx_t + t\gamma T_t v_t)$. Then, by Proposition 1 and the nonexpansivity of T_t, we obtain from (18) that

$$\begin{aligned}\|x_t - p\|^2 &\leq \langle (I - s_t A)(T_t v_t - T_t p), x_t - p\rangle + s_t[\langle (I - t\mu F)Vx_t - (I - t\mu F)Vp, x_t - p\rangle \\ &\quad + t\gamma\langle T_t v_t - T_t p, x_t - p\rangle + t\langle (\gamma I - \mu FV)p, x_t - p\rangle] + s_t\langle (V - A)p, x_t - p\rangle \\ &\leq (1 - s_t\bar{\gamma})\|v_t - p\|\|x_t - p\| + s_t[(1 - t\tau)\|x_t - p\|^2 \\ &\quad + t\gamma\|v_t - p\|\|x_t - p\| + t\langle (\gamma I - \mu FV)p, x_t - p\rangle] + s_t\langle (V - A)p, x_t - p\rangle \\ &\leq [1 - s_t((\bar{\gamma} - 1) + t(\tau - \gamma))]\|x_t - p\|^2 + \langle (V - A)p, x_t - p\rangle + s_t(t\langle (\gamma I - \mu FV)p, x_t - p\rangle).\end{aligned}$$

Hence,

$$\|x_t - p\|^2 \leq \frac{1}{\bar{\gamma} - 1 + t(\tau - \gamma)}(\langle (V - A)p, x_t - p\rangle + t\langle (\gamma I - \mu FV)p, x_t - p\rangle). \tag{34}$$

It follows that $x_{t_n} \to x^*$, where $\{t_n\} \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma}\})$ such that $t_n \to 0$. By Proposition 4, we get $\lim_{n\to\infty}\|x_{t_n} - T_{t_n}x_{t_n}\| = 0$. Observe that

$$\begin{aligned}\|P_C(I - \lambda_{t_n}\nabla f)x_{t_n} - x_{t_n}\| &= \|s_{t_n}x_{t_n} + (1 - s_{t_n})T_{t_n}x_{t_n} - x_{t_n}\| \\ &\leq \|T_{t_n}x_{t_n} - x_{t_n}\|,\end{aligned}$$

where $s_{t_n} = \frac{2-\lambda_{t_n} L}{4} \in (0, \frac{1}{2})$ for $\lambda_{t_n} \in (0, \frac{2}{L})$. Hence we have

$$\begin{aligned}\|P_C(I - \frac{2}{L}\nabla f)x_{t_n} - x_{t_n}\| &\leq \|(I - \frac{2}{L}\nabla f)x_{t_n} - (I - \lambda_{t_n}\nabla f)x_{t_n}\| + \|P_C(I - \lambda_{t_n}\nabla f)x_{t_n} - x_{t_n}\| \\ &\leq (\frac{2}{L} - \lambda_{t_n})\|\nabla f(x_{t_n})\| + \|T_{t_n}x_{t_n} - x_{t_n}\|.\end{aligned}$$

By the boundedness of $\{x_{t_n}\}$, $s_{t_n} \to 0$ ($\Leftrightarrow$ $\lambda_{t_n} \to \frac{2}{L}$) and $\|T_{t_n}x_{t_n} - x_{t_n}\| \to 0$, we deduce

$$\|x^* - P_C(I - \frac{2}{L}\nabla f)x^*\| = \lim_{n\to\infty}\|x_{t_n} - P_C(I - \frac{2}{L}\nabla f)x_{t_n}\| = 0.$$

Therefore, $x^* \in \mathrm{VI}(C, \nabla f) = \Xi$.

Furthermore, from (25), (26) and (28), we have that $\Delta^j_{t_n}x_{t_n} \to x^*$, $\Lambda^m_{t_n}u_{t_n} \to x^*$, $u_{t_n} \to x^*$ and $v_{t_n} \to x^*$. First, we prove that $x^* \in \cap_{m=1}^N \mathrm{I}(B_m, R_m)$. Please note that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$, i.e., $g - B_m v \in R_m v$. Noting that $\Lambda^m_{t_n}u_{t_n} = J_{R_m,\lambda_{m,t_n}}(I - \lambda_{m,t_n}B_m)\Lambda^{m-1}_{t_n}u_{t_n}, m \in \{1, ..., N\}$, we have

$$\Lambda^{m-1}_{t_n}u_{t_n} - \lambda_{m,t_n}B_m\Lambda^{m-1}_{t_n}u_{t_n} \in (I + \lambda_{m,t_n}R_m)\Lambda^m_{t_n}u_{t_n},$$

that is,

$$\frac{1}{\lambda_{m,t_n}}(\Lambda^{m-1}_{t_n}u_{t_n} - \Lambda^m_{t_n}u_{t_n} - \lambda_{m,t_n}B_m\Lambda^{m-1}_{t_n}u_{t_n}) \in R_m\Lambda^m_{t_n}u_{t_n}.$$

