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First Integrals of the May–Leonard Asymmetric System

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Abstract: For the May–Leonard asymmetric system, which is a quadratic system of the Lotka–Volterra type depending on six parameters, we first look for subfamilies admitting invariant algebraic surfaces of degree two. Then for some such subfamilies we construct first integrals of the Darboux type, identifying the systems with one first integral or with two independent first integrals.

Keywords: integrability; invariant surfaces; Lotka–Volterra system; computational algebra

1. Introduction

An important class of mathematical models describing different phenomena in biology, ecology and chemistry are the so-called Lotka–Volterra systems, which are written in the form

$$\dot{x}_i = x_i(\sum_{j=1}^{n} a_{ij}x_j + b_i) \qquad (i = 1, \dots, n). \tag{1}$$

They were introduced independently by Lotka and Volterra in the 1920s to model the interaction among species, see [1,2], and continue being intensively investigated. For the class of systems in Equation (1), most studies are devoted to the case $n = 3$. One of simplest models of such a type describing a competition of three species was introduced by May and Leonard in [3]. It is a model depending on two parameters and is written as the differential system

$$\begin{aligned} \dot{x} &= x(1 - x - \alpha y - \beta z), \\ \dot{y} &= y(1 - \beta x - y - \alpha z), \\ \dot{z} &= z(1 - \alpha x - \beta y - z), \end{aligned} \tag{2}$$

where $x, y, z \geq 0$, $0 < \alpha < 1 < \beta$, and

$$\alpha + \beta > 2. \tag{3}$$

It was shown in [3] that system (2) has four singular points in $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3,\ x, y, z \geq 0\}$—three of them are on the boundary of $\mathbb{R}^3_+$ in

$$E_1 = (1, 0, 0),\ E_2 = (0, 1, 0),\ E_3 = (0, 0, 1)$$

and the fourth one in the interior point

$$C = ((1 + \alpha + \beta)^{-1}, (1 + \alpha + \beta)^{-1}, (1 + \alpha + \beta)^{-1}).$$

There is a separatix cycle F formed by orbits connecting E_1, E_2 and E_3 on the boundary of $\mathbb{R}^3_+$ and every orbit in $\mathbb{R}^3_+$, except of the equilibrium point C, has F as ω-limit. It was shown in [3] that in the degenerate case $\alpha + \beta = 2$, the cycle F becomes a triangle on the invariant plane

$$x + y + z = 1,$$

all orbits inside the triangle are closed and every orbit in the interior of $\mathbb{R}^3_+$ has one of these closed orbits as an ω-limit. Latter on, the dynamics of Equation (2) was studied in more details in [4–6] and some other works.

A generalization of model (2) is the model described by the differential system

$$\begin{aligned} \dot{x} &= x(1 - x - \alpha_1 y - \beta_1 z) = X(x, y, z), \\ \dot{y} &= y(1 - \beta_2 x - y - \alpha_2 z) = Y(x, y, z), \\ \dot{z} &= z(1 - \alpha_3 x - \beta_3 y - z) = Z(x, y, z), \end{aligned} \tag{4}$$

where $x, y, z \geq 0$ and $\alpha_i,\ \beta_i > 0\ (1 \leq i \leq 3)$, which is called the asymmetric May–Leonard model. The dynamics of Equation (4) were studied in [6–9]. In particular, Chi, Hsu and Wu [8] studied (4) under the assumption

$$0 < \alpha_i < 1 < \beta_i\ (1 \leq i \leq 3) \tag{5}$$

and showed that under this assumption the system has a unique interior equilibrium P, which is locally asymptotically stable if

$$A_1 A_2 A_3 > B_1 B_2 B_3,$$

where $A_i = 1 - \alpha_i, \quad B_i = \beta_i - 1, \quad (1 \leq i \leq 3)$, and if

$$A_1 A_2 A_3 < B_1 B_2 B_3,$$

then P is a saddle point with a one-dimensional stable manifold. They also have shown that if $A_1 A_2 A_3 \neq B_1 B_2 B_3$, then the system does not have periodic solutions, and if

$$A_1 A_2 A_3 = B_1 B_2 B_3, \tag{6}$$

then there is a family of periodic solutions. It was shown in [7] that even if assumption (5) is dropped, the system (4) still can have a family of periodic solutions. Moreover, it was shown there, that the periodic solutions of the system do not arise as a result of Hopf bifurcations, but their existence is due to the Lyapunov theorem on holomorphic integrals.

First integrals of the May–Leonard system (2) were studied by Leach and Miritzis [10] (see also [11]), who obtained the following first integrals:

(i) $H_1 = \frac{xyz}{(x+y+z)^3}$ if $\alpha + \beta = 2$ and $\alpha \neq 1$,
(ii) $H_2 = \frac{y(x-z)}{x(y-z)}$ if $\alpha = \beta \neq 1$,
(iii) $H_3 = x/z$ and $H_4 = y/z$, which are two independent first integrals, if $\alpha = \beta = 1$.

It was shown in [4] that system (2) is completely integrable, that is, it admits two independent first integrals, if either $\alpha + \beta = 2$ or $\beta = \alpha$.

In this paper we study integrability of the asymmetric May–Leonard model (4). Using algorithms from elimination theory, we first find systems of the form in Equation (4) admitting invariant planes and invariant surfaces defined by the quadratic polynomials. Then we look for first integrals of the Darboux type constructed using these invariant surfaces and find subfamilies of (4) admitting one or two independent first integrals. As we show, the set of systems with first integrals is much larger for system (4) than for the classical May–Leonard system (2).

2. Preliminaries

In this section we recall some general results from elimination theory and the Darboux theory of integrability, which we shall use in our study.

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= P(x, y, z), \\ \dot{y} &= Q(x, y, z), \\ \dot{z} &= R(x, y, z), \end{aligned} \tag{7}$$

where P, Q and R are polynomials of degree at most m, and let $\mathfrak{X}$ be the corresponding vector field,

$$\mathfrak{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

A C^1 function

$$H : U \to \mathbb{R}$$

with $U \subset \mathbb{R}^3$, non-constant in any open subset of U is a first integral of the differential system (7) if and only if $\mathfrak{X}H \equiv 0$ in U. Let $H_1 : U_1 \to \mathbb{R}$ and $H_2 : U_2 \to \mathbb{R}$ be two first integrals of the system (7). It is said that H_1 and H_2 are functionally independent in $U_1 \cap U_2$ if their gradients are independent in all the points of $U_1 \cap U_2$ except perhaps in a zero Lebesgue measure set. Equivalently, $H_i = H_i(x, y, z),\ i = 1, 2$, are functionally independent if their Jacobian has maximal rank,

$$\text{rank}\frac{\partial(H_1, H_2)}{\partial(x, y, z)} = 2, \tag{8}$$

in all the points of $U_1 \cap U_2$ except perhaps on a zero Lebesgue measure set. System (7) is completely integrable in $\mathbb{R}^3$ if it has two independent first integrals in $\mathbb{R}^3$.

