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Boundary Value Problems for Hybrid Caputo Fractional Differential Equations

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Abstract: In this paper, we discuss the existence of solutions for a hybrid boundary value problem of Caputo fractional differential equations. The main tool used in our study is associated with the technique of measures of noncompactness. As an application, we give an example to illustrate our results.

Keywords: fractional differential equations; boundary value problem; Caputo fractional derivative; measure of noncompactness; Darbo fixed point theorem

1. Introduction

The field of fractional calculus is concerned with the generalization of the integer order differentiation and integration to an arbitrary real or complex order. It has played a significant role in various branches of science such as physics, chemistry, chemical physics, electrical networks, control of dynamic systems, science, engineering, biological science, optics and signal processing (see, for example, [1–4]). Recent development on fractional differential and integral equations are considered in some recent books [5–12]. There exist some papers on the boundary value problems of fractional differential equations (see [13–19]).

Benchohra et al. [15] studied the existence and uniqueness of solutions of the following nonlinear fractional differential equations:

$$\begin{cases} {}^c D_{0+}^{\alpha} y(t) = f(t, y(t)), \text{ for each } t \in J = [0, T], 0 < \alpha < 1, \\ ay(0) + by(T) = c, \end{cases} \quad (1)$$

where ${}^c D_{0+}^{\alpha}$ is the Caputo fractional derivative, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and a, b, c are real constants with $a + b \neq 0$.

The area of differential equations devoted to quadratic perturbations of nonlinear differential equations (called hybrid differential equations) has been considered more important and served as special cases of dynamical systems. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers [20–31].

Dhage and Lakshmikantham [23] discussed the existence and uniqueness theorems of the solution to the ordinary first-order hybrid differential equation with perturbation of first type

$$\begin{cases} \frac{d}{dt} \left(\frac{x(t)}{f(t,x(t))} \right) = g(t, x(t)), \text{ a.e. } t \in J, \\ x(t_0) = x_0 \in \mathbb{R}. \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$, where $J = [t_0, t_0 + a]$ is a bounded interval in \mathbb{R} for some t_0 and $a \in \mathbb{R}$ with $a > 0$.

Zhao et al. [32] studied existence and uniqueness results for the following hybrid differential equations involving Riemann–Liouville differential operators:

$$\begin{cases} D_{0+}^q \left(\frac{x(t)}{f(t,x(t))} \right) = g(t, x(t)), \text{ a.e. } t \in J = [0, T], \\ x(0) = 0, \end{cases}$$

where $0 < q < 1$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. A fixed point theorem in Banach algebras is the main tool used in this work.

Hilal and Kajouni [25] extended the results to the following boundary value problem for fractional hybrid differential equations involving Caputo’s derivative:

$$\begin{cases} {}^cD_{0+}^\alpha \left[\frac{x(t)}{f(t,x(t))} \right] = g(t, x(t)), \text{ a.e. } t \in J = [0, T], \\ a \frac{x(0)}{f(0,x(0))} + b \frac{x(T)}{f(T,x(T))} = c, \end{cases} \tag{2}$$

where $0 < \alpha < 1$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$. They proved the existence of the boundary fractional hybrid differential equations under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities are also established, which are utilized to prove the existence of extremal solutions.

Motivated by some recent studies on hybrid fractional differential equations, we consider the following boundary value problem of nonlinear fractional hybrid differential equations:

$$\begin{cases} {}^cD_{0+}^\alpha \left[\frac{x(t)}{f(t,x(\mu(t)))} \right] = g(t, x(\nu(t))), \text{ } t \in I = [0, 1], \\ a \left[\frac{x(t)}{f(t,x(\mu(t)))} \right]_{t=0} + b \left[\frac{x(t)}{f(t,x(\mu(t)))} \right]_{t=1} = c, \end{cases} \tag{3}$$

where $0 < \alpha \leq 1$, a, b, c are real constants such that $a + b \neq 0$, ${}^cD_{0+}^\alpha$ is the Caputo fractional derivative, $f \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C(I \times \mathbb{R}, \mathbb{R})$, μ and ν are functions from I into itself. Note that, if $\mu(t) = \nu(t) = t$, then the first problem of Equation (3) is reduced to the problem in Equation (2). In addition, if $\mu(t) = \nu(t) = t$, $f(t, x(t)) = 1$, then the problem in Equation (3) is reduced to the problem in Equation (1).