According to the monotonicity of R_m, we have

$$\langle v - \Lambda^m_{t_n}u_{t_n}, g - B_m v - \frac{1}{\lambda_{m,t_n}}(\Lambda^{m-1}_{t_n}u_{t_n} - \Lambda^m_{t_n}u_{t_n} - \lambda_{m,t_n}B_m\Lambda^{m-1}_{t_n}u_{t_n})\rangle \geq 0$$

and hence

$$\begin{aligned}\langle v - \Lambda^m_{t_n}u_{t_n}, g\rangle &\geq \langle v - \Lambda^m_{t_n}u_{t_n}, B_m v + \frac{1}{\lambda_{m,t_n}}(\Lambda^{m-1}_{t_n}u_{t_n} - \Lambda^m_{t_n}u_{t_n} - \lambda_{m,t_n}B_m\Lambda^{m-1}_{t_n}u_{t_n})\rangle \\ &\geq \langle v - \Lambda^m_{t_n}u_{t_n}, B_m\Lambda^m_{t_n}u_{t_n} - B_m\Lambda^{m-1}_{t_n}u_{t_n}\rangle + \langle v - \Lambda^m_{t_n}u_{t_n}, \frac{1}{\lambda_{m,t_n}}(\Lambda^{m-1}_{t_n}u_{t_n} - \Lambda^m_{t_n}u_{t_n})\rangle.\end{aligned}$$

Since $\|\Lambda^m_{t_n}u_{t_n} - \Lambda^{m-1}_{t_n}u_{t_n}\| \to 0$ (due to (25)) and $\|B_m\Lambda^m_{t_n}u_{t_n} - B_m\Lambda^{m-1}_{t_n}u_{t_n}\| \to 0$, we deduce from $\Lambda^m_{t_n}u_{t_n} \to x^*$ and $\{\lambda_{m,t_n}\} \subset [a_m, b_m] \subset (0, 2\eta_m)$ that

$$\lim_{n\to\infty}\langle v - \Lambda^m_{t_n}u_{t_n}, g\rangle = \langle v - x^*, g\rangle \geq 0.$$

By the maximal monotonicity of $B_m + R_m$, we derive $0 \in (R_m + B_m)x^*$, i.e., $x^* \in \mathrm{I}(B_m, R_m)$. Thus, $x^* \in \bigcap_{m=1}^N \mathrm{I}(B_m, R_m)$. Next we prove that $x^* \in \cap_{j=1}^M \mathrm{GMEP}(\Theta_j, \varphi_j, A_j)$. Since $\Delta^j_{t_n}x_{t_n} = T^{(\Theta_j,\varphi_j)}_{r_{j,t_n}}(I - r_{j,t_n}A_j)\Delta^{j-1}_{t_n}x_{t_n}, j \in \{1, ..., M\}$, we have

$$\Theta_j(\Delta^j_{t_n}x_{t_n}, y) + \varphi_j(y) - \varphi_j(\Delta^j_{t_n}x_{t_n}) + \langle A_j\Delta^{j-1}_{t_n}x_{t_n}, y - \Delta^j_{t_n}x_{t_n}\rangle + \langle y - \Delta^j_{t_n}x_{t_n}, \frac{\Delta^j_{t_n}x_{t_n} - \Delta^{j-1}_{t_n}x_{t_n}}{r_{j,t_n}}\rangle \geq 0.$$

By (A2), we have

$$\varphi_j(y) - \varphi_j(\Delta^j_{t_n}x_{t_n}) + \langle A_j\Delta^{j-1}_{t_n}x_{t_n}, y - \Delta^j_{t_n}x_{t_n}\rangle + \langle y - \Delta^j_{t_n}x_{t_n}, \frac{\Delta^j_{t_n}x_{t_n} - \Delta^{j-1}_{t_n}x_{t_n}}{r_{j,t_n}}\rangle \geq \Theta_j(y, \Delta^j_{t_n}x_{t_n}).$$

Let $y \in C$ and $t \in (0,1]$. Set $z_t = ty + (1-t)x^*$. Thus, $z_t \in C$. Hence,

$$\begin{aligned} \langle z_t - \Delta^j_{t_n} x_{t_n}, A_j z_t \rangle \geq \varphi_j(\Delta^j_{t_n} x_{t_n}) - \varphi_j(z_t) + \langle z_t - \Delta^j_{t_n} x_{t_n}, A_j z_t - A_j \Delta^j_{t_n} x_{t_n} \rangle + \Theta_j(z_t, \Delta^j_{t_n} x_{t_n}) \\ + \langle z_t - \Delta^j_{t_n} x_{t_n}, A_j \Delta^j_{t_n} x_{t_n} - A_j \Delta^{j-1}_{t_n} x_{t_n} \rangle - \langle z_t - \Delta^j_{t_n} x_{t_n}, \frac{\Delta^j_{t_n} x_{t_n} - \Delta^{j-1}_{t_n} x_{t_n}}{r_{j,t_n}} \rangle. \end{aligned} \tag{35}$$

By (25), we have $\|A_j \Delta^j_{t_n} x_{t_n} - A_j \Delta^{j-1}_{t_n} x_{t_n}\| \to 0$ as $n \to \infty$. Please note that $\langle z_t - \Delta^j_{t_n} x_{t_n}, A_j z_t - A_j \Delta^j_{t_n} x_{t_n} \rangle \geq 0$. Then,

$$\langle z_t - x^*, A_j z_t \rangle \geq \varphi_j(x^*) - \varphi_j(z_t) + \Theta_j(z_t, x^*). \tag{36}$$

