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Systems of Variational Inequalities with Nonlinear Operators

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Abstract: In this work, we concern ourselves with the problem of solving a general system of variational inequalities whose solutions also solve a common fixed-point problem of a family of countably many nonlinear operators via a hybrid viscosity implicit iteration method in 2 uniformly smooth and uniformly convex Banach spaces. An application to common fixed-point problems of asymptotically nonexpansive and pseudocontractive mappings and variational inequality problems for strict pseudocontractive mappings is also given in Banach spaces.

Keywords: implicit iterations; variational inequalities; lipschitzian pseudocontractions; asymptotically nonexpansive mapping; fixed-point; banach space

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1. Introduction

Let E be a real Banach space whose topological dual space is denoted by E^*. Recall that the normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{\varphi \in E^* : \|\varphi\| = \|x\|, \langle x, \varphi\rangle = \|x\|^2\}, \quad \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ is the duality pair on E and E^*. J is single-valued in a smooth Banach space. In the sequel, we shall denote by j the single-valued duality mapping, that is, $j(x) \in J(x)$. Let C be a convex closed set in E. A mapping $f : C \to C$ is said to be δ-Lipschitzian on C if $\delta \in (0, +\infty)$ and $\|f(x) - f(y)\| \leq \delta\|x - y\|$ for all $x, y \in C$. If $\delta < 1$, then f is called a δ-contraction mapping or a contraction mapping with coefficient δ. Each contraction $f : C \to C$ has a unique fixed point from the well known the Banach contractive principal. A mapping $f : C \to C$ is said to be nonexpansive if it is Lipschitzian with $\delta = 1$. We use $\mathrm{Fix}(f)$ to denote the set of fixed points of f, i.e., $\mathrm{Fix}(f) = \{x \in C : f(x) = x\}$. Moreover, a mapping $T : C \to C$ is said to be asymptotically nonexpansive [1] if there exists a sequence $\{\theta_n\} \subset [0, +\infty)$ with $\lim_{n\to\infty} \theta_n = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \theta_n\|x - y\|, \quad \forall x, y \in C, \ \forall n \geq 0.$$

If

$$\limsup_{n\to\infty} \Big(\sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \Big) \leq 0, \tag{1}$$

and T enjoys the continuity, then T is called an asymptotically nonexpansive mapping in the intermediate sense; see [2]. Throughout this paper, we assume

$$c_n := \max\{0, \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|)\}. \tag{2}$$

Hence, $c_n \geq 0\ \forall n \geq 0,\ c_n \to 0\ (n \to \infty)$, and the definition is reduced to

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n, \quad \forall x, y \in C,\ \forall n \geq 0.$$

Recall that a mapping T with domain $D(T)$ and range $R(T)$ in E is called pseudocontractive if the inequality holds

$$\|x - y\| \leq \|r((I - T)x - (I - T)y) + (x - y)\|, \quad \forall x, y \in D(T), \forall r > 0.$$

From a result of Kato [3], we know that the notion of pseudocontraction is equivalent to the one that for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2.$$

It is well known that the class of pseudocontractive mappings is a crucial generation of nonexpansive mappings. Moreover, focus on pseudocontractive mappings are also from their relation with the class of accretive mappings in Banach spaces (monotone in Hilbert spaces). A mapping A with domain $D(A)$ and range $R(A)$ in E is called accretive if for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0.$$

It will be called a monotone mapping if the space is Hilbert. If for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\alpha\|Ax - Ay\|^2 \leq \langle Ax - Ay, j(x - y)\rangle \quad \text{for some } \alpha > 0,$$

then A is called α-inverse-strongly accretive.

Recently, fixed/zero points of pseudocontraction/accretive operators were investigated by many authors for solving various convex optimization problems; see [4–13] and the references therein.

Let E be a smooth Banach space. Let B_1, B_2 be two non-self-mappings from C to E. The general system of variational inequalities (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* - y^* + x^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* - x^* + y^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{3}$$

where μ_1 and μ_2 are two positive coefficients.