The main tool of our study is a generalization of Darbo’s fixed point theorem for the product of two operators associated to measures of noncompactness. We can use a numerical method to solve the problem in Equation (3) (see [16]).

2. Preliminaries

First, we present the necessary definitions and lemmas from fractional calculus theory.

Definition 1 ([1]). Let $\alpha > 0$, for a function $u : [0, \infty) \rightarrow \mathbb{R}$. The Riemann–Liouville fractional integral of order α of u is defined by

$$I_{0+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Remark 1. The notation $I_{0+}^\alpha u(t) |_{t=0}$ means that the limit is taken at almost all points of the right-sided neighborhood $(0, \varepsilon)$ ($\varepsilon > 0$) of 0 as follows:

$$I_{0+}^\alpha u(t) |_{t=0} = \lim_{t \rightarrow 0+} I_{0+}^\alpha u(t).$$

Generally, $I_{0+}^\alpha u(t) |_{t=0}$ is not necessarily zero. For instance, let $\alpha \in (0, 1)$, $u(t) = t^{-\alpha}$. Then,

$$I_{0+}^\alpha t^{-\alpha} |_{t=0} = \lim_{t \rightarrow 0+} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds = \Gamma(1-\alpha).$$

Definition 2 ([1]). Let $\alpha > 0$. The Caputo fractional derivative of order α of a function $u : (0, \infty) \rightarrow \mathbb{R}$ is given by

$${}^C D_{0+}^\alpha u(t) = I_{0+}^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α , provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 1 ([1]). Let $\alpha, \beta \geq 0$, and $u \in L^1([0, 1])$. Then, $I_{0+}^\alpha I_{0+}^\beta u(t) = I_{0+}^{\alpha+\beta} u(t)$ and ${}^C D_{0+}^\alpha I_{0+}^\alpha u(t) = u(t)$, for all $t \in [0, 1]$

Lemma 2 ([1]). Let $\alpha > 0$, $n = [\alpha] + 1$, then

$$I_{0+}^\alpha {}^C D_{0+}^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} c_k t^k, \quad c_k \in \mathbb{R}.$$

In the sequel, we recall some definitions and basic facts about measures of noncompactness.

Assume that $(E, \|\cdot\|)$ is a real Banach space with zero element 0. By $B(x, r)$, we denote the closed ball in E centered at x with the radius r . By B_r , we denote the ball $B(0, r)$. If X is non-empty subset of E , then \bar{X} and $ConvX$ denote the closure and the closed convex closure of X , respectively. When X is a bounded subset, $diamX$ denotes the diameter of X and $\|X\|$ the quantity given by $\|X\| = \sup\{\|x\| : x \in X\}$. Further, by \mathcal{M}_E , we denote the family of the nonempty and bounded subsets of E and by \mathcal{N}_E its subfamily consisting of the relatively compact subsets.

We accept the following definition of measure of noncompactness.

Definition 3 ([33]). A mapping $\sigma : \mathcal{M}_E \rightarrow [0, \infty)$ is said to be a measure of noncompactness if it satisfies the following conditions:

- (1) The family $Ker\sigma = \{X \in \mathcal{M}_E; \sigma(X) = 0\}$ is non-empty and $Ker\sigma \in \mathcal{N}_E$.
- (2) $X \subset Y \Rightarrow \sigma(X) \leq \sigma(Y)$.
- (3) $\sigma(\bar{X}) = \sigma(ConvX) = \sigma(X)$.
- (4) $\sigma(\lambda X + (1-\lambda)Y) \leq \lambda\sigma(X) + (1-\lambda)\sigma(Y)$ for $\lambda \in [0, 1]$.
- (5) If (X_n) is a sequence of closed subsets of \mathcal{M}_E such that $X_{n+1} \subset X_n$ ($n \geq 1$) and $\lim_{n \rightarrow \infty} \sigma(X_n) = 0$, then $\bigcap_{n=1}^\infty X_n \neq \emptyset$.