Applying (A1), (A4) and (36), we obtain

$$\begin{aligned} 0 &= \Theta_j(z_t, z_t) + \varphi_j(z_t) - \varphi_j(z_t) \\ &\leq t[\Theta_j(z_t, y) + \varphi_j(y) - \varphi_j(z_t)] + (1-t)t\langle y - x^*, A_j z_t \rangle, \end{aligned}$$

and hence

$$0 \leq \Theta_j(z_t, y) + \varphi_j(y) - \varphi_j(z_t) + (1-t)\langle y - x^*, A_j z_t \rangle.$$

Thus,

$$0 \leq \Theta_j(x^*, y) + \varphi_j(y) - \varphi_j(x^*) + \langle y - x^*, A_j x^* \rangle.$$

So, $x^* \in \text{GMEP}(\Theta_j, \varphi_j, A_j)$ and $x^* \in \cap_{j=1}^M \text{GMEP}(\Theta_j, \varphi_j, A_j)$. Therefore, $x^* \in \cap_{j=1}^M \text{GMEP}(\Theta_j, \varphi_j, A_j) \cap \cap_{m=1}^N \text{I}(B_m, R_m) \cap \Xi =: \Omega$.

Next, we prove that $x_t \to \tilde{x}$ as $t \to 0$. First, let us assert that x^* is a solution of the VIP (15). As a matter of fact, since $x_t = x_t - w_t + (I - s_t A)T_t \Lambda^N_t \Delta^M_t x_t + s_t((I - t\mu F)V x_t + t\gamma T_t \Lambda^N_t \Delta^M_t x_t)$, we have

$$x_t - T_t \Lambda^N_t \Delta^M_t x_t = x_t - w_t + s_t(V - A)T_t \Lambda^N_t \Delta^M_t x_t + s_t(V x_t - V T_t \Lambda^N_t \Delta^M_t x_t + t(\gamma T_t \Lambda^N_t \Delta^M_t x_t - \mu F V x_t)).$$

Since T_t, Λ^N_t and Δ^M_t are nonexpansive mappings, $I - T_t \Lambda^N_t \Delta^M_t$ is monotone. By the monotonicity of $I - T_t \Lambda^N_t \Delta^M_t$, we have

$$\begin{aligned} 0 &\leq \langle (I - T_t \Lambda^N_t \Delta^M_t)x_t - (I - T_t \Lambda^N_t \Delta^M_t)p, x_t - p \rangle \\ &\leq s_t \langle (V - A)T_t v_t - (V - A)x_t, x_t - p \rangle + s_t \langle V x_t - V T_t v_t, x_t - p \rangle \\ &\quad + s_t \langle (V - A)x_t, x_t - p \rangle + s_t t \langle (\gamma T_t v_t - \mu F V x_t), x_t - p \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} \langle (A - V)x_t, x_t - p \rangle &\leq \|(V - A)T_t v_t - (V - A)x_t\| \|x_t - p\| + \|V x_t - V T_t v_t\| \|x_t - p\| \\ &\quad + t\|\gamma T_t v_t - \mu F V x_t\| \|x_t - p\| \\ &\leq (2 + \|A\|)\|T_t v_t - x_t\| \|x_t - p\| + t(\gamma \|T_t v_t\| + \mu \|F V x_t\|)\|x_t - p\|. \end{aligned} \tag{37}$$

Now, replacing t in (37) with t_n and noticing the boundedness of $\{\gamma \|T_{t_n} v_{t_n}\| + \mu \|F V x_{t_n}\|\}$ and the fact that $(V - A)T_{t_n} v_{t_n} - (V - A)x_{t_n} \to 0$, we deduce

$$\langle (A - V)x^*, x^* - p \rangle \leq 0.$$

Thus, $x^* \in \Omega$ is a solution of (15); hence $x^* = \tilde{x}$ by uniqueness. Consequently, $x_t \to \tilde{x}$ as $t \to 0$. □

Theorem 2. *Assume that the sequences* $\{\alpha_n\} \subset [0,1]$ *and* $\{s_n\} \subset (0, \frac{1}{2})$ *satisfy:* (C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} s_n = 0$. *Let the sequence* $\{x_n\}$ *be defined by* (16). *Then* $\mathrm{LIM}_n\langle (A - V)\tilde{x}, \tilde{x} - x_n\rangle \leq 0$, *where* $\tilde{x} = \lim_{t\to 0^+} x_t$ *and* x_t *is defined by*

$$x_t = P_C[(I - s_t A)T\Lambda^N\Delta^M x_t + s_t(Vx_t - t(\mu FVx_t - \gamma T\Lambda^N\Delta^M x_t))], \tag{38}$$

where $T, \Lambda^N, \Delta^M : C \to C$ *are defined by* $Tx = P_C(I - \frac{2}{L}\nabla f)x$, $\Lambda^N x = J_{R_N,\lambda_N}(I - \lambda_N B_N)\cdots J_{R_1,\lambda_1}(I - \lambda_1 B_1)x$ *and* $\Delta^M x = T_{r_M}^{(\Theta_M,\varphi_M)}(I - r_M A_M)\cdots T_{r_1}^{(\Theta_1,\varphi_1)}(I - r_1 A_1)x$ *for* $\lambda_i \in [a_i, b_i] \subset (0, 2\eta_i), i = 1, ..., N$ *and* $r_j \in [c_j, d_j] \subset (0, 2\mu_j), j = 1, ..., M$.

