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Optimal Approximate Solution of Coincidence Point Equations in Fuzzy Metric Spaces

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Abstract: The purpose of this paper is to introduce α_f-proximal $\mathcal{H}$-contraction of the first and second kind in the setup of complete fuzzy metric space and to obtain optimal coincidence point results. The obtained results unify, extend and generalize various comparable results in the literature. We also present some examples to support the results obtained herein.

Keywords: fuzzy metric space; t-norm; optimal coincidence point; proximal contraction

MSC: 47H10; 47H04; 47H07

1. Introduction and Preliminaries

Several nonlinear problems arising in various branches of mathematics, engineering, economics, physics, astronomy, biology and economics can be formulated as a fixed point problem of the form $fx = x$, where f is a nonlinear operator defined on a set equipped with some topological structure. Due to an equivalence among fixed point problem and integral and differential equation problem, variational inequality problem and optimization problems attracted the attention of researchers [1–3]. The Banach [4] contraction principle is one of the significant tools for solving such problems.

On the other hand, fixed point equation $Tx = x$ has no solution if $T : A \to B$, where A and B are any nonempty disjoint subsets of a metric space (X, d). It is then natural to find a point $x \in A$ such that the error between x and Tx is minimum. Such a point is called an *approximate solution* of a fixed point equation.

A study of necessary conditions to guarantee the existence of an *approximate solution* of fixed point equations has its due importance in fixed point theory. Among approximate solutions, finding an *optimal solution* is an active research area.

A point x^* in A which satisfies $d(x^*, Tx^*) = d(A, B)$ is called a *best proximity point* of T and the pair (x^*, Tx^*) is called a best proximity pair. A best proximity point x^* in A indeed solves the following optimization problem:

$$\min_{x\in A} d(x, Tx).$$

Best proximity pair theorem deals with the conditions which guarantee the solution of optimization problem given above. Clearly, if sets A and B are not disjoint or identical, then *best proximity point* and *fixed point problem* of mapping T become equivalent and hence *best proximity point results* are a potential generalization of *fixed point results*.

A *classical best approximation* result by K. Fan in [5] reads as follows: *Let $T : A \to X$ be a continuous mapping, where A is nonempty compact convex subset of a Banach space then T has approximate fixed point in A*. For more results, see [6–9].

On the other hand, a framework of probabilistic metric spaces is a matter of great interest for engineers, social scientist and mathematicians, see, for example, [10–13]. Kramosil and Michalek [14] proposed the concept of fuzzy metric space. In [15] using continues t-norm, the concept of fuzzy metric spaces was modified. This modification can be viewed as a generalization of probabilistic metric space to fuzzy case (see [14]).

For some interesting fixed point results in the setup of fuzzy metric space, we refer the reader to [16–18]. Vetro and Salimi [19] studied best proximity point theorems in the framework of non-Archimedean fuzzy metric spaces—see also [20,21].

This paper deals with the problem of finding an optimal approximate solution of coincidence point equation in the framework of fuzzy metric spaces. We study necessary conditions which guarantee the existence and uniqueness of such solutions. The main focus lies on introducing general contractive conditions on operators T and g so that the solution is guaranteed. This paper is divided into four sections: some known definitions, lemmas and important results are discussed in the first section. In the second section, optimal coincidence best proximity point results in complete fuzzy metric space are studied. In the next section, we obtain similar results in complete ordered fuzzy metric space. Section 4 is devoted to the applications of obtained results in fixed point theory. Conclusions are given in the last section.

In this section, some basic definitions and known results are discussed which will be needed in the sequel.

Definition 1. *A commutative and associative binary operation $*$ on $[0,1]$ is called $t-norm$ if $\alpha * 1 = \alpha$ and $\alpha * \beta \leq \gamma * \delta$ whenever $\alpha \leq \gamma$ and $\beta \leq \delta$ holds true, for any $\alpha, \beta, \gamma, \delta \in [0,1]$. Moreover, if $*$ is a continuous mapping, then $*$ is called a continuous t-norm [10].*

Define binary operations $\wedge$, ., and $*_L$ on $[0,1]$ by $a \wedge b = \min\{a,b\}$, $a \cdot b = ab$, and $a *_L b = \max\{a+b-1, 0\}$. Note that $\wedge$, ., and $*_L$ are continuous t-norms, called *min*, *product* and *Lukasiewicz* t-norms, respectively. Furthermore, $*_L \leq \cdot \leq \wedge$.

Definition 2. *A fuzzy set M on $X \times X \times [0,\infty)$ is called a fuzzy metric (compare [22]) if:*

(i) $M(x,y,t)$ *is positive,*
(ii) $M(x,y,t) = 1 \Leftrightarrow x = y$,
(iii) $M(y,x,t) = M(x,y,t)$,
(iv) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$,
(v) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ *is left continuous,*

for any $x,y,z \in X$, and $t,s > 0$, where X is a nonempty set and $*$ is continuous t-norm. The triplet $(X, M, *)$ is said to be a fuzzy metric space.

In the above definition, if

$$M(x,z,t+s) \geq M(x,y,t) * M(y,z,s) \text{ for all } t,s > 0,$$

is replaced with

$$M(x,z,\max\{t,s\}) \geq M(x,y,t) * M(y,z,s) \text{ for all } t,s > 0,$$

then M is called non-Archimedean fuzzy metric on X.