An important theorem about fixed point theorem in the context of measures of noncompactness is the following Darbo's fixed point theorem.

Theorem 1 ([34]). Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathcal{T} : \Omega \rightarrow \Omega$ be a continuous mapping. Suppose that there exists $k \in [0, 1)$ such that

$$\sigma(\mathcal{T}X) \leq k\sigma(X),$$

for any non-empty subset X of Ω , where σ is a measure of noncompactness in E . Then, \mathcal{T} has a fixed point in Ω .

Now, we introduce the following class \mathcal{A} of functions $\varphi : (0, \infty) \rightarrow (1, \infty)$ satisfying the following condition:

For any sequence $(t_n) \subset (0, \infty)$,

$$\lim_{n \rightarrow \infty} \varphi(t_n) = 1 \iff \lim_{n \rightarrow \infty} t_n = 0.$$

A generalization of Theorem 1, which is very useful for our study, is the following theorem (see [35]).

Theorem 2. Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathcal{T} : \Omega \rightarrow \Omega$ be a continuous mapping. Suppose that there exist $\varphi \in \mathcal{A}$ and $k \in [0, 1)$ such that, for any nonempty subset X of Ω with $\sigma(\mathcal{T}X) > 0$,

$$\varphi(\sigma(\mathcal{T}X)) \leq (\varphi(\sigma(X)))^k,$$

for any non-empty subset X of Ω , where σ is a measure of noncompactness in E . Then, \mathcal{T} has a fixed point in Ω .

Next, we assume that the space E has structure of Banach algebra. By xy , we denote the product of two elements $x, y \in X$ and by XY we denote the set defined by $XY = \{xy : x \in X, y \in Y\}$.

Definition 4. Let E be a Banach algebra. A measure of noncompactness σ in E is said to satisfy condition (m) if it satisfies the following condition:

$$\sigma(XY) \leq \|X\|\sigma(Y) + \|Y\|\sigma(X),$$

for any $X, Y \in \mathcal{M}_E$.

This definition appears in [36].

As is known, the family of all real valued and continuous functions defined on interval I is a Banach space with the standard norm

$$\|x\| = \sup\{|x(t)|, t \in I\}.$$

Notice that $(C(I), \|\cdot\|)$ is a Banach algebra, where the multiplication is defined as the usual product of real functions.

In our considerations, we use a measure of noncompactness defined in [33]. To recall the definitions of that measure, let us fix a set $X \in C(I)$. For $x \in X$ and for a given $\varepsilon > 0$, denote by $\omega(x, \varepsilon)$ the modulus of continuity of x , i.e.,

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I, |t - s| \leq \varepsilon\}.$$

Put

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\},$$

and

$$\omega_0(X) = \lim_{\varepsilon \rightarrow 0} \omega(X, \varepsilon).$$

The authors of [33] showed that function ω_0 is a measure of noncompactness in space $C(I)$.

Lemma 3. The measure of noncompactness ω_0 on $C(I)$ satisfies condition (m).

Proof. Let X, Y be a fixed subset of $\mathcal{M}_{C(I)}$, $\varepsilon > 0$ and $t, s \in I$ with $|t - s| \leq \varepsilon$. Then, for $x \in X$ and $y \in Y$, we have

$$\begin{aligned} |x(t)y(t) - x(s)y(s)| &\leq |x(t)y(t) - x(t)y(s)| + |x(t)y(s) - x(s)y(s)| \\ &\leq |x(t)||y(t) - y(s)| + |y(s)||x(t) - x(s)| \\ &\leq \|x\|\omega(y, \varepsilon) + \|y\|\omega(x, \varepsilon). \end{aligned}$$