Proof. We assume, without loss of generality, that $0 < s_n \leq \|A\|^{-1}$ for all $n \geq 0$. Let $\lim_{t\to 0} x_t = \tilde{x}$ and $\tilde{x}$ is the unique solution of (15). By Proposition 4, we deduce that the nets $\{x_t\}$, $\{Vx_t\}$, $\{\Delta^M x_t\}$, $\{\Lambda^N\Delta^M x_t\}$ and $\{FVx_t\}$ are bounded.

Let $p \in \Omega$. Then $T_n p = p$, $\Lambda_n^N p = p$ and $\Delta_n^M p = p$. Applying Lemmas 5 and 7, we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq s_n[(1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\gamma\|v_n - p\| + \alpha_n\|(\gamma I - \mu FV)p\|] \\
&\quad + (1 - s_n\bar{\gamma})\|v_n - p\| + s_n\|(V - A)p\| \\
&\leq s_n[(1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\gamma\|x_n - p\| + \alpha_n\|(\gamma I - \mu FV)p\|] \\
&\quad + s_n\|(V - A)p\| + (1 - s_n\bar{\gamma})\|x_n - p\| \\
&\leq \max\{\frac{\|(V - A)p\| + \|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1}, \|x_n - p\|\}.
\end{aligned}$$

By induction

$$\|x_n - p\| \leq \max\{\frac{\|(V - A)p\| + \|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1}, \|x_0 - p\|\}, \quad \forall n \geq 0.$$

Thus, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{T_n v_n\}$, $\{FVx_n\}$, $\{Vx_n\}$ and $\{y_n\}$ are all bounded. By (C1), we obtain

$$\|x_{n+1} - T_n v_n\| \leq \|(I - s_n A)T_n v_n + s_n y_n - T_n v_n\| = s_n\|y_n - AT_n v_n\| \to 0 \text{ as } n \to \infty.$$

By (31), we also have

$$\|T\Lambda^N\Delta^M x_n - T_n\Lambda_n^N\Delta_n^M x_n\| \leq \|\Lambda^N\Delta^M x_n - \Lambda_n^N\Delta_n^M x_n\| + \widehat{M}|\frac{2}{L} - \lambda_n|, \tag{39}$$

where $\sup_{n\geq 0}\{L\|P_C(I - \frac{2}{L}\nabla f)v_n\| + 4\|\nabla f(v_n)\| + L\|v_n\|\} \leq \widehat{M}$ for some $\widehat{M} > 0$. According to (32), we have

$$\|\Lambda^N\Delta^M x_n - \Lambda_n^N\Delta_n^M x_n\| \leq \widehat{M}_0\sum_{i=1}^{N}|\lambda_i - \lambda_{i,n}| + \|\Delta^M x_n - \Delta_n^M x_n\|, \tag{40}$$

where

$$\begin{aligned}
&\sup_{n\geq 0}\{\frac{1}{\lambda_i}\|J_{R_i,\lambda_i}(I - \lambda_{i,n}B_i)\Lambda^{i-1}\Delta^M x_n - (I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n\| \\
&\quad + \frac{1}{\lambda_{i,n}}\|(I - \lambda_{i,n}B_i)\Lambda^{i-1}\Delta^M x_n - J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n\|\} \leq \widehat{N},
\end{aligned}$$

for some $\widehat{N} > 0$ and $\sup_{n\geq 0}\{\sum_{i=1}^{N}\|B_i\Lambda^{i-1}\Delta^M x_n\| + \widehat{N}\} \leq \widehat{M}_0$ for some $\widehat{M}_0 > 0$. In terms of (33), we have

$$\|\Delta^M x_n - \Delta_n^M x_n\| \leq \widehat{M}_1\sum_{j=1}^{M}|r_j - r_{j,n}|, \tag{41}$$

where $\sup_{n\geq 0}\{\sum_{j=1}^{M}[\|A_j\Delta^{j-1}x_n\| + \frac{1}{r_j}\|T_{r_j}^{(\Theta_j,\varphi_j)}(I-r_jA_j)\Delta^{j-1}x_n - (I-r_jA_j)\Delta^{j-1}x_n\|]\} \leq \widehat{M}_1$ for some $\widehat{M}_1 > 0$. In terms of (39)–(41) we calculate

$$\|T\Lambda^N\Delta^M x_n - T_n\Lambda_n^N\Delta_n^M x_n\| \leq \widehat{M}_1\sum_{j=1}^{M}|r_j - r_{j,n}| + \widehat{M}_0\sum_{i=1}^{N}|\lambda_i - \lambda_{i,n}| + \widehat{M}|\frac{2}{L} - \lambda_n|.$$