In particular, if $B_1 = B_2 = B$, then problem (3) reduces to the following system of variational inequalities (SVI) in Banach spaces:

Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B y^* - y^* + x^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B x^* - x^* + y^*, j(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{4}$$

Furthermore, if $x^* = y^*$, then we obtain the following variational inequality (VI) in Banach spaces: Find $x^* \in C$ such that

$$\langle \mu B x^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C. \tag{5}$$

Let $\mathrm{VI}(C, B)$ denote the set of solutions to problem (5). Whenever $E = H$ a real Hilbert space, it is easy to see that the GSVI (3) reduces to the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* - y^* + x^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* - x^* + y^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{6}$$

which is called a GSVI in Hilbert spaces. In [11], the GSVI (6) was transformed into a fixed-point problem by Ceng, Wang and Yao in the following way.

Lemma 1. *[14] For chosen $x^*, y^* \in C$, x^*-y^* is a solution of GSVI (1.6) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed-point set of the mapping $G := P_C(P_C(I - \eta B_2) - \rho B_1 P_C(I - \eta B_2))$, and $y^* = P_C(I - \eta B_2)x^*$.*

Recently, many authors studied problems (3)–(6) via projection-based methods in Hilbert or Banach spaces; see [15–22] and the references therein. In this paper, we introduce a hybrid viscosity implicit iteration method that is based on Korpelevich's extragradient method, the viscosity approximation method and the Mann iteration method for finding a common solution of the GSVI (3) for two inverse-strongly accretive mappings, a common fixed-point problem (CFPP) of a countable family of uniformly Lipschitzian pseudocontractive mappings and an asymptotically nonexpansive mapping in the intermediate sense. We prove the strong convergence of the proposed method to a common solution of the GSVI (3) and the CFPP, which solves a certain variational inequality on their common solution set in 2-uniformly smooth and uniformly convex Banach spaces. Additionally, we give an application to solve CFPPs of asymptotically nonexpansive and pseudocontractive mappings, and variational inequality problems for strict pseudocontractive mappings in Banach spaces.

2. Preliminaries

Throughout this paper we write $x_n \rightharpoonup x$ (respectively, $x_n \to x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, strongly) to x. Without loss of generality, we assume that E is a real Banach space and the dual will be presented by E^* in this paper.

Definition 1. *Let $\{S_n\}_{n=0}^{\infty}$ be a vector sequence of pseudocontractive continuous self-mappings on C, a convex closed convex subset of Banach space E. Recall that $\{S_n\}_{n=0}^{\infty}$ is said to be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings provided that there exists a constant $\ell > 0$ such that each S_n is a ℓ-Lipschitz continuous mapping.*

In a smooth Banach space E, an operator A is said to be strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property

$$\|aI - bA\| = \sup_{\|x\| \le 1} |\langle (aI - bA)x, j(x)\rangle|, \langle Ax, j(x)\rangle \ge \bar{\gamma}\|x\|^2 \; a \in [0,1], \; b \in [-1,1],$$

where I is the identity mapping and $j(\cdot)$ is the single-valued normalized duality mapping.

Recall that a Banach space E is said to be strictly convex if for any $x, y \in \{x \in E : \|x\| = 1\}$, $x \neq y \Rightarrow \|\frac{x+y}{2}\| < 1$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in \{x \in E : \|x\| = 1\}$, $\|x - y\| \ge \epsilon \Rightarrow \|x + y\| \le 2 - 2\delta$. Clearly, if E is uniformly convex, then it is strictly convex. A Banach space E is said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in \{x \in E : \|x\| = 1\}$ and in this case we call E smooth; E is said to have a uniformly Gâteaux differentiable norm if for each $y \in \{x \in E : \|x\| = 1\}$, the above limit is attained uniformly for $x \in \{x \in E : \|x\| = 1\}$. Moreover, it is said to have a uniformly Fréchet differentiable norm if the above limit is attained uniformly for $x, y \in \{x \in E : \|x\| = 1\}$ and in this case we call E uniformly smooth. The norm of E is said to be the Fréchet differentiable if for each $x \in \{x \in E : \|x\| = 1\}$, the above limit is attained uniformly for $y \in \{x \in E : \|x\| = 1\}$. The modulus of smoothness of E is defined by

$$\varrho(\tau) = \sup\{\frac{(\|x+y\| + \|x-y\|) - 2}{2} : x, y \in E,\ \|x\| = 1,\ \|y\| = \tau\},$$

where $\varrho : [0, \infty) \to [0, \infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau \to 0} \dfrac{\varrho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $\kappa > 0$ such that $\dfrac{\varrho(\tau)}{\kappa} \leq \tau^q$ for all $\tau > 0$. From [23], we know the following relation. Let q be a fixed number with $1 < q \leq 2$ and E be a Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c > 0$ such that

$$\|x\|^q + \|\kappa y\|^q \geq \frac{\|x+y\|^q + \|x-y\|^q}{2}, \quad \forall x, y \in E.$$