Since M is a fuzzy set on $X^2 \times [0,\infty)$ and $M(x,y,t)$ is regarded as the degree of closeness of x and y with respect to $t \geq 0$.

Furthermore, $M(x,y,\cdot)$ is a nondecreasing function on $(0,\infty)$, for each $x,y \in X$ [23].

The set

$$B_M(x,\varepsilon,t) = \{y \in X : M(x,y,t) > 1-\varepsilon\}$$

is an open $M-$ ball in X, where $x \in X, \varepsilon \in (0,1), t > 0$. Note that fuzzy metric M induces Hausdorff topology on X. A sequence $\{x_n\}$ converges to an element x in fuzzy metric space X (with respect to τ_M) if and only if $\lim\limits_{n\to\infty} M(x_n,x,t) = 1$ for all $t > 0$. A sequence $\{x_n\}$ is a Cauchy sequence in a fuzzy metric space X if, for each $t > 0$ and $\varepsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\varepsilon$ for all $n,m \geq n_0$. If every Cauchy sequence in a fuzzy metric space X is convergent, then X is called a complete. If the limit of any convergent sequence in A belongs to A, then A is closed. If each sequence in A has a convergent subsequence, then A is compact.

Define a fuzzy metric M_d on a given metric space (X,d) by

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}.$$

Then, $(X, M_d, \cdot)$ is called standard fuzzy metric space [15].

Let A and B be nonempty subsets of a fuzzy metric space X. Then,

$$M(x,A,t) = \sup_{a\in A} M(x,a,t), \text{ for } t > 0$$

gives distance of a point $x \in X$ from A. Moreover,

$$M(A,B,t) = \sup\{M(a,b,t) : a \in A, b \in B\}.$$

is the distance between A and B. Consider a *coincidence point equation* $gx = Tx$. A point x in A is said to be an *optimal solution* of *coincidence point equation* $Tx = gx$, if

$$M(gx,Tx,t) = M(A,B,t)$$

holds [20].

Definition 3. *[19] Let $x \in X$ and $t > 0$. Define $A_0(t)$ and $B_0(t)$ as follows:*

$$\begin{aligned} A_0(t) &= \{x \in A : M(x,y,t) = M(A,B,t) \text{ for some } y \in B\}, \text{ and} \\ B_0(t) &= \{y \in B : M(x,y,t) = M(A,B,t) \text{ for some } x \in A\}. \end{aligned}$$

Definition 4. *[20] Let f : be a self mapping on A if $M(fx,fy,t) \leq M(x,y,t)$ for any $x,y \in A$ and $t > 0$ then f is called fuzzy expansive.*

If, in the above definition, inequality is replaced with equality, then f is called fuzzy isometry.

Definition 5. *[20] A set B is said to be fuzzy approximately compact with respect to A if for every sequence $\{y_n\}$ in B and for some $x \in A$, $M(x,y_n,t) \longrightarrow M(x,B,t)$ implies that $x \in A_0(t)$.*

Wardowski [18] defined a class $\mathcal{H}$ of mapping that consists upon the mappings $\eta : (0,1] \to [0,\infty)$, where η is continuous and strictly decreasing on $[0,1]$. It follows from the definition of $\mathcal{H}$ that $\eta(1) = 0$ for any $\eta \in \mathcal{H}$.

Definition 6. *[18] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be M-Cauchy if, for every $\epsilon \in (0, 1)$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\eta(M(x_m, x_n, t)) < \epsilon$$

for all $m, n \geq n_0$, where $\eta \in \mathcal{H}$.

Definition 7. *A mapping $T : X \to X$ is said to be (a) $\alpha-$admissible if $\alpha(x, y, t) \geq 1$ implies that $\alpha(Tx, Ty, t) \geq 1$; (b) α_R-admissible if*

$$\alpha(Tx, Ty, t) \geq 1 \text{ that implies } \alpha(x, y, t) \geq 1.$$

Definition 8. *A sequence $\{x_n\}$ in X converging to an element $x \in X$ is said to be α-regular if for $\alpha(x_n, x_{n+1}, t) \geq 1$, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x, t) \geq 1$ holds for all $k \in \mathbb{N}$.*

Proposition 1. *A set A is said to be α-complete, if, for any sequence, $\{y_n\}$ in A with $\alpha(y_n, y_{n+1}, t) \geq 1$ and $y_n \to y_0$ as $n \to \infty$ implies $\alpha(y_n, y_0, t) \geq 1$.*

Definition 9. *A mapping $T : A \to B$ is said to be α_f-proximal admissible mapping if for any $x, y, u, v \in A$ and $t > 0$,*

$$\left.\begin{array}{r} \alpha(x, y, t) \geq 1 \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \Longrightarrow \alpha(u, v, t) \geq 1.$$

Definition 10. *A mapping $T : A \longrightarrow B$ is said to be an α_f-proximal $\mathcal{H}$-contraction of first kind if for any u, v, x, y in A and $t > 0$, there exists a function $\eta \in \mathcal{H}$ such that*

$$\left.\begin{array}{r} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\}$$
$$\Longrightarrow \quad \alpha(x, y, t)\eta(M(u, v, t)) \leq k[\eta(M(x, y, t))].$$

Definition 11. *Let $g : A \to A$. A mapping $T : A \longrightarrow B$ is said to be an α_f-proximal $\mathcal{H}$-contraction of second kind if for any u, v, x, y in A and $t > 0$, there exists a function $\eta \in \mathcal{H}$ such that*

$$\left.\begin{array}{r} M(gu, Tx, t) = M(A, B, t) \\ M(gv, Ty, t) = M(A, B, t) \end{array}\right\}$$
$$\Longrightarrow \quad \alpha(gx, gy, t)\eta(M(gu, gv, t)) \leq k[\eta(M(x, y, t)]).$$

In the above definition, if we take $g = I_A$, then α_f-proximal $\mathcal{H}-$contraction of second kind becomes α_f-proximal $\mathcal{H}-$contraction of first kind.