A Darboux polynomial of system (7) is a polynomial $f(x, y, z)$ such that

$$\mathfrak{X}f = \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = Kf, \tag{9}$$

where $K(x, y, z)$ is a polynomial of degree at most $m - 1$. The polynomial $K(x, y, z)$ is called the cofactor of f. It easy to see that if f is a Darboux polynomial of Equation (7), then the equation $f = 0$ defines an algebraic surface which is invariant under the flow of system (7). For this reason, f often is referred as an invariant algebraic surface of Equation (7).

A simple computation shows that if there are Darboux polynomials $f_1, f_2, ..., f_k$ with the cofactors $K_1, K_2, ..., K_k$ satisfying

$$\sum_{i=1}^{k} \lambda_i K_i = 0, \tag{10}$$

where $\lambda_1, \ldots, \lambda_k$ are some non-zero real numbers, then

$$H = f_1^{\lambda_1} \cdots f_k^{\lambda_k}, \tag{11}$$

is a first integral of (7). An integral of the form in Equation (11) is called a Darboux integral of system (7).

The ideas of the method go back to the works of Darboux [12,13]. Further developments of the approach were presented in the works of Prelle and Singer [14] and Schlomiuk [15,16]. In [14], the authors did not use the term "Darboux polynomials", but they proposed an algorithm to find first integrals using them. This algorithm was put in relation with the Darboux method in the work of Schlomiuk [15,16]. See also [17,18] for more details on the method.

To find Darboux polynomials (algebraic invariant surfaces) of system (4) we will use the following result from computational commutative algebra. Let I be an ideal in the polynomial ring $k[x_1, \ldots, x_n]$, where k is a field, and ℓ be a fixed number from the set $\{0, 1, \ldots, n-1\}$. The ℓ-th elimination ideal of I is the ideal

$$I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n].$$

According to the Elimination Theorem (see, for example, [19,20]) in order to compute (for any $0 \leqslant \ell \leqslant n-1$) the ℓ-th elimination ideal I_ℓ of an ideal I in $k[x_1, \ldots, x_n]$, one can choose the lexicographic term order with

$$x_1 > x_2 > \cdots > x_n$$

on the ring $k[x_1, \ldots, x_n]$ and compute a Gröbner basis G for the ideal I with respect to this order. Then, by the Elimination theorem, the set

$$G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$$

is a Gröbner basis for the ℓ-th elimination ideal I_ℓ. Geometrically, the elimination means projecting the variety $\mathbf{V}(I)$ of the ideal I to the affine space $k^{n-\ell}$ corresponding to the variables $x_{\ell+1}, \ldots, x_n$.

3. Darboux Polynomials of System (4)

In this section, using the Elimination Theorem, we look for Darboux polynomials of degree two for system (4). A general form of a polynomial of degree two is

$$\begin{aligned} f(x,y,z) = h_{000} + h_{100}x + h_{010}y + h_{001}z + h_{200}x^2 + h_{110}xy \\ + h_{101}xz + h_{020}y^2 + h_{011}yz + h_{002}z^2. \end{aligned} \tag{12}$$

A cofactor of any Darboux polynomials of system (7) is a polynomial of degree one which we write in the form

$$K(x,y,z) = c_0 + c_1x + c_2y + c_3z. \tag{13}$$

Polynomial (12) will be a Darboux polynomial of system (4) with cofactor (13) if

$$\mathfrak{X} f = Kf, \tag{14}$$

where now

$$\mathfrak{X} f := \frac{\partial f}{\partial x} X + \frac{\partial f}{\partial y} Y + \frac{\partial f}{\partial z} Z,$$

with X, Y and Z defined in (4).

Comparing the coefficients of the monomials on both sides of (14) we obtain the polynomial system

$$g_1 = g_2 = \ldots = g_{19} = g_{20} = 0,$$

where

$$
\begin{aligned}
g_1 &= -c_0 h_{000}, \\
g_2 &= -c_3 h_{000} + h_{001} - c_0 h_{001}, \\
g_3 &= -h_{001} - c_3 h_{001} + 2h_{002} - c_0 h_{002}, \\
g_4 &= -2h_{002} - c_3 h_{002}, \\
g_5 &= -c_2 h_{000} + h_{010} - c_0 h_{010}, \\
g_6 &= -\beta_3 h_{001} - c_2 h_{001} - \alpha_2 h_{010} - c_3 h_{010} + 2h_{011} - c_0 h_{011}, \\
g_7 &= -2\beta_3 h_{002} - c_2 h_{002} - h_{011} - \alpha_2 h_{011} - c_3 h_{011}, \\
g_8 &= -h_{010} - c_2 h_{010} + 2h_{020} - c_0 h_{020}, \\
g_9 &= -2h_{020} - c_2 h_{020}, \\
g_{10} &= -h_{011} - \beta_3 h_{011} - c_2 h_{011} - 2\alpha_2 h_{020} - c_3 h_{020}, \\
g_{11} &= -c_1 h_{000} + h_{100} - c_0 h_{100}, \\
g_{12} &= -\alpha_3 h_{001} - c_1 h_{001} - \beta_1 h_{100} - c_3 h_{100} + 2h_{101} - c_0 h_{101}, \\
g_{13} &= -2\alpha_3 h_{002} - c_1 h_{002} - h_{101} - \beta_1 h_{101} - c_3 h_{101}, \\
g_{14} &= -\beta_2 h_{010} - c_1 h_{010} - \alpha_1 h_{100} - c_2 h_{100} + 2h_{110} - c_0 h_{110}, \\
g_{15} &= -2\beta_2 h_{020} - c_1 h_{020} - h_{110} - \alpha_1 h_{110} - c_2 h_{110}, \\
g_{16} &= -\alpha_3 h_{011} - \beta_2 h_{011} - c_1 h_{011} - \alpha_1 h_{101} - \beta_3 h_{101} \\
&\quad - c_2 h_{101} - \alpha_2 h_{110} - \beta_1 h_{110} - c_3 h_{110}, \\
g_{17} &= -h_{100} - c_1 h_{100} + 2h_{200} - c_0 h_{200}, \\
g_{18} &= -2h_{200} - c_1 h_{200}, \\
g_{19} &= -h_{110} - \beta_2 h_{110} - c_1 h_{110} - 2\alpha_1 h_{200} - c_2 h_{200}, \\
g_{20} &= -h_{101} - \alpha_3 h_{101} - c_1 h_{101} - 2\beta_1 h_{200} - c_3 h_{200}.
\end{aligned} \tag{15}
$$