The best constant κ in the above inequality is called the q-uniformly smooth constant of E; see [23] for more details. In addition, no Banach space is q-uniformly smooth for $q > 2$; see [24] for more details. If E be a 2-uniformly smooth Banach space. Then

$$\|x+y\|^2 - \|x\|^2 \leq 2\langle y, j(x)\rangle + 2\|\kappa y\|^2, \quad \forall x, y \in E,$$

where κ is the 2-uniformly smooth constant of E.

In particular, if E is a Hilbert space, then the duality pairing $\langle \cdot, \cdot \rangle$ reduces to the inner product, $j = I$ the identity mapping of E, and $\kappa = \sqrt{2}/2$.

For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\varphi \in E^* : \|\varphi\| = \|x\|^{q-1}, \langle x, \varphi\rangle = \|x\|^q\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J(x) = \dfrac{J_q(x)}{\|x\|^{q-2}}$ for all $x \in E$. If E is a Hilbert space, then $J = I$ (the identity mapping). Recall that the following statements hold:

(1) if E is smooth, then J is norm-to-weak* continuous single-valued on E;
(2) if E is uniformly smooth, then J is norm-to-norm uniformly continuous single-valued on bounded subsets of E;
(3) if E has a uniformly Gáteaux differentiable norm, then J is norm-to-weak* uniformly continuous single-valued on bounded subsets of E;

Proposition 1. *(see [25]). Let C be a convex nonempty closed set in a Banach space E. Let $S_0, S_1, \ldots$ be a sequence of mappings of C into itself. Suppose $\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1}x\| : x \in C\} < \infty$. For each $y \in C$, $\{S_n y\}$ converges in norm to some point of C. Moreover, let S be a mapping defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. $\{\|Sx - S_n x\| : x \in C\} \to 0$ as $n \to \infty$.*

Proposition 2. *(see [26]). Let C be a convex closed set in a Banach space E and $T : C \to C$ be a strong continuous pseudocontraction mapping. Then, T has a fixed point. Indeed, it is the unique fixed point in C for T.*

Let D be a nonempty set in C and let Π be a mapping from C to D. Then Π is said to be a sunny if $\Pi[(1-t)\Pi(x) + tx] = \Pi(x)$, when $(1-t)\Pi(x) + tx \in C$ for all $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

In a smooth Banach space E, a duality mapping J is said to be weakly sequentially continuous [27], if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then $\{j(x_n)\}$ converges weakly* to $j(x)$. In [27], Gossez and Lami

Dozo showed that a space with a weakly continuous duality mapping satisfies Opial's condition. Conversely, we know from [28] that if a space satisfies Opial's condition and has a uniformly Gáteaux differentiable norm, then it has a weakly continuous duality mapping.

Proposition 3. *(see [29]). Let C be a nonempty closed convex subset of a smooth Banach space E, D be a nonempty subset of C and Π be a retraction of C onto D. Then the following are equivalent:*

(i) *Π is sunny and nonexpansive;*
(ii) $\langle x-y, j(\Pi(x)-\Pi(y))\rangle \geq \|\Pi(x)-\Pi(y)\|^2, \forall x, y \in C$;
(iii) $\langle x-\Pi(x), j(y-\Pi(x))\rangle \leq 0, \forall x \in C, y \in D$.

If E is a Hilbert space, then a sunny nonexpansive retraction Π_C of E onto C coincides with the nearest projection of E onto C and it is well known that if C is a convex closed set in a reflexive Banach space E with a uniformly Gáteaux differentiable norm and D is a nonexpansive retract of C, then it is a sunny nonexpansive retract of C; see, e.g., [30,31] and the references therein.

To prove our main results, we need to use some lemmas in the sequel. The following Lemma is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$.