2. Optimal Coincidence Point Solution in Fuzzy Metric Spaces

We start with the following result.

Lemma 1. *Let $T : A \longrightarrow B$ be an α_f-proximal admissible mapping. Suppose that $A_0(t) \neq \phi$ and $T(A_0(t)) \subseteq B_0(t)$. If there exists $x_0, x_1 \in A_0(t)$ such that $M(x_1, Tx_0, t) = M(A, B, t)$ and $\alpha(x_0, x_1, t) \geq 1$, then starting with x_0 in $A_0(t)$, we may find a sequence $\{x_n\} \subset A_0(t)$ such that*

$$\left.\begin{array}{l} M(x_{n+1}, Tx_n, t) = M(A, B, t) \\ \text{and } \alpha(x_{n+1}, x_n, t) \geq 1 \text{ for all } n \in \mathbb{N}. \end{array}\right\} \tag{1}$$

Proof. By given assumption, $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, there exists $x_2 \in A$ such that $M(x_2, Tx_1, t) = M(A, B, t)$ and hence $x_2 \in A_0(t)$. Thus, we have

$$\begin{aligned} \alpha(x_0, x_1, t) &\geq 1, \\ M(x_1, Tx_0, t) &= M(A, B, t) \text{ and} \\ M(x_2, Tx_1, t) &= M(A, B, t). \end{aligned}$$

As T is α_f-proximal admissible mapping, we obtain that $\alpha(x_1, x_2, t) \geq 1$. Continuing this way, we obtain a sequence $\{x_n\} \subset A_0(t)$ which satisfies condition (1). □

Definition 12. *A sequence $\{x_n\} \subset A_0(t)$ satisfying condition (1) is called (α, T)-proximal fuzzy sequence starting with $x_0 \in A_0(t)$.*

Definition 13. *A set $A_0(t)$ is called proximal (α, T)-complete if and only if every (α, T)-proximal fuzzy Cauchy sequence starting with some $x_0 \in A_0(t)$ converges to an element in $A_0(t)$.*

We also need following Lemma in the sequel.

Lemma 2. *Let $T : A \to B$, where A and B are nonempty closed subsets of a complete fuzzy metric space X, if $A_0(t) \neq \emptyset$ and $T(A_0(t)) \subseteq B_0(t)$. Then, the set $A_0(t)$ is proximal (α, T)-complete provided that B is approximately compact with respect to A.*

Proof. Let x_0 be a given point in $A_0(t)$ and $\{x_n\}$ a (α, T)-proximal fuzzy Cauchy sequence starting with some $x_0 \in A_0(t)$, that is,

$$M(x_n, Tx_{n-1}, t) = M(A, B, t) \text{ with } \alpha(x_n, x_{n-1}, t) \geq 1.$$

Since $(X, M, *)$ is complete and A is closed, there exist an element x^* in A such that $\lim_{n\to\infty} M(x_n, x^*, t) = 1$. Furthermore,

$$M(x_n, B, t) \geq M(x_n, Tx_{n-1}, t) = M(A, B, t) \geq M(x_n, B, t).$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we have

$$M(x^*, B, t) \geq M(x^*, Tx^*, t) \geq M(x^*, B, t),$$

which implies that

$$M(x^*, B, t) = M(x^*, Tx^*, t).$$

Taking $y_n = Tx^*$ for all $n \in \mathbb{N}$ and using the assumption that B is approximately compact with respect to A, we have $x^* \in A_0(t)$. □

Theorem 1. *Let $g : A \to A$ be a one to one fuzzy expansive and α_R-admissible mapping with $\emptyset \neq A_0(t) \subseteq g(A_0(t))$ for any $t > 0$. Suppose that a continuous mapping $T : A \to B$ is α_f-proximal $\mathcal{H}$-contraction of second kind and α_f-proximal admissible mapping with $T(A_0(t)) \subseteq B_0(t)$, where B is fuzzy approximately compact with respect to A. If there exists $x_0, x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$ and $\alpha(x_0, x_1, t) \geq 1$. Then, mappings g and T have a unique optimal coincidence point x^* in $A_0(t)$.*

Proof. Let x_0, x_1 be a given point in $A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$ and $\alpha(x_0, x_1, t) \geq 1$. Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$ for any $t > 0$, it follows that there exists an element x_2 in $A_0(t) \subseteq g(A_0(t))$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Since T is α_f-proximal admissible mapping and g is α_R-admissible mapping, $\alpha(gx_1, gx_2, t) \geq 1$ implies that $\alpha(x_1, x_2, t) \geq 1$. Continuing this way, we can obtain a sequence $\{gx_n\}$ in $A_0(t)$ such that the following holds true:

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t) = M(gx_{n+2}, Tx_{n+1}, t) \text{ with } \alpha(x_n, x_{n+1}, t) \geq 1 \text{ and } \alpha(x_{n+1}, x_{n+2}, t) \geq 1,$$

which implies that

$$\begin{aligned} \eta(M(x_{n+1}, x_{n+2}, t)) \leq \eta(M(gx_{n+1}, gx_{n+2}, t)) &\leq \alpha(x_n, x_{n+1}, t)\eta(M(gx_{n+1}, gx_{n+2}, t)) \\ &\leq k\eta(M(x_n, x_{n+1}, t)). \end{aligned} \quad (2)$$

In addition,

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) = M(gx_{n+1}, Tx_n, t)$$

implies that

$$\begin{aligned} \eta(M(x_n, x_{n+1}, t)) \leq \eta(M(gx_n, gx_{n+1}, t)) &\leq \alpha(x_{n-1}, x_n, t)\eta(M(gx_n, gx_{n+1}, t)) \\ &\leq k\eta(M(x_{n-1}, x_n, t)). \end{aligned} \quad (3)$$

From Labels (3) and (2), we have

$$\begin{aligned} \eta(M(x_{n+1}, x_{n+2}, t)) \leq \eta(M(gx_{n+1}, gx_{n+2}, t)) &\leq \alpha(x_n, x_{n+1}, t)\eta(M(gx_{n+1}, gx_{n+2}, t)) \\ &\leq k\eta(M(x_n, x_{n+1}, t)) \\ &\leq k^2\eta(M(x_{n-2}, x_{n-1}, t)). \end{aligned}$$

Continuing on the same lines, we obtain

$$\begin{aligned} \eta(M(x_{n+1}, x_{n+2}, t)) \leq \eta(M(gx_{n+1}, gx_{n+2}, t)) &\leq \alpha(x_n, x_{n+1}, t)\eta(M(gx_{n+1}, gx_{n+2}, t)) \\ &\leq k\eta(M(x_n, x_{n+1}, t)) \\ &\leq k^2\eta(M(x_{n-2}, x_{n-1}, t)) \\ &\leq \cdots \\ &\leq k^{n+1}\eta(M(x_0, x_1, t)). \end{aligned}$$

Since $k \in (0, 1)$ and η is strictly decreasing, we have

$$\begin{aligned} \eta(M(x_{n+1}, x_{n+2}, t)) &\leq \eta(M(gx_{n+1}, gx_{n+2}, t)) \\ &\leq k^{n+1}\eta(M(x_0, x_1, t)) \\ &< \eta(M(x_0, x_1, t)) \end{aligned}$$

and

$$M(x_{n+1}, x_{n+2}, t) \geq M(x_0, x_1, t) > 0, n \in \mathbb{N}, t > 0.$$

Now, consider any $n, m \in \mathbb{N},\ m < n$, and $\{a_i\}_{i \in \mathbb{N}}$ be a strictly decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i = 1$. Then, we have

$$\begin{aligned} M(x_m, x_n, t) \geq M(gx_m, gx_n, t) &\geq M(gx_m, gx_m, t - \textstyle\sum_{i=m}^{n-1} a_i t) * M(gx_m, gx_n, \textstyle\sum_{i=m}^{n-1} a_i t) \\ &\geq 1 * M(gx_m, gx_{m+1}, a_m t) * M(gx_{m+1}, gx_{m+2}, a_{m+1} t) * \cdots * M(gx_{n-1}, gx_n, a_{n-1} t) \\ &\geq \prod_{i=m}^{n-1} M(gx_i, gx_{i+1}, a_i t). \end{aligned}$$

Thus,

$$\begin{aligned}\eta(M(gx_m, gx_n, t)) &\leq \eta(\prod_{i=m}^{n-1} M(gx_i, gx_{i+1}, a_i t)) \\ &\leq \sum_{i=m}^{n-1} \eta(M(gx_i, gx_{i+1}, a_i t)) \\ &\leq \sum_{i=m}^{n-1} k^i \eta(M(gx_0, gx_1, t)), m, n \in \mathbb{N}, m < n, t > 0.\end{aligned}$$

The above sum is finite, and $\eta(M(gx_0, gx_1, a_i t)_{i\in\mathbb{N}}$ is non-decreasing and $\eta(M(gx_i, gx_j, t_i))$ is bounded, hence the series $\sum_{i=1}^{\infty} k^i \eta(M(gx_0, gx_1, t))$ is convergent. Consequently, for some $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} k^i \eta(M(gx_0, gx_1, t)) < \epsilon$ and

$$\eta(M(gx_m, gx_n, t)) \leq \sum_{i=m}^{n-1} k^i \eta(M(gx_0, gx_1, t) < \epsilon,\ m, n \geq n_0, m < n, t > 0.$$

Hence, $\{gx_n\}$ is a M-Cauchy sequence in $A_0(t)$. Furthermore, $A_0(t)$ is closed. As $A_0(t)$ is proximal (α, T)-complete (Lemma 2), the sequence $\{gx_n\}$ converges to some element gx^* in $A_0(t)$, that is,

$$\lim_{n\to\infty} M(gx_n, gx^*, t) = 1.$$

Now,

$$\begin{aligned}M(gx_n, B, t) &\geq M(gx_n, Tx_{n+1}, t) \\ &= M(A, B, t) \geq M(gx_n, B, t)\end{aligned}$$

implies that

$$M(gx^*, B, t) \geq M(gx^*, Tx^*, t) \geq M(gx^*, B, t).$$

Take $y_n = Tx^*$ (say) in B. As g is continuous, the sequence $\{gx_n\}$ converges to gx^*, and $M(gx^*, y_n, t) \to M(gx^*, B, t)$. Since B is fuzzy approximately compact with respect to A, $gx^* \in A_0(t)$. Since $A_0 \subseteq g(A_0)$, there exist some $u \in A_0(t)$ such that

$$M(gu, Tx^*, t) = M(A, B, t) = M(gx_{n+1}, Tx_n, t) \text{ for all } n \in \mathbb{N}.$$