We denote by $I = \langle g_1, g_2, ..., g_{19}, g_{20} \rangle$ the ideal generated by polynomials (15). Since computations based on the Elimination Theorem are very laborious, to simplify them we consider separately the cases $h_{000} = 1$ and $h_{000} = 0$, that is, we look separately for invariant curves $f = 0$ not passing and passing through the origin, so from now on in this section we assume that $h_{000} = 1$.

To find Darboux polynomials of system (4) of degree two, we have to determine for which values of parameters α_i, β_i $(i = 1, 2, 3)$ system (15) has a solution with at least one of coefficient $h_{200}, h_{002}, h_{011}, h_{020}, h_{101}, h_{110}$ different from zero. To satisfy this condition we have six options that can be written in polynomial forms as

$$
\begin{aligned}
&1 - wh_{200} = 0, \;\; 1 - wh_{110} = 0, \;\; 1 - wh_{101} = 0, \\
&1 - wh_{020} = 0, \;\; 1 - wh_{011} = 0, \;\; 1 - wh_{002} = 0,
\end{aligned} \tag{16}
$$

respectively (where w is a new variable). For instance, to find systems of the form (4) which have surfaces with $h_{200} \neq 0$, we can compute (for example, with the routine eliminate of the computer algebra system SINGULAR [21]) the 13th elimination ideal of the ideal $I^{(1)} = \langle I, 1 - wh_{200} \rangle$, in the ring $\mathbb{Q}[w, c_0, c_1, c_2, c_3, h_{001}, h_{002}, h_{010}, h_{011}, h_{020}, h_{100}, h_{101}, h_{110}, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3]$. Denote this elimination ideal by $I_{13}^{(1)}$ and its variety by V_1 (that is, $V_1 = \mathbf{V}(I_{13}^{(1)})$). Proceeding analogously we find the other five elimination ideals $I_{13}^{(2)}, \ldots, I_{13}^{(6)}$ corresponding to the other cases of Equation (16). Denote the corresponding varieties $V_2 = \mathbf{V}(I_{13}^{(2)}), \ldots, \mathbf{V}(I_{13}^{(6)})$. It is clear that the union $V = V_1 \cup ... \cup V_6$ of these six varieties contains the set of all systems (4) having invariant surfaces of the form (12) not passing through the origin. To compute the irreducible decomposition of the variety V it is sufficient to compute the ideal $J = I_{13}^{(1)} \cap \cdots \cap I_{13}^{(6)}$, which defines the variety $V = V_1 \cup ... \cup V_6$ and then to find the irreducible decomposition of V. The intersection of ideals can be computed with the routine intersect of SINGULAR, and the irreducible decomposition of V can be found with the routine minAssGTZ [22], which is based on the algorithm of [23]. Theoretically, such computations should give all systems in family (4) having invariant surfaces of degree two. However all the routines eliminate, intersect and minAssGTZ rely on computations of many Gröbner bases, and such computations can be rarely completed when computing over the field $\mathbb{Q}$ of rational numbers for polynomials in many variables. To be able to complete our computations, we computed in the field of the finite characteristic 32003 and then lifted the resulting ideals to the ring of polynomials with rational coefficients using the rational reconstruction algorithm of [24] (a MATHEMATICA code for the algorithm can be found in [25]).

The primary decomposition of the radical of the ideal

$$J = \bigcap_{i=1}^{6} I_{13}^{(i)} \tag{17}$$

computed using the routine minAssGTZ in the field of characteristic 32003 consists of 88 ideals, that is, we have 88 irreducible components of the variety $\mathbf{V}(J)$ given in Appendix A. It means there are 88 conditions on the parameters α_i, β_i of system (4) for existence of an invariant surface of degree two not passing through the origin.

However some of these conditions give systems with the same dynamics in the phase space, since system (4) has a symmetry with respect to simple linear transformations. Namely, it is easily seen that the transformations

$$x \to z,\ y \to x,\ z \to y, \tag{18}$$

$$x \to y,\ y \to z,\ z \to x, \tag{19}$$

$$x \to y,\ y \to x,\ z \to z, \tag{20}$$

$$x \to z,\ y \to y,\ z \to z, \tag{21}$$

$$x \to x,\ y \to z,\ z \to y, \tag{22}$$

which correspond to re-labeling of the coordinate axes, do not change the shape of the system. For instance, under transformation (19) system (4) is changed into the system

$$\begin{aligned} \dot{x} &= x(1 - x - \alpha_2 y - \beta_2 z), \\ \dot{y} &= y(1 - \beta_3 x - y - \alpha_3 z), \\ \dot{z} &= z(1 - \alpha_1 x - \beta_1 y - z), \end{aligned} \tag{23}$$

which can be obtained from system (4) by the change of parameters

$$\alpha_1 \to \alpha_3,\ \beta_1 \to \beta_3,\ \alpha_2 \to \alpha_1,\ \beta_2 \to \beta_1,\ \alpha_3 \to \alpha_2,\ \beta_3 \to \beta_2. \tag{24}$$

Thus, if we have a condition on the parameters of Equation (4) under which the system has an algebraic invariant surface, another condition will be obtained by the transformation of the parameters according to rule (24). For example, as we will see below, system (4) has the invariant surface

$$f = 2 - 4x + 2x^2 - 2y + yz$$

if condition (5) of Theorem 1 is fulfilled, that is, if

$$\beta_3 = \beta_1 = \alpha_3 + 1 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 1/2 = 0.$$