Lemma 2. *[32] Let E be a real Banach space and J be the normalized duality mapping on E. Then for any given $x, y \in E$, the following inequality holds:*

$$\|x+y\|^2 - \|x\|^2 \leq 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

If C is a convex closed set in a smooth Banach space E and Π_C a sunny nonexpansive retraction from E onto C, we have

$$\mathrm{VI}(C, B) = \mathrm{Fix}(\Pi_C(I-\lambda B)).$$

where $B: C \to E$ be an accretive mapping and $\lambda > 0$,

Using Proposition 3, we immediately obtain the following lemmas.

Lemma 3. *Let C be a nonempty closed convex subset of a smooth Banach space E and $B_1, B_2 : C \to E$ be two nonlinear mappings. Let Π_C be a sunny nonexpansive retraction from E onto C. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the GSVI (3) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$ where $\mathrm{GSVI}(C, B_1, B_2)$ is the set of fixed points of the mapping*

$$G := \Pi_C(\Pi_C(I-\mu_2 B_2) - \mu_1 B_1 \Pi_C(I-\mu_2 B_2))$$

and $y^* = \Pi_C(I-\mu_2 B_2)x^*$.

Lemma 4. *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E. Let the mapping $A: C \to E$ be α-inverse-strongly accretive. Then, for any given $\lambda \geq 0$,*

$$\|(I-\lambda A)x - (I-\lambda A)y\|^2 - \|x-y\|^2 \leq 2\lambda(\kappa^2\lambda - \alpha)\|Ax - Ay\|^2.$$

In particular, if $0 \leq \lambda\kappa^2 \leq \alpha$, then $I-\lambda A$ is a nonexpansive operator. Let Π_C be a sunny nonexpansive retraction from E onto C. Let the mappings $B_1, B_2 : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let the mapping $G: C \to C$ be defined as $G := \Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)$. If $0 \leq \mu_1\kappa^2 \leq \alpha$ and $0 \leq \mu_2\kappa^2 \leq \beta$, then $G: C \to C$ is nonexpansive.

Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T: C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Given any bounded subset

$\mathcal{K} \subset C$. For every $\varepsilon > 0$ and every integer $n \geq 2$ there exist an integer $N_\varepsilon \geq 1$ and $\delta_\varepsilon > 0$, where both N_ε and δ_ε are independent of n, such that if $k \geq N_\varepsilon$, $z_1, z_2, ..., z_n \in \mathcal{K}$ and if

$$\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_\varepsilon$$

for $1 \leq i, j \leq n$, then

$$\|T^k(\sum_{i=1}^{n} \lambda_i z_i) - \sum_{i=1}^{n} \lambda_i T^k z_i\| < \varepsilon$$

for all $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ such that $\lambda_i \geq 0$ for $i = 1, 2, ..., n$ and $\sum_{i=1}^{n} \lambda_i = 1$; see ([33], Lemma 4) for details.

From the above results, we know that if $\{x_m\}_{m=0}^{\infty}$ is a sequence in C converging weakly to x and if $\lim_{m\to\infty} \|x_m - Tx_m\| = 0$, then $Tx = x$, where $T : C \to C$ is a uniformly continuous self-mapping on C, which is asymptotically nonexpansive in the intermediate sense.

Lemma 5. *(see [34]). Let E be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbf{R}$, $g(0) = 0$ such that*

$$g(\|x - y\|) + 2\langle x, j(y)\rangle \leq \|x\|^2 + \|y\|^2, \quad \forall x, y \in \{x \in E : \|x\| \leq r\}.$$

Lemma 6. *(see [35]). Let E be a reflexive Banach space, C be a convex nonempty, closed subset of E, and $T : C \to E$ be a nonexpansive mapping. Suppose that E admits a weakly sequentially continuous duality mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping.*

Lemma 7. *(see [36]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq a_n + s_n t_n + \nu_n - s_n a_n, \quad \forall n \geq 0,$$

where $\{s_n\}, \{t_n\}$ and $\{\nu_n\}$ satisfy the conditions:

(i) $\limsup_{n\to\infty} t_n \leq 0$;
(ii) $\{s_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} s_n = \infty$;
(iii) $\nu_n \geq 0$, $\forall n \geq 0$, *and* $\sum_{n=0}^{\infty} \nu_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we suggest and analyze a hybrid viscosity implicit iteration method for solving the GSVI (3) with the hierarchical variational inequality (HVI) constraint for countably many uniformly Lipschitzian pseudocontractive mappings and an asymptotically nonexpansive mapping in the intermediate sense in a 2-uniformly smooth and uniformly convex Banach space.