It is clear that

$$\begin{aligned}\|T\Lambda^N\Delta^M x_t - x_{n+1}\| &\leq \|T\Lambda^N\Delta^M x_t - T\Lambda^N\Delta^M x_n\| + \|T\Lambda^N\Delta^M x_n - T_n\Lambda_n^N\Delta_n^M x_n\| \\ &\quad + \|T_n\Lambda_n^N\Delta_n^M x_n - x_{n+1}\| \\ &\leq \|x_t - x_n\| + \widehat{M}_1\sum_{j=1}^{M}|r_j - r_{j,n}| + \widehat{M}_0\sum_{i=1}^{N}|\lambda_i - \lambda_{i,n}| + \widehat{M}|\frac{2}{L} - \lambda_n| \\ &\quad + \|T_n v_n - x_{n+1}\| \\ &= \|x_t - x_n\| + \epsilon_n,\end{aligned} \tag{42}$$

where $\epsilon_n = \widehat{M}_1\sum_{j=1}^{M}|r_j - r_{j,n}| + \widehat{M}_0\sum_{i=1}^{N}|\lambda_i - \lambda_{i,n}| + \widehat{M}|\frac{2}{L} - \lambda_n| + \|T_n v_n - x_{n+1}\| \to 0$ as $n \to \infty$. Please note that

$$\langle Ax_t - Ax_n, x_t - x_n\rangle \geq \bar{\gamma}\|x_t - x_n\|^2. \tag{43}$$

Furthermore, for simplicity, we write $w_t = (I - s_tA)T\Lambda^N\Delta^M x_t + s_t((I - t\mu F)Vx_t + t\gamma T\Lambda^N\Delta^M x_t)$. By (38), we get $x_t = P_C w_t$ and

$$\begin{aligned}x_t - x_{n+1} &= (I - s_tA)T\Lambda^N\Delta^M x_t - (I - s_tA)x_{n+1} + s_t[(I - t\mu F)Vx_t - (I - t\mu F)Vx_{n+1} \\ &\quad + t(\gamma T\Lambda^N\Delta^M x_t - \mu FVx_{n+1}) + (V - A)x_{n+1}] + x_t - w_t.\end{aligned}$$

Applying Lemma 1 and Proposition 1, we have

$$\begin{aligned}\|x_t - x_{n+1}\|^2 &\leq 2s_t t\langle\gamma T\Lambda^N\Delta^M x_t - \mu FVx_{n+1}), x_t - x_{n+1}\rangle + 2s_t\langle(V - A)x_{n+1}, x_t - x_{n+1}\rangle \\ &\quad + 2s_t\langle(I - t\mu F)Vx_t - (I - t\mu F)Vx_{n+1}, x_t - x_{n+1}\rangle \\ &\quad + \|(I - s_tA)T\Lambda^N\Delta^M x_t - (I - s_tA)x_{n+1}\|^2 \\ &\leq (1 - s_t\bar{\gamma})^2\|T\Lambda^N\Delta^M x_t - x_{n+1}\|^2 + 2s_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle \\ &\quad - 2s_t t\mu\langle FVx_t - FVx_{n+1}, x_t - x_{n+1}\rangle + 2s_t\langle(V - A)x_{n+1}, x_t - x_{n+1}\rangle \\ &\quad + 2s_t t\|\gamma T\Lambda^N\Delta^M x_t - \mu FVx_{n+1}\|\|x_t - x_{n+1}\| \\ &\leq (1 - s_t\bar{\gamma})^2\|T\Lambda^N\Delta^M x_t - x_{n+1}\|^2 + 2s_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle \\ &\quad + 2s_t t(\mu\|FVx_t - FVx_{n+1}\| + \|\gamma T\Lambda^N\Delta^M x_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\ &\quad + 2s_t\langle(V - A)x_{n+1}, x_t - x_{n+1}\rangle.\end{aligned} \tag{44}$$

Using (42) and (43) in (44), we obtain

$$\begin{aligned}\|x_t - x_{n+1}\|^2 &\leq (1 - s_t\bar{\gamma})^2(\|x_t - x_n\| + \epsilon_n)^2 + 2s_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle \\ &\quad + 2s_t t(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma T\Lambda^N\Delta^M x_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\ &\quad + 2s_t\langle(V - A)x_{n+1}, x_t - x_{n+1}\rangle \\ &\leq s_t^2\bar{\gamma}\langle A(x_t - x_n), x_t - x_n\rangle + \|x_t - x_n\|^2 + (1 - s_t\bar{\gamma})^2(2\|x_t - x_n\|\epsilon_n + \epsilon_n^2) \\ &\quad + 2s_t t(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma T\Lambda^N\Delta^M x_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\ &\quad + 2s_t[\langle(V - A)x_t, x_t - x_{n+1}\rangle + \langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle \\ &\quad - \langle A(x_t - x_n), x_t - x_n\rangle].\end{aligned} \tag{45}$$

Applying the Banach limit LIM to (45), we have

$$\begin{aligned}\mathrm{LIM}_n\|x_t-x_{n+1}\|^2 \le\ & s_t^2\bar{\gamma}\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle+\mathrm{LIM}_n\|x_t-x_n\|^2\\ &+2s_t t\mathrm{LIM}_n(\mu\kappa\|x_t-x_{n+1}\|+\|\gamma T\Lambda^N\Delta^M x_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|\\ &+2s_t[\mathrm{LIM}_n\langle (V-A)x_t,x_t-x_{n+1}\rangle+\mathrm{LIM}_n\langle A(x_t-x_{n+1}),x_t-x_{n+1}\rangle\\ &-\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle].\end{aligned} \tag{46}$$