Applying to Equation (4) the transformation (19) we obtain that system (4) has the invariant surface

$$f = 2 - 4z + 2z^2 - 2x + xy$$

if the condition
$$\beta_2 = \beta_3 = \alpha_2 + 1 = \beta_1 - 3 = \alpha_1 + 1 = \alpha_3 - 1/2 = 0$$
holds, that is, condition (3) is changed according to Equation (24). Similarly, after substitutions (20)–(22) the conditions for existence of invariant surfaces are changed according to the rules

$$\alpha_1 \to \alpha_2,\ \beta_1 \to \beta_2,\ \alpha_2 \to \alpha_3,\ \beta_2 \to \beta_3,\ \alpha_3 \to \alpha_1,\ \beta_3 \to \beta_1, \tag{25}$$
$$\alpha_1 \to \beta_2,\ \beta_1 \to \alpha_2,\ \alpha_2 \to \beta_1,\ \beta_2 \to \alpha_1,\ \alpha_3 \to \beta_3,\ \beta_3 \to \alpha_3, \tag{26}$$
$$\alpha_1 \to \beta_3,\ \beta_1 \to \alpha_3,\ \alpha_2 \to \beta_2,\ \beta_2 \to \alpha_2,\ \alpha_3 \to \beta_1,\ \beta_3 \to \alpha_1, \tag{27}$$
$$\alpha_1 \to \beta_1,\ \beta_1 \to \alpha_1,\ \alpha_2 \to \beta_3,\ \beta_2 \to \alpha_3,\ \alpha_3 \to \beta_2,\ \beta_3 \to \alpha_2, \tag{28}$$

respectively.

We say, that two conditions for existence of invariant planes are conjugate if one can be obtained from another by means of one of transformations (24), (25)–(28). For instance, condition (3) (which is the same as condition (7) from Appendix A) and conditions (10), (19), (25), (33), (47) from Appendix A can be obtained from each other by one of the transformations (24), (25)–(28), so all these conditions are conjugate.

Note that some of the obtained 88 conditions give Darboux polynomials of degree two which are not irreducible, but they are products of two polynomials of degree one. Namely, if

(i) $\alpha_1 = \beta_1 = 0$ (condition 1 of the Appendix), then system (4) has the Darboux polynomial $(-1+x)^2$ (and the conjugate conditions are 22 and 36 from Appendix A);
(ii) $\alpha_2 = \beta_1 = \beta_2 + \alpha_1 - 2 = 0$ (condition 5 of Appendix A), then system (4) has the Darboux polynomial $(-1+x+z)^2$ (and the conjugate conditions are 44 and 78 from Appendix);
(iii) $\beta_1 + \alpha_3 - 2 = \beta_2 + \alpha_1 - 2 = \beta_3 + \alpha_2 - 2 = 0$ (condition 88 of Appendix), then system (4) has the Darboux polynomial $(-1+x+y+z)^2$.

From the analysis of the obtained 88 conditions we obtain the following result.

Theorem 1. *System* (4) *has an irreducible invariant surface not passing through the origin if its parameters have the values given in the following Table 1 or are conjugate to them.*

Table 1. Parameter values for systems with invariant surfaces not passing through the origin.

	α_1	α_2	α_3	β_1	β_2	β_3	Condition in Appendix A
1.	3	0	α_3	0	1/2	β_3	2
2.	3	0	α_3	0	3	β_3	4
3.	$1+\alpha_3$	-1	α_3	0	$1-\alpha_3$	0	6
4.	1/2	-1	-1	0	3	0	7
5.	-1	3/2	3	0	3	0	8
6.	1	1/2	3	0	1	3	11
7.	3	-1	1/2	0	1/2	3	12
8.	1/2	3/2	3	0	3	1/2	14
9.	3	-1	-3	0	3	3	15
10.	1/2	3/2	2	0	3	1/2	16
11.	$1-\beta_3$	$2-\beta_3$	0	0	$1+\beta_3$	β_3	17
12.	α_3-2	-1	α_3	0	$4-\alpha_3$	3	18
13.	$2-\beta_2$	3	1/2	3	β_2	1/2	53
14.	1/2	3	3	3	3	1/2	54
15.	1/2	3	α_3	$2-\alpha_3$	3	1/2	55
16.	α_1	3	3	3	$2-\alpha_1$	3	65
17.	α_3-2	3	α_3	$2-\alpha_3$	$4-\alpha_3$	3	67

Remark 1. *For instance, the first row of the table means that the parameters α_3 and β_3 can be chosen arbitrary, the other parameters satisfy the condition*

$$\alpha_1 = 3,\ \alpha_2 = \beta_1 = 0,\ \beta_2 = \frac{1}{2},$$

and this is condition 2 from Appendix A.

Proof of Theorem 1. For each case of the theorem we give below the irreducible Darboux polynomial f of degree two which defines the invariant quadratic invariant surface $f = 0$ not passing through the origin and the corresponding cofactor:

1. $f = 1 - x - 2y + y^2;\quad K = -x - 2y;$
2. $f = 1 - 2x + x^2 - 2y - 2xy + y^2;\quad K = -2(x + y);$
3. $f = 2 - 2x - 2y + yz;\quad K = -x - y;$
4. $f = 2 - 4x + 2x^2 - 2y + yz;\quad K = -2x - y;$
5. $f = 2 - 4x + 2x^2 + 2xy - 2z + xz;\quad K = -2x - z;$
6. $f = 2 - 4x + 2x^2 - 4y + 4xy + 2y^2 - 2z + xz;\quad K = -2x - 2y - z;$
7. $f = 1 - x - 2y + y^2 + yz;\quad K = -x - 2y;$
8. $f = 2 - 4x + 2x^2 - 2y - 2z + xz;\quad K = -2x - y - z;$
9. $f = 1 - 2x + x^2 - 2y - 2xy + y^2 + yz;\quad K = -2(x + y);$
10. $f = 1 - 2x + x^2 - y - z + xz;\quad K = -2x - y - z;$
11. $f = 1 - x - y - z + xz;\quad K = -x - y - z;$
12. $f = 1 - 2x + x^2 - 2y + 2xy + y^2 + yz;\quad K = -2(x + y);$
13. $f = 1 - x - y - 2z + z^2;\quad K = -x - y - 2z;$
14. $f = 1 - 2x + x^2 - y - 2z - 2xz + z^2;\quad K = -2x - y - 2z;$
15. $f = 1 - 2x + x^2 - y - 2z + 2xz + z^2;\quad K = -2x - y - 2z;$
16. $f = 1 - 2x + x^2 - 2y + 2xy + y^2 - 2z - 2xz - 2yz + z^2;\quad K = -2(x + y + z);$
17. $f = 1 - 2x + x^2 - 2y + 2xy + y^2 - 2z + 2xz - 2yz + z^2;\quad K = -2(x + y + z).$

□

4. First Integrals of System (4)

In this section we look for Darboux first integrals of the system (4), which can be constructed using the invariant surfaces obtained in the previous section.