Since $\alpha(x_n, x_{n-1}, t) \geq 1$ and T is α_f-proximal admissible mapping, hence $\alpha(gx_{n+1}, gx_n, t) \geq 1$ and $\{gx_n\}$ converges to gx^*. Since $A_0(t)$ is proximal (α, T)-complete, we therefore have $\alpha(gx^*, gx_n, t) \geq 1$. In addition, g is α_R-admissible mapping, which implies that $\alpha(x^*, x_n, t) \geq 1$. As $\{g, T\}$ is α_f-proximal $\mathcal{H}-$contraction of second kind and g is a fuzzy expansive mapping, we have

$$\alpha(x^*, x_n, t)\eta(M(gu, gx_{n+1}, t)) \leq k(\eta(M(x^*, x_n, t))).$$

Taking limit as $n \to \infty$ on both sides of the above inequality, we obtain $gu = gx^*$. Furthermore, g is one to one and hence $u = x^*$. Thus,

$$M(gx^*, Tx^*, t) = M(gu, Tx^*, t) = M(A, B, t)$$

gives that x^* is the optimal coincidence point of the pair $\{g, T\}$.

Uniqueness: Let y^* be another optimal coincidence point of mappings g, and T in $A_0(t)$, then

$$\begin{aligned} M(gx^*, Tx^*, t) &= M(A, B, t) \text{ and} \\ M(gy^*, Ty^*, t) &= M(A, B, t),\ \alpha(x^*, y^*, t) \geq 1. \end{aligned}$$

Since $\{g, T\}$ is α_f-proximal $\mathcal{H}$-contraction of second kind and g is fuzzy expansive, so

$$\begin{aligned} \eta(M(gx^*, gy^*, t)) &\leq \alpha(x^*, y^*, t)\eta(M(gx^*, gy^*, t)) \\ &\leq k(\eta(M(x^*, y^*, t))) < \eta(M(x^*, y^*, t)), \end{aligned}$$

a contradiction—hence the result. □

Example 1. *Let $X = [0,1] \times \mathbb{R}$, $A = \{(0,x) : x \geq 0$ and $x \in \mathbb{R}\}$ and $B = \{(1,y) : y \geq 0$ and $y \in \mathbb{R}\}$. Then,*

$$M_d(A, B, t) = \frac{t}{t+1},\ A_0(t) = \{(0,0)\} \text{ and } B_0(t) = \{(1,0)\}.$$

Define $T : A \to B$ and $g : A \to A$ by:

$$T(x, 0) = (1, \frac{x}{4}) \text{ and } g(0, x) = (0, 4x).$$

In addition, consider $\eta(t) = \frac{1}{t} - 1, t \in (0,1]$ and $\alpha : X \times X \times (0, \infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1, & x, y \in [0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $T(A_0(t)) = B_0(t)$ and $A_0(t) = g(A_0(t))$. Note that the points $u = (0, x_1), v = (0, x_2)$, $x = (0, y_1)$ and $y = (0, y_2)$ in A satisfies $M(gu, Tx, t) = M(A, B, t)$ and $M(gv, Ty, t) = M(A, B, t)$ if $x_1 = \frac{y_1}{16}$ and $x_2 = \frac{y_2}{16}$. Under these circumstances, T becomes α_f-proximal $\mathcal{H}$-contraction of second kind. Thus, all of the conditions of the Theorem (1) are satisfied. Moreover, $(0,0)$ is an unique optimal coincidence point of (g, T) in $A_0(t)$.

Theorem 2. *Let $T : A \to B$ be a continuous α_f-proximal $\mathcal{H}$-contraction of first kind and α_f-proximal admissible mapping with $T(A_0(t)) \subseteq B_0(t)$ for any $t > 0$. If there exists $x_0, x_1 \in A_0(t)$ such that $M(x_1, Tx_0, t) = M(A, B, t)$ and $\alpha(x_0, x_1, t) \geq 1$, then the mapping T has a unique best proximity point x^* in $A_0(t)$ provided that $A_0(t)$ is proximal (α, T)-complete and B is fuzzy approximately compact with respect to A.*

Proof. By taking $g = I_A$ in Theorem (1). In this case, α_f-proximal $\mathcal{H}$-contraction of second kind becomes an α_f-proximal $\mathcal{H}$-contraction of first kind and the result follows. □

Corollary 1. *Let $g : A \to A$ be a one to one fuzzy non-expansive mapping and $T : A \to B$ with $A_0(t) \neq \phi$, $T(A_0(t)) \subseteq B_0(t)$, $A_0(t) \subseteq g(A_0(t))$ for any $t > 0$. If $A_0(t)$ is proximal (α, T)-complete and B is fuzzy approximately compact with respect to A, the pair (g, T) further satisfies the following implication:*

$$\left. \begin{array}{l} M(gu, Tx, t) = M(A, B, t) \\ M(gv, Ty, t) = M(A, B, t) \end{array} \right\} \text{implies that } \eta(M(gu, gv, t)) \leq k\eta(M(x, y, t)),$$

where $\eta \in \mathcal{H}$. Then, the pair (g, T) has a unique optimal coincidence point x^* in $A_0(t)$.