Theorem 1. *Let C be a convex closed set in a 2-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous duality mapping. Let Π_C be a sunny nonexpansive retraction from E onto C. Let the mappings $B_1, B_2 : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $f : C \to C$ be a contraction mapping with coefficient $\gamma \in [0, 1)$ and $F : E \to E$ be a strongly positive linear bounded operator with the coefficient $\bar{\gamma}$ such that $0 < \gamma < \bar{\gamma}\theta$ and $0 < \theta \leq \|F\|^{-1}$. Let $T : C \to C$ be uniformly continuous and asymptotically nonexpansive mapping in the intermediate sense, and $\{S_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C such that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T) \neq \emptyset$ where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed-point set of the mapping $G := \Pi_C(\Pi_C(I - \mu_2 B_2) - \mu_1 B_1 \Pi_C(I - \mu_2 B_2))$ with $0 < \mu_1 \kappa^2 < \alpha$ and $0 < \mu_2 \kappa^2 < \beta$ for κ the*

2-uniformly smooth constant of E. Assume that $\sum_{n=0}^{\infty} c_n < \infty$, where c_n is defined by (2). For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} u_n = \beta_n(x_n - S_n u_n) + S_n u_n, \\ z_n = \Pi_C(u_n - \mu_2 B_2 u_n), \\ y_n = \Pi_C(z_n - \mu_1 B_1 z_n), \\ x_{n+1} = \Pi_C[(T^n y_n - \alpha_n \theta F T^n y_n) + \alpha_n f(x_n)], \quad \forall n \geq 0, \end{cases} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $[0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1} x\| < \infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$. If $\sum_{n=0}^{\infty} \|T^{n+1} y_n - T^n y_n\| < \infty$, then $\{x_n\}$ converges strongly to $x^* \in \Omega$. In this case,

(a) $x^* \in \Omega$ *solves the VI:* $\langle f(x^*) - \theta F(x^*), j(x - x^*)\rangle \leq 0$, $\forall x \in \Omega$;
(b) (x^*, y^*) *is a solution of GSVI (1.3) with* $y^* = \Pi_C(I - \mu_2 B_2)x^*$.

Proof. First of all, from $\alpha_n \to 0$ $(n \to \infty)$, we may assume, without loss of generality, that $\alpha_n \leq \frac{\|F\|^{-1}}{\theta}$ for all $n \geq 0$. Since F is strongly positive linear bounded, it follows that $1 - \alpha_n \bar{\gamma}\theta \geq \|I - \alpha_n \theta F\|$. Taking into account that $\{\beta_n\}$ is bounded away from 0 and 1, we may assume that $\{\beta_n\} \subset [a,b] \subset (0,1)$ for some $a, b \in (0,1)$. Please note that the mapping $G : C \to C$ is defined as $G := \Pi_C(\Pi_C(I - \mu_2 B_2) - \mu_1 B_1 \Pi_C(I - \mu_2 B_2))$, where $0 < \mu_1 \kappa^2 \leq \alpha$ and $0 < \mu_2 \kappa^2 \leq \beta$ for κ the 2-uniformly smooth constant of E. Therefore, by Lemma 3, we obtain that G is nonexpansive. It is easy to see that for each $n \geq 0$ there exists a unique element $u_n \in C$ such that

$$u_n = \beta_n(x_n - S_n u_n) + S_n u_n. \tag{8}$$

As a matter of fact, consider the mapping

$$F_n x = \beta_n(x_n - S_n x) + S_n x, \quad \forall x \in C.$$

Since $S_n : C \to C$ is a continuous pseudocontraction mapping, we deduce that

$$(1 - \beta_n)\|x - y\|^2 \geq (1 - \beta_n)\langle S_n x - S_n y, j(x - y)\rangle = \langle F_n x - F_n y, j(x - y)\rangle, \quad \forall x, y \in C.$$

Also, from $\{\beta_n\} \subset [a,b] \subset (0,1)$ we get $0 < 1 - \beta_n < 1$ for all $n \geq 0$. Thus, F_n is a continuous and strong pseudocontraction mapping of C into itself. By Proposition 2, we know that for each $n \geq 0$ there exists a unique element $u_n \in C$, satisfying (8). Therefore, it can be readily seen that the hybrid viscosity implicit iterative scheme (7) can be rewritten as

$$\begin{cases} u_n = \beta_n(x_n - S_n u_n) + S_n u_n, \\ y_n = G u_n, x_{n+1} = \Pi_C[(T^n y_n - \alpha_n \theta F T^n y_n) + \alpha_n f(x_n)], \quad \forall n \geq 0, \end{cases} \tag{9}$$

Next, we divide the rest of the proof into several steps.