Theorem 2. *(a) If one of conditions 1–3, 11, 12, 17 of Theorem 1 holds, then the corresponding system (4) admits at least one Darboux first integral. (b) If one of conditions 4–10, 13–16 of Theorem 1 holds, then the corresponding system (4) is completely integrable, that is, it admits two independents Darboux first integrals.*

Proof. First note that system (4) always has the following three invariant surfaces of degree one, with the respective cofactors,

$$\begin{aligned} f_1 &= x;\quad K_1 = 1 - x - \alpha_1 y - \beta_1 z; \\ f_2 &= y;\quad K_2 = 1 - \beta_2 x - y - \alpha_2 z; \\ f_3 &= z;\quad K_3 = 1 - \alpha_3 x - \beta_3 y - z. \end{aligned} \tag{29}$$

However, in most cases it is impossible to construct Darboux first integrals using just these invariant planes and the surfaces given by Theorem 1. To find the integrals we additionally look for invariant surfaces of the form (12) with $h_{000} = 0$ using the procedure described at the beginning of Section 3. For each considered case we have to solve system (15) with $h_{000} = 0$ and parameters α_i, β_i $(i = 1, 2, 3)$ given by Theorem 1. Since some parameters are fixed the corresponding systems (15) are easily solved with MATHEMATICA (no need for computations with SINGULAR now).

Case (a). To prove statement (a) of the theorem, we present the Darboux first integrals for each case mentioned in the statement.

If condition (1) of Theorem 1 is satisfied the system has the form

$$\dot{x} = x(1 - x - 3y), \quad \dot{y} = y(1 - x/2 - y)y, \quad \dot{z} = z(1 - \alpha_3 x - \beta_3 y - z) \tag{30}$$

Besides the invariant surfaces f_1, f_2 and f_3 given above and the invariant surface

$$f = 1 - x - 2y + y^2 \tag{31}$$

system (30) has the following surfaces f_4, f_5 (with cofactors K_4, K_5, respectively),

$$\begin{aligned} f_4 &= x + 4y; \quad K_4 = 1 - x - y; \\ f_5 &= x + 2y - 2y^2; \quad K_5 = 1 - x - 2y. \end{aligned} \tag{32}$$

From the corresponding Equation (10) we find that $\lambda_1 = \lambda_3/2, \lambda_2 = \lambda_4, \lambda_5 = -\lambda_3 - 2\lambda_4, \lambda_6 = 0$. Thus, for arbitrary λ_3, λ_4 not both equal to zero system (30) has a Darboux first integral

$$\widetilde{H} = x^{\lambda_4} y^{\lambda_3} (x + 4y)^{\lambda_4} \left(x - 2y^2 + 2y\right)^{-\lambda_3 - 2\lambda_4} \left(-x + y^2 - 2y + 1\right)^{\frac{\lambda_3}{2}}.$$

In particular, taking $\lambda_4 = 1$ and $= \lambda_3 = 0$ we have the Darboux first integral

$$H = \frac{x(x + 4y)}{(x + 2y - 2y^2)^2}.$$

Using the same approach we obtain the following Darboux first integrals for the remaining cases:

(2) $H = \dfrac{xy}{(-x + x^2 - y - 2xy + y^2)^2}$;

(3) $H = \dfrac{(x + y - yz)^2}{x^2 + 2xy + y^2 - 2yz}$;

(11) $H = \dfrac{xz(1 - x - y - z + xz)}{(-x - y - z + 2xz)^2}$;

(12) $H = \dfrac{yz}{(-x + x^2 - y + 2xy + y^2 + yz)^2}$;

(17) $H = \dfrac{yz}{(-x + x^2 - y + 2xy + y^2 - z + 2xz - 2yz + z^2)^2}$.

Case (b). For each system of this case we present two independent Darboux first integrals.

Case (4). Besides the invariant surface f_1, f_2, f_3 given above and f of the previous theorem, we have the invariant surfaces $f_4 = 4x + y - 2z$ with the cofactor $K_4 = 1 - x - y - z$. Using these polynomials we can find the following two Darboux first integrals:

$$\begin{aligned} H_1 &= \frac{z(2 - 4x + 2x^2 - 2y + yz)}{(4x + y - 2z)}, \\ H_2 &= \frac{yz}{x^2}. \end{aligned}$$

To check if these first integrals are independent, we compute their gradients and obtain that they are

$$\begin{aligned} G_1 = \{ &\frac{4(-2 + 2x + y)(1 + x - z)z}{(4x + y - 2z)^2}, \quad -\frac{2(1 + x - z)^2 z}{(4x + y - 2z)^2}, \\ &\frac{2(4x - 8x^2 + 4x^3 + y - 6xy + x^2y - y^2 + 4xyz + y^2z - yz^2)}{(4x + y - 2z)^2} \}, \\ G_2 = \{ &-\frac{2yz}{x^3}, \quad \frac{z}{x^2}, \quad \frac{y}{x^2} \}, \end{aligned}$$

respectively. Then we verify if for the Jacobian

$$J = [G_1, G_2].$$

condition (8) holds. One of 2×2 minors of the matrix J is

$$m_1 = -\frac{2z^2(x-z+1)\left(2x^2-2x+yz-y\right)}{x^3(4x+y-2z)}.$$

Clearly, m_1 is different from zero on the neighborhood of the origin except the set of the points where

$$xz(x-z+1)(2x^2-2x+yz-y)(4x+y-2z)^2 = 0. \tag{33}$$

Since the set defined by Equation (33) has Lebesgue measure zero, the Darboux first integrals H_1 and H_2 are independent.