Using the property $\mathrm{LIM}_n a_n=\mathrm{LIM}_n a_{n+1}$, we have

$$\begin{aligned}&\mathrm{LIM}_n\langle (A-V)x_t,x_t-x_n\rangle=\mathrm{LIM}_n\langle (A-V)x_t,x_t-x_{n+1}\rangle\\ &\le \frac{s_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle+\frac{1}{2s_t}[\mathrm{LIM}_n\|x_t-x_n\|^2-\mathrm{LIM}_n\|x_t-x_{n+1}\|^2]\\ &\quad+t\mathrm{LIM}_n(\mu\kappa\|x_t-x_{n+1}\|+\|\gamma T\Lambda^N\Delta^M x_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|\\ &\quad+\mathrm{LIM}_n\langle A(x_t-x_{n+1}),x_t-x_{n+1}\rangle-\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle\\ &=\frac{s_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle\\ &\quad+t\mathrm{LIM}_n(\mu\kappa\|x_t-x_{n+1}\|+\|\gamma T\Lambda^N\Delta^M x_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|.\end{aligned} \tag{47}$$

Since

$$s_t\langle A(x_t-x_n),x_t-x_n\rangle\le s_t\|A\|\|x_t-x_n\|^2\le s_t\|A\|K^2\to 0 \text{ as } t\to 0, \tag{48}$$

where $\|x_t-x_n\|+\|\gamma T\Lambda^N\Delta^M x_t-\mu FVx_n\|\le K$,

$$t\|x_t-x_{n+1}\|^2\to 0 \text{ and } t\|\gamma T\Lambda^N\Delta^M x_t-\mu FVx_{n+1}\|\|x_t-x_{n+1}\|\to 0 \text{ as } t\to 0, \tag{49}$$

we conclude from (47)–(49) that

$$\begin{aligned}\mathrm{LIM}_n\langle (A-V)\tilde{x},\tilde{x}-x_n\rangle &\le \limsup_{t\to 0}\mathrm{LIM}_n\langle (A-V)x_t,x_t-x_n\rangle\\ &\le \limsup_{t\to 0} t\mathrm{LIM}_n(\mu\kappa\|x_t-x_{n+1}\|+\|\gamma T\Lambda^N\Delta^M x_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|\\ &\quad+\limsup_{t\to 0}\frac{s_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle\\ &=0.\end{aligned}$$

This completes the proof. □

Theorem 3. *Let the sequences* $\{\alpha_n\}\subset[0,1]$ *and* $\{s_n\}\subset(0,\frac{1}{2})$ *satisfy: (C1)* $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}s_n=0$ *and (C2)* $\sum_{n=0}^{\infty}s_n=\infty$. *Let the sequence* $\{x_n\}$ *be defined by* (16). *If* $x_{n+1}-x_n\rightharpoonup 0$, *then* x_n *converges strongly to* $\tilde{x}\in\Omega$, *which solves* (15).

Proof. We assume, without loss of generality, that $\alpha_n\tau<1$ and $\frac{2s_n(\bar{\gamma}-1)}{1-s_n}<1$ for all $n\ge 0$. Let x_t be defined by (38). Then, $\lim_{t\to 0}x_t:=\tilde{x}\in\Omega$ (due to Theorem 1). We divide the rest of the proof into several steps.

Step 1. It is easy to show that

$$\|x_n-p\|\le\max\{\|x_0-p\|,\frac{\|(V-A)p\|+\|(\gamma I-\mu FV)p\|}{\bar{\gamma}-1}\},\ \forall n\ge 0.$$

Hence $\{x_n\},\{u_n\},\{v_n\},\{T_nv_n\},\{FVx_n\},\{Vx_n\}$ and $\{y_n\}$ are all bounded.
Step 2. We show that $\limsup_{n\to\infty}\langle (V-A)\tilde{x},x_n-\tilde{x}\rangle\le 0$. To this end, let

$$a_n:=\langle (A-V)\tilde{x},\tilde{x}-x_n\rangle,\quad \forall n\ge 0.$$

According to Theorem 2, we deduce $\text{LIM}_n a_n \leq 0$. Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \limsup_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $x_{n_j} \rightharpoonup v \in H$. This indicates that $x_{n_j+1} \rightharpoonup v$. Hence,

$$w - \lim_{j\to\infty}(\tilde{x} - x_{n_j+1}) = w - \lim_{j\to\infty}(\tilde{x} - x_{n_j}) = (\tilde{x} - v),$$

and therefore

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}\langle (A - V)\tilde{x}, (\tilde{x} - x_{n_j+1}) - (\tilde{x} - x_{n_j})\rangle = 0.$$

Then, by Lemma 8, we derive

$$\limsup_{n\to\infty}\langle (V - A)\tilde{x}, x_n - \tilde{x}\rangle = \limsup_{n\to\infty}\langle (A - V)\tilde{x}, \tilde{x} - x_n\rangle \leq 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. Set $w_n = (I - s_n A)T_n v_n + s_n y_n$ for all $n \geq 0$. Then $x_{n+1} = P_C w_n$. Utilizing (16) and $T_n \Lambda_n^N \Delta_n^M \tilde{x} = T_n \tilde{x} = \tilde{x}$, we have

$$\begin{aligned} x_{n+1} - \tilde{x} = {} & s_n[(I - \alpha_n \mu F)V x_n - (I - \alpha_n \mu F)V\tilde{x} + \alpha_n \gamma (T_n v_n - T_n \tilde{x}) + \alpha_n(\gamma I - \mu F V)\tilde{x}] \\ & + (I - s_n A)(T_n v_n - T_n \tilde{x}) + s_n(V - A)\tilde{x} + x_{n+1} - w_n. \end{aligned}$$