Step 1. We claim that $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{f(x_n)\}, \{T^n y_n\}$ and $\{F(T^n y_n)\}$ are bounded vector sequences. Indeed, take an element $p \in \Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T)$ arbitrarily. Then we have $S_n p = p$, $Gp = p$ and $Tp = p$. Since each $S_n : C \to C$ is a pseudocontraction mapping, it follows that

$$\begin{aligned}
\|p - u_n\|^2 &= \langle \beta_n(p - x_n) + (1 - \beta_n)(p - S_n u_n), j(p - u_n)\rangle \\
&= (1 - \beta_n)\langle p - S_n u_n, j(p - u_n)\rangle + \beta_n\langle p - x_n, j(p - u_n)\rangle \\
&\le (1 - \beta_n)\|p - u_n\|^2 + \beta_n\|x_n - p\|\|p - u_n\|,
\end{aligned}$$

which hence yields

$$\|p - u_n\| \le \|p - x_n\|, \quad \forall n \ge 0. \tag{10}$$

Then we observe

$$\|y_n - p\| = \|Gu_n - p\| \le \|u_n - p\| \le \|x_n - p\|. \tag{11}$$

Combining (9) and (11), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\le \|\alpha_n(f(x_n) - \theta F(p)) + (I - \alpha_n\theta F)T^n y_n - (I - \alpha_n\theta F)p\| \\
&\le \alpha_n\gamma\|x_n - p\| + \alpha_n\|f(p) - \theta F(p)\| + (1 - \alpha_n\bar{\gamma}\theta)\|T^n y_n - p\| \\
&\le \alpha_n\gamma\|x_n - p\| + \alpha_n\|f(p) - \theta F(p)\| + (1 - \alpha_n\bar{\gamma}\theta)(\|y_n - p\| + c_n) \\
&\le c_n + \alpha_n\|f(p) - \theta F(p)\| + \alpha_n\gamma\|x_n - p\| + (1 - \alpha_n\bar{\gamma}\theta)\|x_n - p\| \\
&\le c_n + \max\{\|x_n - p\|, \frac{\|f(p) - \theta F(p)\|}{\bar{\gamma}\theta - \gamma}\}.
\end{aligned}$$

By induction, we get

$$\|x_n - p\| \le \sum_{n=0}^{\infty} c_n + \max\{\|x_0 - p\|, \frac{\|f(p) - \theta F(p)\|}{\bar{\gamma}\theta - \gamma}\}, \quad \forall n \ge 0.$$

It immediately follows that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}, \{u_n\}, \{f(x_n)\}, \{T^n y_n\}$ and $\{F(T^n y_n)\}$ (due to (10), (11) and the Lipschitz continuity of f). Taking into account that $\{S_n\}$ is ℓ-uniformly Lipschitzian on C, we know that

$$\ell\|u_n - p\| + \|p\| \ge |S_n u_n - p\| + \|p\| \ge \|S_n u_n\|,$$

which implies that $\{S_n u_n\}$ is bounded. In addition, from Lemma 3 and $p \in \Omega \subset \mathrm{GSVI}(C, B_1, B_2)$, it also follows that (p, q) is a solution of GSVI (3) where $q = \Pi_C(p - \mu_2 B_2 p)$. Please note that $z_n = \Pi_C(I - \mu_2 B_2)u_n$ for all $n \ge 0$. Then by Lemma 4 we get

$$\begin{aligned}
\|z_n\| &\le \|\Pi_C(u_n - \mu_2 B_2 u_n) - \Pi_C(p - \mu_2 B_2 p)\| + \|q\| \\
&\le \|(u_n - \mu_2 B_2 u_n) - (p - \mu_2 B_2 p)\| + \|q\| \\
&\le \|q\| + \|u_n - p\|.
\end{aligned}$$

This shows that $\{z_n\}$ is bounded.