Case (5). Besides the invariant surfaces f_1, f_2, f_3 given in Equation (29) and f of the previous theorem, we have the following invariant surfaces passing through the origin (with the respective cofactors):

$$\begin{aligned}
&f_4 = -4xy + 2xz + z^2; \quad K_4 = -2(-1+2x+z);\\
&f_5 = 2y + z; \quad K_5 = 1 - 3x - y - z;\\
&f_6 = 2x + 2y + z; \quad K_6 = 1 - x - y - z;\\
&f_7 = -2x + 2x^2 + 2xy - z + xz; \quad K_7 = 1 - 2x - z.
\end{aligned}$$

Using these polynomials we can find the following two Darboux first integrals:

$$\begin{aligned}
H_1 &= \frac{xy^2}{z^2(2x+2y+z)},\\
H_2 &= \frac{xy^2}{(2y+z)(-2x+2x^2+2xy-z+xz)^2}.
\end{aligned}$$

The gradients of them are

$$\begin{aligned}
G_1 = \{&\frac{y^2(2y+z)}{z^2(2x+2y+z)^2}, \ \frac{2xy(2x+y+z)}{z^2(2x+2y+z)^2}, \ -\frac{xy^2(4x+4y+3z)}{z^3(2x+2y+z)^2}\},\\
G_2 = \{&-\frac{y^2(-2x+6x^2+2xy+z+xz)}{(2y+z)(-2x+2x^2+2xy-z+xz)^3},\\
&\frac{2xy(-2xy+2x^2y-2xy^2-2xz+2x^2z-yz+xyz-z^2+xz^2)}{(2y+z)^2(-2x+2x^2+2xy-z+xz)^3},\\
&-\frac{xy^2(-2x+2x^2-4y+6xy-3z+3xz)}{(2y+z)^2(-2x+2x^2+2xy-z+xz)^3}\},
\end{aligned}$$

respectively. Similarly as in the previous case, computing the minors of the Jacobian we check H_1 and H_2 are independent.

Using similar computations we get the following pairs of independent Darboux first integrals for the remaining cases:

(6) $$H_1 = \frac{z}{x(2-4x+2x^2-4y+4xy+2y^2-2z+xz)},$$
$$H_2 = \frac{z(2x+4y+z)}{y^2(2-4x+2x^2-4y+4xy+2y^2-2z+xz)};$$

(7) $$H_1 = -\frac{x^2}{yz(-1+x+2y-y^2-yz)},$$

$$H_2 = -\frac{(x-2z)^2}{z(y+z)(-1+x+2y-y^2-yz)};$$

(8) $$H_1 = \frac{xz(2-4x+2x^2-2y-2z+xz)}{(y+z)^2},$$

$$H_2 = \frac{(2x+z)(y+z)^2}{xy^2};$$

(9) $$H_1 = \frac{yz(1-2x+x^2-2y-2xy+y^2+yz)}{x^2},$$

$$H_2 = \frac{y^2z(4x-z)}{x^2(x^2-2xy+y^2+yz)};$$

(10) $$H_1 = -\frac{x(y-z+xz+z^2)}{z(-2x+2x^2-y-z+2xz)},$$

$$H_2 = \frac{y^2(x+z)}{z(2x-2x^2+y+z-2xz)(y-z+xz+z^2)};$$

(13) $$H_1 = \frac{(-x-y-2z+2z^2)^2}{z^2(1-x-y-2z+z^2)},$$

$$H_2 = \frac{xz^{2\alpha_1-2}(x+y+4z)^{2-2\alpha_1}}{y};$$

(14) $$H_1 = \frac{(-2x+2x^2-y-2z-4xz+2z^2)^2}{(x-z)^2(1-2x+x^2-y-2z-2xz+z^2)},$$

$$H_2 = \frac{x(y+4z)^2(-2x+2x^2-y-2z-4xz+2z^2)^2}{z(x-z)^4(1-2x+x^2-y-2z-2xz+z^2)^2};$$

(15) $$H_1 = \frac{y(4x+y+4z)}{(2x-2x^2+y+2z-4xz-2z^2)^2},$$

$$H_2 = \frac{x(4x+y+4z)^{1-\alpha_3}(-2x+2x^2-y-2z+4xz+2z^2)^{1+\alpha_3}}{z(x+z)^2(1-2x+x^2-y-2z+2xz+z^2)};$$

(16) $$H_1 = \frac{(x+y)z}{(-x+x^2-y+2xy+y^2-z-2xz-2yz+z^2)^2},$$

$$H_2 = \frac{1}{y^4}x^4z^{2-2\alpha_1}(1-2x+x^2-2y+2xy+y^2-2z-2xz-2yz+z^2)^{1-\alpha_1}$$
$$(-x+x^2-y+2xy+y^2-z-2xz-2yz+z^2)^{2\alpha_1-2}.$$

□

Remark 2. *Note that first two equations of* (30) *are independent of z, they are,*

$$\dot{x} = x(1-x-3y), \quad \dot{y} = y(1-x/2-y)y. \tag{34}$$

Therefore we cannot construct another independent first integral $H_2(x,y,z)$ of Equation (30) *using only the planes $x=0$, $y=0, z=0$ and the surfaces defined by Equations* (31) *and* (32)*. Indeed, since the equations of all surfaces of this case, except of $z=0$, are independent on z, if such integral would exist, it would be independent of z, but then two-dimensional system* (34) *would have two independent first integrals, which is impossible.*

In case (2), similarly as in case (1), the system is separable into a two-dimensional system and a single first order equation, and we can construct only one Darboux first integral using the found invariant surfaces.

5. Conclusions

To summarize, we have found some Darboux first integrals of the May–Leonard system (4) which are constructed using Darboux polynomials of degree one and two. We do not know if we found all independent first integrals of system (4) which can be constructed from Darboux polynomials of degree one and two. To verify if the list is complete, we have to find Darboux polynomials of Equation (4), which define invariant algebraic surfaces passing through the origin, that is, polynomials (12) with $h_{000} = 0$. A naïve expectation is that this case should be simpler, than the case $h_{000} = 1$, which we have successfully investigated in this paper. However it turns out that the case $h_{000} = 0$ is computationally much more difficult and we were not able to complete computations for this case using our computational facilities. We believe that a reason for this difficulty is that since the origin is a singular point there are many invariant surfaces passing through the origin and it implies a complicated structure of the elimination ideals which we have to compute using our approach.