Thus,

$$\omega(xy, \varepsilon) \leq \|x\|\omega(y, \varepsilon) + \|y\|\omega(x, \varepsilon),$$

and

$$\omega(XY, \varepsilon) \leq \|X\|\omega(Y, \varepsilon) + \|Y\|\omega(X, \varepsilon).$$

Therefore, we get

$$\omega_0(XY) = \lim_{\varepsilon \rightarrow 0} \omega(XY, \varepsilon) \leq \|X\|\omega_0(Y) + \|Y\|\omega_0(X).$$

This completes the proof. \square

Lemma 4. For any $y \in C(I)$, the unique solution of the hybrid fractional differential equation,

$${}^cD_{0+}^\alpha \left[\frac{x(t)}{f(t, x(\mu(t)))} \right] = y(t), \quad 0 < t < 1, \tag{4}$$

with boundary conditions

$$a \left[\frac{x(t)}{f(t, x(\mu(t)))} \right]_{t=0} + b \left[\frac{x(t)}{f(t, x(\mu(t)))} \right]_{t=1} = c, \tag{5}$$

is given by

$$x(t) = f(t, x(\mu(t))) \left\{ \int_0^1 G(t, s)y(s)ds + \frac{c}{a+b} \right\}, \tag{6}$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1} - \frac{b}{a+b}(1-s)^{\alpha-1} & , \quad 0 \leq s < t \leq 1, \\ -\frac{b}{a+b}(1-s)^{\alpha-1} & , \quad 0 \leq t \leq s < 1. \end{cases} \tag{7}$$

Here, $G(t, s)$ is called the Green function of the boundary value problem in Equations (4) and (5).

Proof. We may apply Lemma 2 to reduce Equation (4) to an equivalent integral equation

$$\frac{x(t)}{f(t, x(\mu(t)))} = I_{0+}^\alpha y(t) + c_0, \quad c_0 \in \mathbb{R}. \tag{8}$$

Consequently, the general solution of Equation (4) is

$$x(t) = f(t, x(\mu(t)))(I_{0+}^\alpha y(t) + c_0). \tag{9}$$

Applying the boundary conditions in Equation (5) to Equation (8), we find that

$$ac_0 + b(I_{0+}^\alpha y(1) + c_0) = c.$$

Therefore, we have

$$c_0 = \frac{1}{a+b}(c - bI_{0+}^\alpha y(1)).$$

Substituting the value of c_0 into Equation (9), we get Equation (6).

$$x(t) = f(t, x(\mu(t))) \left\{ I_{0+}^\alpha y(t) - \frac{b}{a+b} I_{0+}^\alpha y(1) + \frac{c}{a+b} \right\}. \tag{10}$$

that can be written as

$$x(t) = f(t, x(\mu(t))) \left\{ \int_0^1 G(t,s)y(s)ds + \frac{c}{a+b} \right\},$$

where G is defined by Equation (7). The proof is complete. \square

Remark 2. From the expression of $G(t, s)$, the function $G(t, s)$ is not continuous on $I \times I$, however the function $t \mapsto \int_0^1 G(t, s)ds$ is continuous on I , so it is bounded.

Thanks to Lemma 4, the proposed problem is equivalent to the following integral equation

$$x(t) = f(t, x(\mu(t))) \left\{ \int_0^1 G(t,s)g(s, x(\nu(s)))ds + \frac{c}{a+b} \right\}.$$

3. Proof of Main Results

Firstly, we list some assumptions:

- (H1) The functions $\mu, \nu : I \rightarrow I$ are continuous.
- (H2) $f \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(I \times \mathbb{R}, \mathbb{R})$.
- (H3) There exists a constant $k \in (0, 1)$ such that

$$|f(t, x_1) - f(t, x_2)| \leq (|x_1 - x_2| + 1)^k - 1, \quad t \in I, \quad x_1, x_2 \in \mathbb{R}.$$