Thus, utilizing Proposition 1 and Lemma 1, we get

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 \leq {} & [(1 - s_n\bar{\gamma})\|T_n v_n - T_n\tilde{x}\| + s_n((1 - \alpha_n\tau)\|x_n - \tilde{x}\| + \alpha_n\gamma\|T_n v_n - T_n\tilde{x}\|)]^2 \\ & + 2s_n[\alpha_n\|(\gamma I - \mu F V)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\ \leq {} & [1 - s_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))]\|x_n - \tilde{x}\|^2 \\ & + 2s_n[\alpha_n\|(\gamma I - \mu F V)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\ = {} & [1 - s_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))]\|x_n - \tilde{x}\|^2 + s_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma)) \\ & \times \frac{2}{\bar{\gamma} - 1 + \alpha_n(\tau - \gamma)}[\alpha_n\|(\gamma I - \mu F V)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\ = {} & (1 - \omega_n)\|x_n - \tilde{x}\|^2 + \omega_n\delta_n, \end{aligned}$$

where $\omega_n = s_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))$ and

$$\delta_n = \frac{2}{\bar{\gamma} - 1 + \alpha_n(\tau - \gamma)}[\alpha_n\|(\gamma I - \mu F V)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle].$$

It is easy to check that $\omega_n \to 0$, $\sum_{n=0}^{\infty}\omega_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. By Lemma 6 with $r_n = 0$, we deduce that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. This completes the proof. □

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References and Note

1. Korpelevich, G.-M. The extragradient method for finding saddle points and other problems. *Matecon* **1976**, *12*, 747–756.
2. Peng, J.-W.; Yao, J.-C. A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems. *Taiwan. J. Math.* **2008**, *12*, 1401–1432. [CrossRef]
3. Ceng, L.-C.; Petrusel, A.; Lee, C.; Wong, M.-M. Two extragradient approximation methods for variational inequalities and fixed point problems of strict pseudo-contractions. *Taiwan. J. Math.* **2009**, *13*, 607–632.
4. Ceng, L.-C.; Ansari, Q.-H.; Yao, J.-C. Relaxed extragradient iterative methods for variational inequalities. *Appl. Math. Comput.* **2011**, *218*, 1112–1123. [CrossRef]
5. Ceng, L.-C.; Guu, S.-M.; Yao, J.-C. Finding common solutions of a variational inequality, a general system of variational inequalities, and a fixed-point problem via a hybrid extragradient method. *Fixed Point Theory Appl.* **2011**, *2011*, 626159. [CrossRef]
6. Ceng, L.-C.; Al-Homidan, S. Algorithms of common solutions for generalized mixed equilibria, variational inclusions, and constrained convex minimization. *Abstr. Appl. Anal.* **2014**, *2014*, 132053. [CrossRef]
7. Ceng, L.-C.; Yao, J.-C. A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem. *Nonlinear Anal.* **2010**, *72*, 1922–1937. [CrossRef]
8. Yao, Y.; Chen, R.; Xu, H.-K. Schemes for finding minimum-norm solutions of variational inequalities. *Nonlinear Anal.* **2010**, *72*, 3447–3456. [CrossRef]
9. Yao, Y.; Liou, Y.-C.; Yao, J.-C. Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations. *J. Nonlinear Sci. Appl.* **2017**, *10*, 843–854. [CrossRef]
10. Yao, Y.; Postolache, M.; Liou, Y.-C.; Yao, Z.-S. Construction algorithms for a class of monotone variational inequalities. *Optim. Lett.* **2016**, *10*, 1519–1528. [CrossRef]
11. Yao, Y.; Qin, X.; Yao, J.-C. Projection methods for firmly type nonexpansive operators. *J. Nonlinear Convex Anal.* **2018**, *19*, 407–415.
12. Yao, Y.; Postolache, M.; Yao, J.-C. An iterative algorithm for solving the generalized variational inequalities and fixed points problems. *Mathematics* **2019**, *7*, 61. [CrossRef]
13. Ceng, L.-C.; Ansari, Q.-H.; Schaible, S.; Yao, J.-C. Iterative methods for generalized equilibrium problems, systems of general generalized equilibrium problems and fixed point problems for nonexpansive mappings in Hilbert spaces. *Fixed Point Theory* **2011**, *12*, 293–308.
14. Ceng, L.-C.; Ansari, Q.-H.; Yao, J.-C. Some iterative methods for finding fixed points and for solving constrained convex minimization problems. *Nonlinear Anal.* **2011**, *74*, 5286–5302. [CrossRef]
15. Ceng, L.-C.; Guu, S.-M.; Yao, J.-C. A general composite iterative algorithm for nonexpansive mappings in Hilbert spaces. *Comput. Math. Appl.* **2011**, *61*, 2447–2455. [CrossRef]
16. Takahashi, S.; Takahashi, W. Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space. *Nonlinear Anal.* **2008**, *69*, 1025–1033. [CrossRef]
17. Ceng, L.-C.; Yao, J.-C. A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. *J. Comput. Appl. Math.* **2008**, *214*, 186–201. [CrossRef]
18. Combettes, P.-L. Solving monotone inclusions via compositions of nonexpansive averaged operators. *Optimization* **2004**, *53*, 475–504. [CrossRef]
19. Zeng, L.-C.; Yao, J.-C. Modified combined relaxation method for general monotone equilibrium problems in Hilbert spaces. *J. Optim. Theory Appl.* **2006**, *131*, 469–483. [CrossRef]
20. Yao, Y.; Postolache, M.; Zhu, Z. Gradient methods with selection technique for the multiple-sets split feasibility problem. *Optimization*, in press.
21. Yao, Y.; Shahzad, N. Strong convergence of a proximal point algorithm with general errors. *Optim. Lett.* **2012**, *6*, 621–628. [CrossRef]
22. Zegeye, H.; Shahzad, N.; Yao, Y.-H. Minimum-norm solution of variational inequality and fixed point problem in Banach spaces. *Optimization* **2015**, *64*, 453–471. [CrossRef]
23. Yao, Y.H.; Chen, R.-D.; Yao, J.-C. Strong convergence and certain control conditions for modified Mann iteration. *Nonlinear Anal.* **2008**, *68*, 1687–1693. [CrossRef]
24. Yao, Y.; Postolache, M.; Yao, J.-C. Iterative algorithms for the generalized variational inequalities. *U.P.B. Sci. Bull. Ser. A* **2019**, in press.