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Appendix A

Here we list the irreducible components of the variety of ideals in (17), which give conditions for existence in system (4) of invariant surfaces of degree two not passing through the origin of the system:

1. $\alpha_1 = \beta_1 = 0$
2. $-(1/2) + \beta_2 = \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
3. $-3 + \beta_2 = \alpha_2 = \beta_1 = -(1/2) + \alpha_1 = 0$
4. $-3 + \beta_2 = \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
5. $\alpha_2 = \beta_1 = \beta_2 + \alpha_1 - 2 = 0$
6. $\beta_3 = -1 + \alpha_3 + \beta_2 = 1 + \alpha_2 = \beta_1 = -1 + \alpha_1 - \alpha_3 = 0$
7. $\beta_3 = 1 + \alpha_3 = -3 + \beta_2 = 1 + \alpha_2 = \beta_1 = -(1/2) + \alpha_1 = 0$
8. $\beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(3/2) + \alpha_2 = \beta_1 = 1 + \alpha_1 = 0$
9. $-3 + \beta_3 = -(3/2) + \alpha_3 = \beta_2 = 1 + \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
10. $-3 + \beta_3 = 1 + \alpha_3 = \beta_2 = -(1/2) + \alpha_2 = \beta_1 = 1 + \alpha_1 = 0$
11. $-3 + \beta_3 = -3 + \alpha_3 = -1 + \beta_2 = -(1/2) + \alpha_2 = \beta_1 = -1 + \alpha_1 = 0$
12. $-3 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = 1 + \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
13. $1 + \alpha_3 = \beta_2 = -2 + \alpha_2 + \beta_3 = \beta_1 = -1 + \alpha_1 + \beta_3 = 0$
14. $-(1/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(3/2) + \alpha_2 = \beta_1 = -(1/2) + \alpha_1 = 0$
15. $-3 + \beta_3 = 3 + \alpha_3 = -3 + \beta_2 = 1 + \alpha_2 = \beta_1 = -3 + \alpha_1 = 0$
16. $-(1/2) + \beta_3 = -2 + \alpha_3 = -3 + \beta_2 = -(3/2) + \alpha_2 = \beta_1 = -(1/2) + \alpha_1 = 0$
17. $\alpha_3 = -1 + \beta_2 - \beta_3 = -2 + \alpha_2 + \beta_3 = \beta_1 = -1 + \alpha_1 + \beta_3 = 0$
18. $-3 + \beta_3 = -4 + \alpha_3 + \beta_2 = 1 + \alpha_2 = \beta_1 = 2 + \alpha_1 - \alpha_3 = 0$
19. $1 + \beta_3 = -3 + \alpha_3 = 1 + \beta_2 = \alpha_2 = -(1/2) + \beta_1 = \alpha_1 = 0$
20. $1 + \beta_3 = -1 + \alpha_3 + \beta_2 = \alpha_2 = -2 + \alpha_3 + \beta_1 = \alpha_1 = 0$
21. $-(3/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = \alpha_2 = 1 + \beta_1 = \alpha_1 = 0$
22. $\alpha_2 = \beta_2 = 0$
23. $-2 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = \alpha_2 = -(3/2) + \beta_1 = -3 + \alpha_1 = 0$
24. $-3 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = \alpha_2 = -(3/2) + \beta_1 = -3 + \alpha_1 = 0$
25. $1 + \beta_3 = \alpha_3 = -(1/2) + \beta_2 = \alpha_2 = 1 + \beta_1 = -3 + \alpha_1 = 0$
26. $3 + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = \alpha_2 = 1 + \beta_1 = -3 + \alpha_1 = 0$
27. $-(1/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = \alpha_2 = 1 + \beta_1 = -(1/2) + \alpha_1 = 0$
28. $\alpha_3 = -1 + \beta_2 - \beta_3 = \alpha_2 = 1 + \beta_1 = -1 + \alpha_1 + \beta_3 = 0$
29. $-3 + \beta_3 = \alpha_3 = 1 + \beta_2 = \alpha_2 = -(3/2) + \beta_1 = -3 + \alpha_1 = 0$
30. $-3 + \alpha_3 = 2 + \beta_2 - \beta_3 = \alpha_2 = 1 + \beta_1 = -4 + \alpha_1 + \beta_3 = 0$
31. $-3 + \beta_3 = -3 + \alpha_3 = -1 + \beta_2 = \alpha_2 = -(1/2) + \beta_1 = -1 + \alpha_1 = 0$
32. $\beta_3 = -1 + \alpha_3 + \beta_2 = \alpha_2 = -2 + \alpha_3 + \beta_1 = -1 + \alpha_1 - \alpha_3 = 0$
33. $\beta_3 = -(1/2) + \alpha_3 = \beta_2 = 1 + \alpha_2 = -3 + \beta_1 = 1 + \alpha_1 = 0$
34. $\beta_3 = \beta_2 = 1 + \alpha_2 - \alpha_3 = -2 + \alpha_3 + \beta_1 = 1 + \alpha_1 = 0$
35. $\beta_3 = 1 + \alpha_3 = \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(3/2) + \alpha_1 = 0$
36. $\alpha_3 = \beta_3 = 0$