- (H4) There exists a continuous nondecreasing function $\psi : \mathbb{R}_+ \rightarrow (0, +\infty)$ such that

$$|g(t, x)| \leq \psi(|x|), \quad t \in I, \quad x \in \mathbb{R}_+.$$

- (H5) There exists $r > 0$ such that

$$[(r + 1)^k - 1 + M] \left\{ \frac{|a| + 2|b|}{|a + b|\Gamma(\alpha + 1)} \psi(r) + \frac{|c|}{|a + b|} \right\} \leq r,$$

and

$$\frac{|a| + 2|b|}{|a + b|\Gamma(\alpha + 1)} \psi(r) + \frac{|c|}{|a + b|} \leq 1,$$

where

$$M = \sup\{|f(t, 0, 0)| : t \in I\}.$$

Now, we are in a position to state and prove our main result in this paper.

Theorem 3. Assume that Assumptions (H1)–(H5) hold. Then, the problem in Equation (3) has at least one solution in the Banach algebra $C(I)$.

Proof. To prove this result using Theorem 2, we consider the operator \mathcal{T} on the Banach algebra $C(I)$ as follows

$$\mathcal{T}x(t) = f(t, x(\mu(t))) \left\{ \int_0^1 G(t,s)g(s, x(\nu(s)))ds + \frac{c}{a+b} \right\}$$

for $t \in I$. By virtue of Lemma 4, a fixed point of \mathcal{T} gives us the desired result.

We define operators \mathcal{F} and \mathcal{G} on the Banach algebra $C(I)$ in the following way:

$$\mathcal{F}x(t) = f(t, x(\mu(t))),$$

and

$$\mathcal{G}x(t) = \int_0^1 G(t,s)g(s,x(v(s)))ds + \frac{c}{a+b}$$

for $t \in I$. Then, $\mathcal{T}x = (\mathcal{F}x).(\mathcal{G}x)$ for any $x \in C(I)$.

We divide the proof into five steps.

Step 1: \mathcal{T} transforms $C(I)$ into itself.

In fact, since the product of continuous functions is a continuous function, it is sufficient to prove that $\mathcal{F}x, \mathcal{G}x \in C(I)$ for any $x \in C(I)$. Now, from Assumptions (H1) and (H2), it follows that if $x \in C(I)$ then $\mathcal{F}x \in C(I)$. Next, we prove that if $x \in C(I)$ then $\mathcal{G}x \in C(I)$. To do this, let $\varepsilon > 0$ be fixed; take $x \in C(I)$ and $t_1, t_2 \in I$ with $t_2 - t_1 \leq \varepsilon$; and assume that $t_1 \leq t_2$. Then, in view of Assumption (H4), we get

$$\begin{aligned} |\mathcal{G}x(t_2) - \mathcal{G}x(t_1)| &\leq \int_0^{t_1} |G(t_1,s) - G(t_2,s)||g(s,x(v(s)))|ds \\ &\quad + \int_{t_1}^{t_2} |G(t_1,s) - G(t_2,s)||g(s,x(v(s)))|ds \\ &\quad + \int_{t_2}^1 |G(t_1,s) - G(t_2,s)||g(s,x(v(s)))|ds \\ &\leq \psi(\|x\|) \left(\int_0^{t_1} |G(t_1,s) - G(t_2,s)|ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} |G(t_1,s) - G(t_2,s)|ds + \int_{t_2}^1 |G(t_1,s) - G(t_2,s)|ds \right) \\ &\leq \frac{\psi(\|x\|)}{\Gamma(\alpha+1)} \left(2(t_2 - t_1)^\alpha + (t_1^\alpha - t_2^\alpha) \right) \\ &\leq \frac{2\psi(\|x\|)}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha \\ &\leq \frac{2\psi(\|x\|)}{\Gamma(\alpha+1)} \varepsilon^\alpha. \end{aligned}$$

From the above inequality, we conclude that $|\mathcal{G}x(t_2) - \mathcal{G}x(t_1)| \rightarrow 0$ when $\varepsilon \rightarrow 0$. Therefore, $\mathcal{G}x \in C(I)$. This proves that if $x \in C(I)$ then $\mathcal{T}x \in C(I)$.