Proof. Take $\alpha(x, y, t) = 1$ for all $x, y \in A_0(t)$ and $t > 0$ in Theorem (1). The proof follows under the same lines as in Theorem (1). □

3. Optimal Coincidence Point and Approximation Results in Ordered Structures

In this section, we will provide results in ordered metric spaces.

Let $(X, M, *)$ is a fuzzy metric space and $(X, \preceq)$ is a partially ordered set. Then, $(X, M, *, \preceq)$ is known as a partially ordered fuzzy metric space. In the sequel sets, A and B are assumed to be nonempty closed subsets of $(X, M, *, \preceq)$.

A nonempty set X is called partially ordered fuzzy metric space if $(X, M, *)$ is a fuzzy metric space and $\preceq$ is a partial order on X. Suppose that A and B are subsets of a partially ordered fuzzy metric space X.

Definition 14. *[24] A mapping $T : A \longrightarrow B$ is called (a) nondecreasing or order preserving if for any x, y in A with $x \preceq y$, we have $Tx \preceq Ty$ (b) an ordered reversing if, for any x, y in A with $x \preceq y$, we have $Tx \succeq Ty$ (c) monotone if it is order preserving or order reversing.*

Definition 15. *[21] A mapping $T : A \longrightarrow B$ is called proximal fuzzy order preserving (proximal fuzzy order reversing), if:*

$$\left.\begin{array}{r} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \Longrightarrow u \preceq v \ (u \succeq v), \text{for any } u, v, x \text{ and } y \text{ in } A.$$

If $A = B$ in the above definition, then proximal fuzzy order preserving (proximal fuzzy order reversing) mapping will become order preserving (order reversing).

Lemma 3. *Let $A_0(t)$ and $T(A_0(t)) \subseteq B_0(t)$.Then, for $a \in A_0(t)$, there exists a sequence $\{x_n\} \subset A_0(t)$ such that*

$$\left.\begin{array}{l} x_0 = a, \\ M(x_{n+1}, Tx_n, t) = M(A, B, t), \text{for all } n \in \mathbb{N} \text{ with } (x_n, x_{n+1}) \in \Delta. \end{array}\right\} \quad (4)$$

Proof. Define the function $\alpha : A \times A \times [0, \infty) \to [0, \infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1, & \text{if } (x, y) \in \Delta, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

By the set Δ, we mean the collection of $x, y \in X$ such that either $x \preceq y$ or $y \preceq x$. From Equation (1), by taking the above function, we obtain a sequence which satisfies the condition Equation (4). □

Definition 16. *[21] A sequence $\{x_n\} \subset A_0(t)$ satisfying the condition (4) is called ordered proximal Picard sequence starting with $a \in A_0(t)$.*

Definition 17. *[21] A set $A_0(t)$ is ordered proximal T-orbitally complete if and only if every ordered proximal Picard Cauchy sequence $\{x_n\}$ starting with $x_0 \in A_0(t)$ converges to an element in the set $A_0(t)$.*

Lemma 4. *Let $T : A \to B$ be continuous, fuzzy proximally monotone and α_f-proximal $\mathcal{H}$-contraction of first kind with $A_0(t) \neq \emptyset$ and $T(A_0(t)) \subseteq B_0(t)$. Suppose that each pair of elements in partially ordered complete fuzzy metric spaces $(X, M, *, \preceq)$ has a lower and upper bound. Then, $A_0(t)$ is fuzzy proximal T-orbitally complete provided that T is one to one on $A_0(t)$ and there exists a function $\alpha : A \times A \times [0, \infty) \to [0, \infty)$ such that $(x, y) \in \Delta$ and $\alpha(x, y, t) \geq 1$ for all $x, y \in A$.*

Proof. Consider a function $\alpha : A \times A \times [0, \infty) \to [0, \infty)$ defined in Equation (5). Let x_0 be a given point in $A_0(t)$ and $\{x_n\}$ be an ordered *proximal Picard* Cauchy sequence starting with x_0. As $(X, M, *, \preceq)$ is complete ordered fuzzy metric space and A is closed, there exist some x^* in A such that $\lim_{n\to\infty} M(x_n, x^*, t) = 1$. By definition of ordered *proximal Picard sequence* $\{x_n\}$, we have

$$M(x_n, Tx_{n-1}, t) = d(A, B) \text{ and } M(x_{n+1}, Tx_n, t) = M(A, B, t) \text{ with } (x_{n-1}, x_n) \in \Delta$$

for all $n \in \mathbb{N}$. Since T is a α_f-proximal $\mathcal{H}$-contraction of first kind and function $\alpha(x, y, t)$ defined in Equation (5) agrees with the α_f-proximal admissible mapping defined on $A \times A$, the rest of the proof follows on the same lines given in Equation (2). □

Theorem 3. *Let $T : A \to B$ be continuous, proximally monotone and α_f-proximal $\mathcal{H}$-contraction of first kind with $A_0(t) \neq \phi$ and $T(A_0(t)) \subseteq B_0(t)$. Suppose that each pair of elements in partially ordered complete fuzzy metric spaces $(X, M, *, \preceq)$ has a lower and upper bound. If B is approximately fuzzy compact with respect to A, then T has a unique best proximity point x^* in $A_0(t)$ provided that T is one-to-one on $A_0(t)$ and for all $(x, y) \in \Delta$ such that $\alpha(x, y, t) \geq 1$ for all $x, y \in A$.*