37. $\beta_3 = -(1/2) + \alpha_3 = -3 + \beta_1 = \alpha_1 = 0$
38. $\beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = -2 + \alpha_2 = -3 + \beta_1 = -(3/2) + \alpha_1 = 0$
39. $\beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(3/2) + \alpha_1 = 0$
40. $\beta_3 = -3 + \alpha_3 = -(1/2) + \beta_1 = \alpha_1 = 0$
41. $\beta_3 = -3 + \alpha_3 = -3 + \beta_1 = \alpha_1 = 0$
42. $\beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = 1 + \alpha_1 = 0$
43. $\beta_3 = -3 + \alpha_3 = -3 + \beta_2 = 3 + \alpha_2 = -3 + \beta_1 = 1 + \alpha_1 = 0$
44. $\alpha_1 = \beta_1 + \alpha_3 - 2 = \beta_3 = 0$
45. $\beta_3 = -1 + \alpha_3 = -3 + \beta_2 = -3 + \alpha_2 = -1 + \beta_1 = -(1/2) + \alpha_1 = 0$
46. $\beta_3 = -3 + \beta_2 = -2 + \alpha_2 - \alpha_3 = -2 + \alpha_3 + \beta_1 = 1 + \alpha_1 = 0$
47. $-(1/2) + \beta_3 = \alpha_3 = 1 + \beta_2 = -3 + \alpha_2 = 1 + \beta_1 = \alpha_1 = 0$
48. $-(1/2) + \beta_3 = -1 + \alpha_3 = -3 + \beta_2 = -3 + \alpha_2 = -1 + \beta_1 = \alpha_1 = 0$
49. $-(1/2) + \beta_3 = \alpha_3 = \beta_2 = -3 + \alpha_2 = 0$
50. $-(1/2) + \beta_3 = 1 + \alpha_3 = \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(1/2) + \alpha_1 = 0$
51. $-(1/2) + \beta_3 = \alpha_3 = -(3/2) + \beta_2 = -3 + \alpha_2 = -2 + \beta_1 = -(1/2) + \alpha_1 = 0$
52. $-(1/2) + \beta_3 = \alpha_3 = -(3/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(1/2) + \alpha_1 = 0$
53. $-(1/2) + \beta_3 = -(1/2) + \alpha_3 = -3 + \alpha_2 = -3 + \beta_1 = -2 + \alpha_1 + \beta_2 = 0$
54. $-(1/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -(1/2) + \alpha_1 = 0$
55. $-(1/2) + \beta_3 = -3 + \beta_2 = -3 + \alpha_2 = -2 + \alpha_3 + \beta_1 = -(1/2) + \alpha_1 = 0$
56. $-3 + \beta_3 = \alpha_3 = \beta_2 = -(1/2) + \alpha_2 = 0$
57. $-3 + \beta_3 = \alpha_3 = \beta_2 = -3 + \alpha_2 = 0$
58. $-3 + \beta_3 = -(3/2) + \alpha_3 = \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -2 + \alpha_1 = 0$
59. $-3 + \beta_3 = -(3/2) + \alpha_3 = \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -3 + \alpha_1 = 0$
60. $-3 + \beta_3 = 1 + \alpha_3 = \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = 3 + \alpha_1 = 0$
61. $-3 + \beta_3 = \alpha_3 = 1 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -3 + \alpha_1 = 0$
62. $-3 + \beta_3 = \alpha_3 = 1 + \beta_2 = -3 + \alpha_2 = 3 + \beta_1 = -3 + \alpha_1 = 0$
63. $-3 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = -3 + \alpha_1 = 0$
64. $-3 + \beta_3 = -3 + \alpha_3 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -2 + \alpha_1 + \beta_2 = 0$
65. $-3 + \beta_3 = -3 + \alpha_3 = -3 + \alpha_2 = -3 + \beta_1 = -2 + \alpha_1 + \beta_2 = 0$
66. $-3 + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = -3 + \alpha_1 = 0$
67. $-3 + \beta_3 = -4 + \alpha_3 + \beta_2 = -3 + \alpha_2 = -2 + \alpha_3 + \beta_1 = 2 + \alpha_1 - \alpha_3 = 0$
68. $-3 + \beta_3 = -(1/2) + \beta_2 = -(1/2) + \alpha_2 = -2 + \alpha_3 + \beta_1 = -3 + \alpha_1 = 0$
69. $-3 + \beta_3 = -3 + \beta_2 = -3 + \alpha_2 = -2 + \alpha_3 + \beta_1 = -3 + \alpha_1 = 0$
70. $-3 + \alpha_3 + \beta_3 = \beta_2 = -2 + \alpha_2 + \beta_3 = 1 + \beta_1 - \beta_3 = \alpha_1 = 0$
71. $\alpha_3 = 1 + \beta_2 = -2 + \alpha_2 + \beta_3 = 1 + \beta_1 - \beta_3 = \alpha_1 = 0$
72. $1 + \beta_3 = \alpha_3 = -(3/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = \alpha_1 = 0$
73. $1 + \beta_3 = -(1/2) + \alpha_3 = -(1/2) + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = \alpha_1 = 0$
74. $-(3/2) + \beta_3 = -3 + \alpha_3 = -2 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = \alpha_1 = 0$
75. $-(3/2) + \beta_3 = -3 + \alpha_3 = -3 + \beta_2 = -(1/2) + \alpha_2 = -(1/2) + \beta_1 = \alpha_1 = 0$
76. $1 + \beta_3 = -3 + \alpha_3 = 3 + \beta_2 = -3 + \alpha_2 = -3 + \beta_1 = \alpha_1 = 0$
77. $1 + \beta_3 = -4 + \alpha_3 + \beta_2 = -3 + \alpha_2 = -2 + \alpha_3 + \beta_1 = \alpha_1 = 0$
78. $\alpha_3 = \beta_2 = \beta_3 + \alpha_1 - 2 = 0$
79. $1 + \alpha_3 = \beta_2 = -2 + \alpha_2 + \beta_3 = -3 + \beta_1 = -4 + \alpha_1 + \beta_3 = 0$
80. $-1 + \beta_3 = -(1/2) + \alpha_3 = \beta_2 = -1 + \alpha_2 = -3 + \beta_1 = -3 + \alpha_1 = 0$
81. $-1 + \beta_3 = \alpha_3 = -(1/2) + \beta_2 = -1 + \alpha_2 = -3 + \beta_1 = -3 + \alpha_1 = 0$
82. $-(1/2) + \alpha_3 = -(1/2) + \beta_2 = -2 + \alpha_2 + \beta_3 = -3 + \beta_1 = -3 + \alpha_1 = 0$
83. $-3 + \alpha_3 = -3 + \beta_2 = -2 + \alpha_2 + \beta_3 = -(1/2) + \beta_1 = -(1/2) + \alpha_1 = 0$
84. $-3 + \alpha_3 = -3 + \beta_2 = -2 + \alpha_2 + \beta_3 = -3 + \beta_1 = -3 + \alpha_1 = 0$
85. $\alpha_3 + \beta_3 = -3 + \beta_2 = -2 + \alpha_2 + \beta_3 = -2 + \beta_1 - \beta_3 = -3 + \alpha_1 = 0$
86. $\alpha_3 = 1 + \beta_2 = -2 + \alpha_2 + \beta_3 = -2 + \beta_1 - \beta_3 = -3 + \alpha_1 = 0$
87. $-3 + \alpha_3 = 2 + \beta_2 - \beta_3 = -2 + \alpha_2 + \beta_3 = -3 + \beta_1 = -4 + \alpha_1 + \beta_3 = 0$
88. $\beta_1 + \alpha_3 - 2 = \beta_2 + \alpha_1 - 2 = \beta_3 + \alpha_1 - 2 = 0.$

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