Step 2: An estimate of $\|\mathcal{T}x\|$ for $x \in C(I)$.

Now, let us fix $x \in C(I)$, then using our assumptions for $t \in I$, we obtain

$$\begin{aligned} |(\mathcal{T}x)(t)| &= \left| f(t,x(\mu(t))) \left\{ I_{0+}^\alpha g(t,x(v(t))) - \frac{b}{a+b} I_{0+}^\alpha g(1,x(v(1))) + \frac{c}{a+b} \right\} \right| \\ &\leq (|f(t,x(\mu(t))) - f(t,0)| + |f(t,0)|) \\ &\quad \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|g(s,x(v(s)))|}{(t-s)^{1-\alpha}} ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^1 \frac{|g(s,x(v(s)))|}{(1-s)^{1-\alpha}} ds + \frac{|c|}{|a+b|} \right\} \\ &\leq [(|x(\mu(t))| + 1)^k - 1 + M] \\ &\quad \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi(|x(v(s))|)}{(t-s)^{1-\alpha}} ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^1 \frac{\psi(|x(v(s))|)}{(1-s)^{1-\alpha}} ds + \frac{|c|}{|a+b|} \right\} \\ &\leq [(\|x\| + 1)^k - 1 + M] \left\{ \frac{|a| + 2|b|}{|a+b|\Gamma(\alpha+1)} \psi(\|x\|) + \frac{|c|}{|a+b|} \right\}. \end{aligned}$$

Therefore,

$$\|\mathcal{T}x\| \leq [(\|x\| + 1)^k - 1 + M] \left\{ \frac{|a| + 2|b|}{|a+b|\Gamma(\alpha+1)} \psi(\|x\|) + \frac{|c|}{|a+b|} \right\}.$$

By Assumption (H5), we deduce that the operator \mathcal{T} maps the ball $B_r \subset C(I)$ into itself. Moreover, let us observe that, from the last estimates, we obtain

$$\begin{cases} \|\mathcal{F}B_r\| &\leq (r + 1)^k - 1 + M, \\ \|\mathcal{G}B_r\| &\leq \frac{|a| + 2|b|}{|a+b|\Gamma(\alpha+1)} \psi(r) + \frac{|c|}{|a+b|}. \end{cases} \tag{11}$$

Step 3: The operators \mathcal{F} and \mathcal{G} are continuous on the ball B_r .

In fact, firstly we prove that the operator \mathcal{F} is continuous on the ball B_r . To do this, fix $\varepsilon > 0$ and take arbitrary $x, y \in B_r$ such that $\|x - y\| \leq \varepsilon$. Then, for $t \in I$, we have

$$\begin{aligned} |(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| &= |f(t, x(\mu(t))) - f(t, y(\mu(t)))| \\ &\leq (|x(\mu(t)) - y(\mu(t))| + 1)^k - 1 \\ &\leq (\|x - y\| + 1)^k - 1 \\ &\leq (\varepsilon + 1)^k - 1, \end{aligned}$$

and, since $(\varepsilon + 1)^k - 1 \rightarrow 0$ when $\varepsilon \rightarrow 0$, from the above inequality, the operator \mathcal{F} is continuous on the ball B_r .

Next, we prove that the operator \mathcal{G} is continuous on the ball B_r . To do this, we take a sequence $\{x_n\} \subset B_r$ and $x \in B_r$ such that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$, and we have to prove that $\|\mathcal{G}x_n - \mathcal{G}x\| \rightarrow 0$ as $n \rightarrow \infty$. Since $g(t, x)$ is uniformly continuous on the compact $I \times [-r, r]$, we may denote

$$H = \sup\{|g(t, x)| : t \in I, x \in [-r, r]\}.$$

Since $\mu : I \rightarrow I$ is continuous, then for any n and $t \in I$, we have $|x_n(\mu(t))| \leq r$. Thus, for any n and $t \in I$, we get

$$|G(t, s)| |g(t, x_n(\mu(t)))| \leq H |G(t, s)|, \quad (t, s) \in I \times I.$$

By applying Lebesgue dominated convergence theorem, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} (\mathcal{G}x_n)(t) &= \lim_{n \rightarrow \infty} \int_0^1 G(t, s) g(s, x_n(\mu(s))) ds + \frac{c}{a+b} \\ &= \int_0^1 G(t, s) g(s, x(\mu(s))) ds + \frac{c}{a+b} \\ &= (\mathcal{G}x)(t). \end{aligned}$$