Proof. Let x_0 be a given point in $A_0(t)$. From Lemma (1), the ordered *proximal Picard sequence* $\{x_n\}$ in $A_0(t)$ satisfies

$$M(x_n, Tx_{n-1}, t) = M(A, B, t), M(x_{n+1}, Tx_n, t) = M(A, B, t) \quad (6)$$

for all $n \in \mathbb{N}$. In addition, define a function $\alpha : A \times A \times [0, \infty) \to [0, \infty)$ which satisfies Equation (5), since T is a α_f-proximal $\mathcal{H}$-contraction of first kind. The following arguments are similar to those in the proof of Lemma (2) and Theorem (1) by taking $g = I_A$. In addition, the function $\alpha(x, y, t)$ agrees with the α_f-proximal admissible mapping defined on $A \times A$. Following the same lines of the proof of Theorem (1), the result follows. □

Theorem 4. *Let $g : A \to A$ be an expansive mapping, $\alpha : A \times A \times [0, \infty) \to [0, \infty)$ and $T : A \to B$ with $A_0(t) \neq \phi$, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$ for any $t > 0$. If B is approximately compact with respect to A and the pair (g, T) is α_f-proximal $\mathcal{H}-$contraction of second kind. Suppose that each pair of elements in partially ordered complete fuzzy metric spaces $(X, M, *, \preceq)$ has a lower and an upper bound. Then, the pair (g, T) has a unique optimal coincidence point x^* in $A_0(t)$ provided that $\alpha(x, y, t) \geq 1$ such that $(x, y) \in \Delta$ for all $x, y \in A$.*

Proof. Let x_0 be a given point in $A_0(t)$. As $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can choose an element x_1 in $A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$ where $(x_0, x_1) \in \Delta$. In addition, $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$, there exists an element $x_2 \in A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Since g is ordered, where $(gx_1, gx_2) \in \Delta$, hence $\alpha(gx_1, gx_2, t) = 1$. Continuing this way, we can obtain a sequence $\{gx_n\}$ in $A_0(t)$ such that it satisfies

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) \text{ and } M(gx_{n+1}, Tx_n, t) = M(A, B, t), \text{ where } (gx_{n-1}, gx_n) \in \Delta \text{ and } (gx_n, gx_{n+1}) \in \Delta.$$

Define a function $\alpha : A \times A \times [0, \infty) \to [0, \infty)$ as in Equation (4) which agrees with α_f-proximal admissible mapping. Following the arguments similar to those in Equation (1), the result follows. □

Corollary 2. *If $T : A \to B$ is a α_f-proximal $\mathcal{H}$-contraction of first kind with $A_0(t) \neq \phi$ and $T(A_0(t)) \subseteq B_0(t)$ for any $t > 0$. Then, T has a unique best proximity point x^* in $A_0(t)$ provided that B is approximative compact with respect to A.*

Corollary 3. *Let $T : A \to B$ be α_f-proximal $\mathcal{H}$-contraction of first kind with $A_0(t) \neq \phi$, $T(A_0(t)) \subseteq B_0(t)$ for any $t > 0$. Suppose that each pair of elements in the partially ordered complete metric space $(X, M, *, \preceq)$ has a lower and upper bound. If B is approximately compact with respect to A, then T has a unique best proximity point x^* in $A_0(t)$ provided that $\alpha(x, y, t) \geq 1$ such that $(x, y) \in \Delta$ for all $x, y \in A$.*

Example 2. *Suppose that* $X = [0,1] \times \mathbb{R}$*,* $A = \{(0,x) : x \leq 0$ *and* $x \in \mathbb{R}\}$ *and* $B = \{(1,y) : y \leq 0$ *and* $y \in \mathbb{R}\}$*. Note that*

$$M_d(A,B,t) = \frac{t}{t+1},\ A_0(t) = \{(0,0)\} \text{ and } B_0(t) = \{(1,0)\}.$$

Define $T : A \to B$ *by*

$$T(x,0) = (1, \frac{x}{5})\text{and} x \preceq y \text{ is defined as } x \leq y.$$

Obviously, $T(A_0(t)) = B_0(t)$ *and* $A_0(t) = g(A_0(t))$*. Note that the points* $u = (0,x_1), v = (0,x_2), x = (0,y_1)$ *and* $y = (0,y_2)$ *in* A *satisfy* $M(u,Tx,t) = M(A,B,t)$ *and* $M(v,Ty,t) = M(A,B,t)$ *if* $x_1 = \frac{y_1}{5}$*,* $x_2 = \frac{y_2}{5}$ *and* $\alpha(x,y,t) = 1$ *as* $(x,y) \in \Delta$*. In addition,* $\eta(M(u,v,t)) \leq k\eta(M(x,y,t))$ *holds true, where* $\eta(t) = \frac{1}{t} - 1$*. Thus, all the conditions of the corollary (3) are satisfied. Moreover,* $(0,0)$ *is the best proximity point of* T *in* $A_0(t)$ *if* $k \geq \frac{1}{5}$*.*

4. Application:

As an application of obtained results, we prove some new fixed point theorems as follows. We start with the following:

Theorem 5. *Let* $(X,M,*)$ *be a complete fuzzy metric space, and* $\alpha : A \times A \times [0,\infty) \to [0,\infty)$*. If* $T : X \to X$ *is* α*-admissible mapping such that the following hold:*

(i) $\alpha(x,y,t)\eta(M(Tx,Ty,t)) \leq k\eta(M(x,y,t)$, where $\eta \in \mathcal{H}$.
(ii) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,t) \geq 1$.
(iii) Either T is continuous or $\{x_n\}$ is $\alpha-$ordered regular.