25. Yao, Y.; Leng, L.; Postolache, M.; Zheng, X. Mann-type iteration method for solving the split common fixed point problem. *J. Nonlinear Convex Anal.* **2017**, *18*, 875–882.
26. Thakur, B.-S.; Postolache, M. Existence and approximation of solutions for generalized extended nonlinear variational inequalities. *J. Inequal. Appl.* **2013**, *2013*, 590. [CrossRef]
27. Dadashi, V.; Postolache, M. Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators. *Arab. J. Math.* **2019**. [CrossRef]
28. Dong, Q.-L.; Lu, Y.-Y.; Yang, J. The extragradient algorithm with inertial effects for solving the variational inequality. *Optimization* **2016**, *65*, 2217–2226. [CrossRef]
29. Zhao, X.-P.; Ng, K.-F.; Li, C.; Yao, J.-C. Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems. *Appl. Math. Optim.* **2018**, *78*, 613–641. [CrossRef]
30. Yao, Y.-H.; Postolache, M.; Liou, Y.-C. Strong convergence of a self-adaptive method for the split feasibility problem. *Fixed Point Theory Appl.* **2013**, *2013*, 201. [CrossRef]
31. Yao, Y.; Yao, J.-C.; Liou, Y.-C.; Postolache, M. Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms. *Carpathian J. Math.* **2018**, *34*, 459–466.
32. Yao, Y.; Liou, Y.-C.; Postolache, M. Self-adaptive algorithms for the split problem of the demicontractive operators. *Optimization* **2018**, *67*, 1309–1319. [CrossRef]
33. Rockafellar, R.-T. Monotone operators and the proximal point algorithms. *SIAM J. Control Optim.* **1976**, *14*, 877–898. [CrossRef]
34. Huang, N.J. A new completely general class of variational inclusions with noncompact valued mappings. *Comput. Math. Appl.* **1998**, *35*, 9–14. [CrossRef]
35. Zeng, L.-C.; Guu, S.-M.; Yao, J.C. Characterization of H-monotone operators with applications to variational inclusions. *Comput. Math. Appl.* **2005**, *50*, 329–337. [CrossRef]
36. Ceng, L.-C.; Ansari, Q.-H.; Wong, M.-M.; Yao, J.-C. Mann type hybrid extragradient method for variational inequalities, variational inclusions and fixed point problems. *Fixed Point Theory* **2012**, *13*, 403–422.
37. Ceng, L.-C.; Guu, S.-M.; Yao, J.-C. Iterative algorithm for finding approximate solutions of mixed quasi-variational-like inclusions. *Comput. Math. Appl.* **2008**, *56*, 942–952. [CrossRef]
38. Jung, J.-S. A general composite iterative method for strictly pseudocontractive mappings in Hilbert spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 173. [CrossRef]
39. Yao, Y.; Liou, Y.-C.; Kang, S.-M. Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method. *Comput. Math. Appl.* **2010**, *59*, 3472–3480. [CrossRef]
40. Shioji, N.; Takahashi, W. Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces. *Proc. Am. Math. Soc.* **1997**, *125*, 3641–3645. [CrossRef]
41. Byrne, C. A unified treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Probl.* **2004**, *20*, 103–120. [CrossRef]
42. Xu, H.-K.; Kim, T.-H. Convergence of hybrid steepest-descent methods for variational inequalities. *J. Optim. Theory Appl.* **2003**, *119*, 185–201. [CrossRef]
43. Barbu, V. *Nonlinear Semigroups and Differential Equations in Banach Spaces*; Springer: Heidelberg, Germany, 1976.
44. Marino, G.; Xu, H.-K. A general iterative method for nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* **2006**, *318*, 43–56. [CrossRef]
45. Marino, G.; Xu, H.-K. Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. *J. Math. Math. Appl.* **2007**, *329*, 336–346. [CrossRef]
46. Xu, H.-K. Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **2002**, *66*, 240–256. [CrossRef]