Thus, the above inequality shows that the operator \mathcal{G} is continuous in B_r . Hence, we conclude that \mathcal{T} is continuous operator on B_r .

Step 4: We estimate $\omega_0(\mathcal{F}X)$ and $\omega_0(\mathcal{G}X)$ for $\emptyset \neq X \subset B_r$.

Firstly, we estimate $\omega_0(\mathcal{F}X)$. Let $\varepsilon > 0$ be fixed; since $\mu : I \rightarrow I$ is uniformly continuous, we can find $\delta > 0$ (which can be taken with $\delta < \varepsilon$) such that, for $|t_1 - t_2| < \delta$, we have $|\mu(t_1) - \mu(t_2)| < \varepsilon$. Let $x \in X$ and $t_1, t_2 \in I$ with $|t_1 - t_2| \leq \delta < \varepsilon$. Then, in view of Assumption (H3), we have

$$\begin{aligned} |(\mathcal{F}x)(t_1) - (\mathcal{F}x)(t_2)| &= |f(t_1, x(\mu(t_1))) - f(t_2, x(\mu(t_2)))| \\ &\leq |f(t_1, x(\mu(t_1))) - f(t_1, x(\mu(t_2)))| + |f(t_1, x(\mu(t_2))) - f(t_2, x(\mu(t_2)))| \\ &\leq \left[(|x(\mu(t_1)) - x(\mu(t_2))| + 1)^k - 1 \right] + \omega(f, \varepsilon) \\ &\leq \left[(\omega(X, \varepsilon) + 1)^k - 1 \right] + \omega(f, \varepsilon), \end{aligned}$$

where

$$\omega(f, \varepsilon) = \sup\{|f(t_1, x) - f(t_2, x)| : t_1, t_2 \in I, |t_1 - t_2| \leq \varepsilon, x \in [-r, r]\}.$$

Thus,

$$\omega(\mathcal{F}X, \varepsilon) \leq \left[(\omega(X, \varepsilon) + 1)^k - 1 \right] + \omega(f, \varepsilon).$$

Observe that the function $f(t, x)$ is uniformly continuous on the set $I \times [-r, r]$. Hence, we deduce that $\omega(f, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus, from the above inequality, we conclude

$$\omega_0(\mathcal{F}X) \leq (\omega_0(X) + 1)^k - 1. \tag{12}$$

Next, we estimate $\omega_0(\mathcal{G}X)$. Fix $\varepsilon > 0$; since $\int_0^1 G(t, s)ds$ is uniformly continuous on I , there exists $\delta > 0$ (which can be taken with $\delta < \varepsilon$) such that, for any $t_1, t_2 \in I$ with $|t_2 - t_1| \leq \delta < \varepsilon$,

$$\int_0^1 |G(t_1, s) - G(t_2, s)|ds \leq \frac{\varepsilon}{H}.$$

Thus,

$$\begin{aligned} |\mathcal{G}x(t_2) - \mathcal{G}x(t_1)| &= \int_0^1 |G(t_1, s) - G(t_2, s)| |g(s, x(v(s)))| ds \\ &\leq H \int_0^1 |G(t_1, s) - G(t_2, s)| ds < \varepsilon. \end{aligned}$$

Thus,

$$\omega(\mathcal{G}x, \varepsilon) \leq \varepsilon.$$

Taking $\varepsilon \rightarrow 0$, we get

$$\omega_0(\mathcal{G}X) = 0. \tag{13}$$

Step 5: We estimate $\omega_0(\mathcal{T}X)$ for $\emptyset \neq X \subset B_r$.