Then, T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof. Let $A = B = X$. We prove that T is α_f-proximal $\mathcal{H}-$contraction of first kind. Let $x,y,u,v \in X$ such that the following conditions hold:

$$\begin{cases} (x,y) \in \Delta, \\ M(u,Tx,t) = M(A,B,t), \\ M(v,Ty,t) = M(A,B,t). \end{cases}$$

As $M(A,B,t) = 1$, we have $u = Tx$ and $v = Ty$. Since T satisfies the condition (i), therefore

$$\alpha(x,y,t)\eta(M(u,v,t)) = \alpha(x,y,t)\eta(M(Tx,Ty,t)) \leq k\eta(M(x,y,t))$$

implies that T is α_f-proximal $\mathcal{H}$-contraction of first kind. Consider

$$\begin{cases} \alpha(x,y,t) \geq 1, \\ M(u,Tx,t) = M(A,B,t), \\ M(v,Ty,t) = M(A,B,t). \end{cases}$$

Then, α-admissible property of T implies that $\alpha(u,v,t) = \alpha(Tx,Ty,t) \geq 1$. Therefore, T is α-ordered regular admissible mapping. Applying condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,t) \geq 1$. If we choose $x_1 = Tx_0$, then

$$\alpha(x_0,x_1,t) \geq 1 \text{ and } M(A,B,t) = M(x_1,Tx_0,t) = M(Tx_0,Tx_0,t).$$

Since set B is approximately compact with respect to A. All the conditions of Theorem (2) are satisfied, so there exists $x^* \in X$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$ and hence $Tx^* = x^*$. In the following remark, we compared the already existing results in literature. □

Remark 1. *Latif et al. [25] defined α-proximal fuzzy contraction of type-I and type-II. If we define $\eta(t) = 1 - \psi(t^\alpha)$, where $\psi \in \Psi$ (as defined in [25]) and $\alpha \in [0,1]$, then $\eta \in \mathcal{H}$. Then, α_f-proximal contraction of first and second kind will reduce to α-proximal fuzzy contraction of type-I and type-II in [25]. If we take $\alpha(x, y, t) = 1$ for all $x, y \in A$, $t > 0$ and $\eta(t) = 1 - \psi(t^\alpha)$ in Theorem (1), (2) and simplify our results along with some minor conditions on involved mappings, we obtain Theorems 2.2, 3.2, 3.5, and 3.8 in [25].*

Explanation: Take $\eta(t) = 1 - \psi(t^\alpha)$, where $\psi \in \Psi$ (as defined in [25]) and $\alpha \in (0, 1)$ in α_f-proximal contraction of first kind defined in Equation (2) as $\eta(M(u, v, t)) \le k[\eta(M(x, y, t))]$, then we have

$$\begin{aligned}
1 - \psi([M(u,v,t)]^\alpha) &\le k(1 - \psi([(M(x,y,t)]^\alpha) \\
-\psi([M(u,v,t)]^\alpha) &\le k - k\psi([M(x,y,t)]^\alpha) \le -k\psi([(M(x,y,t)]^\alpha), \text{ since } k \in (0,1) \\
-\psi([M(u,v,t)]^\alpha) &\le -k\psi([(M(x,y,t)]^\alpha) \\
\psi([M(u,v,t)]^\alpha) &\ge k\psi([(M(x,y,t)]^\alpha).
\end{aligned}$$

Furthermore,

$$\psi(M(u,v,t)) \ge \psi([M(u,v,t)]^\alpha) \ge k\psi([(M(x,y,t)]^\alpha).$$

We have

$$\psi(M(u,v,t)) \ge k\psi([(M(x,y,t)]^\alpha).$$

If $\psi(M(u,v,t)) \ge M(u,v,t) \ge k\psi([(M(x,y,t)]^\alpha)$ happens, then we have $M(u,v,t) \ge k\psi([(M(x,y,t)]^\alpha)$, which is an α-proximal fuzzy contraction of type-I defined in [25]. A similar explanation exist for α-proximal fuzzy contraction of type-II.

5. Conclusions

In this paper, we introduced α_f-proximal contraction of first and second kind in complete fuzzy metric space and some optimal coincidence point results are obtained. Some examples are provided to show that the results presented in this paper generalize comparable existing results in the sense of nonself mapping. Furthermore, we obtained optimal coincidence point results of such contractions in ordered structures along with some examples. If we restrict ourselves to self mapping only, results in [16,18] are extended. We provided an application in fixed point theory, if we restrict non-self mappings to self mappings in the framework of a complete fuzzy metric space.

Though techniques to prove best proximity point results are not new but an introduction of a new class of mappings in the framework of fuzzy metric spaces extends the scope of the study of best proximity point theory. Moreover, there is not much work done in fuzzy metric spaces. Our results will open new avenues of research in this direction. It will be interesting to study the same problem for a pair of non-self mappings in fuzzy metric spaces. Moreover, the study of coupled best proximity point in such spaces will also be a valuable contribution towards best proximity point theory.

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