From Lemma 3 and the estimates in Equations (11)–(13), we have

$$\begin{aligned} \omega_0(\mathcal{T}X) &= \omega_0(\mathcal{F}X, \mathcal{G}X) \leq \|\mathcal{F}X\| \omega_0(\mathcal{G}X) + \|\mathcal{G}X\| \omega_0(\mathcal{F}X) \\ &\leq \|\mathcal{F}B_{r_0}\| \omega_0(\mathcal{G}X) + \|\mathcal{G}B_{r_0}\| \omega_0(\mathcal{F}X) \\ &\leq [(\omega_0(X) + 1)^k - 1] \left[\frac{|a| + 2|b|}{|a + b|\Gamma(\alpha + 1)} \psi(r_0) + \frac{|c|}{|a + b|} \right]. \end{aligned}$$

By Assumption (H5), we know that

$$\frac{|a| + 2|b|}{|a + b|\Gamma(\alpha + 1)} \psi(r_0) + \frac{|c|}{|a + b|} \leq 1.$$

Hence,

$$\omega_0(\mathcal{T}X) + 1 \leq (\omega_0(X) + 1)^k.$$

Thus, the contractive condition appearing in Theorem 1 is satisfied with $\varphi(t) = t + 1$, where $\varphi \in A$. By applying Theorem 2, we get that the operator \mathcal{T} has at least one fixed point in the ball B_r . Consequently, the problem in Equation (3) has at least one solution in B_r . This completes the proof. \square

4. An Example

Consider the following fractional hybrid problem

$$\begin{aligned} {}^cD_{0^+}^{\frac{1}{2}} \left[\frac{x(t)}{\sqrt{1+|x(e^{t-1})|}} \right] &= \frac{1}{3} \sin x(\sqrt{t}), \quad t \in I = [0, 1], \\ \left[\frac{x(t)}{\sqrt{1+|x(e^{t-1})|}} \right]_{t=0} &+ \left[\frac{x(t)}{\sqrt{1+|x(e^{t-1})|}} \right]_{t=1} = 0. \end{aligned} \tag{14}$$

Corresponding to the problem in Equation (3), we have that $f(t, x) = \sqrt{1 + |x|}$, $|g(t, x)| = \frac{1}{3} \sin x$, $\mu(t) = e^{t-1}$, $\nu(t) = \sqrt{t}$, $\alpha = \frac{1}{2}$, $a = b = 1$, $c = 0$, $M = \sup_{t \in I} |f(t, 0, 0)| = 1$. It is clear that Assumptions (H1)–(H2) hold.

On the other hand, since the function $\beta(x) = \sqrt{1+|x|} - 1$ is concave (because $\beta''(x) \leq 0$) and $\beta(0) = 0$, we infer that β is subadditive and, therefore, for any $t \in I$ and $x_1, x_2 \in \mathbb{R}$, we have

$$\begin{aligned} |f(t, x_1) - f(t, x_2)| &= |\beta(x_1) - \beta(x_2)| \leq \beta(x_1 - x_2) \\ &= \sqrt{1+|x_1 - x_2|} - 1. \end{aligned}$$

Thus, Assumption (H3) holds, with $k = 1/2$.

Moreover, for any $t \in I$ and $x \in \mathbb{R}$, we have

$$|g(t, x)| = \frac{1}{3} |\sin x| \leq \frac{1}{3} |x|.$$

Hence, Assumption (H4) holds, where $\psi(x) = \frac{1}{3}x$.

Observe that Assumption (H5) is equivalent to

$$\frac{\sqrt{r+1}}{\sqrt{\pi}} \leq 1 \quad \text{and} \quad \frac{r}{\sqrt{\pi}} \leq 1,$$

Thus, Assumption (H5) is satisfied for all $0 < r \leq \sqrt{\pi}$.

Thus, all assumptions of Theorem 3 are satisfied, and consequently the problem in Equation (14) has at least one solution in $C(I)$.

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