Step 2. We claim that $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$. Indeed, from (9) we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\le \|\alpha_n f(x_n) + (I - \alpha_n \theta F)T^n y_n - \alpha_{n-1} f(x_{n-1}) - (I - \alpha_{n-1}\theta F)T^{n-1}y_{n-1}\| \\
&\le \alpha_n \gamma \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - \theta F T^n y_{n-1}\| + (1 - \alpha_n \bar{\gamma}\theta)(\|y_n - y_{n-1}\| + c_n) \\
&\quad + (1 - \alpha_{n-1}\bar{\gamma}\theta)\|T^n y_{n-1} - T^{n-1} y_{n-1}\| \\
&\le \alpha_n \gamma \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M_1 + (1 - \alpha_n \bar{\gamma}\theta)\|u_n - u_{n-1}\| \\
&\quad + \|T^n y_{n-1} - T^{n-1} y_{n-1}\| + c_n,
\end{aligned} \tag{12}$$

where

$$M_1 = \sup_{n \ge 1} \|\theta F T^n y_{n-1} - f(x_{n-1})\|.$$

Also, simple calculations show that

$$\begin{aligned}
&\|u_n - u_{n-1}\|^2 \\
&\le \beta_n \|x_n - x_{n-1}\| \|u_n - u_{n-1}\| + (1 - \beta_n)[\|S_n u_n - S_{n-1} u_n\| \|u_n - u_{n-1}\| \\
&\quad + \|u_n - u_{n-1}\|^2] + |\beta_n - \beta_{n-1}| \|x_{n-1} - S_{n-1} u_{n-1}\| \|u_n - u_{n-1}\|.
\end{aligned} \tag{13}$$

So, it follows from (13) that

$$\begin{aligned}
\|u_n - u_{n-1}\| \quad &\le \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n)[\|S_n u_n - S_{n-1} u_n\| \\
&\quad + \|u_n - u_{n-1}\|] + |\beta_n - \beta_{n-1}| \|x_{n-1} - S_{n-1} u_{n-1}\|,
\end{aligned}$$

which immediately leads to

$$\begin{aligned}
\|u_n - u_{n-1}\| - \|x_n - x_{n-1}\| \quad &\le \frac{1 - \beta_n}{\beta_n} \|S_n u_n - S_{n-1} u_n\| + |\beta_n - \beta_{n-1}| \frac{\|x_{n-1} - S_{n-1} u_{n-1}\|}{\beta_n} \\
&\le \frac{1}{a} \|S_n u_n - S_{n-1} u_n\| + |\beta_n - \beta_{n-1}| \frac{\|x_{n-1} - S_{n-1} u_{n-1}\|}{a}.
\end{aligned} \tag{14}$$

Putting $D = \{u_n : n \ge 0\}$, we know that D is a bounded subset of C. Then by the assumption we get $\sum_{n=1}^{\infty} \sup_{x \in D} \|S_n x - S_{n-1} x\| < \infty$. Noticing that

$$\sup_{x \in D} \|S_n x - S_{n-1} x\| \ge \|S_n u_n - S_{n-1} u_n\|, \quad \forall n \ge 1,$$

we have

$$\sum_{n=1}^{\infty} \|S_n u_n - S_{n-1} u_n\| < \infty. \tag{15}$$

Therefore, from (12) and (14) we deduce that

$$\begin{aligned}
\|x_{n+1} - x_n\| \quad &\le \alpha_n \gamma \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M_1 + (1 - \alpha_n \bar{\gamma}\theta)\{\|x_n - x_{n-1}\| \\
&\quad + \frac{1}{a} \|S_n u_n - S_{n-1} u_n\| + |\beta_n - \beta_{n-1}| \frac{\|x_{n-1} - S_{n-1} u_{n-1}\|}{a}\} \\
&\quad + c_n + \|T^n y_{n-1} - T^{n-1} y_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}| M_1 + \frac{1}{a} \|S_n u_n - S_{n-1} u_n\| + [1 - \alpha_n(\bar{\gamma}\theta - \gamma)] \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \frac{\|x_{n-1} - S_{n-1} u_{n-1}\|}{a} + c_n + \|T^n y_{n-1} - T^{n-1} y_{n-1}\| \\
&\le M_2(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + [1 - \alpha_n(\bar{\gamma}\theta - \gamma)] \|x_n - x_{n-1}\| \\
&\quad + \|S_n u_n - S_{n-1} u_n\| + c_n + \|T^n y_{n-1} - T^{n-1} y_{n-1}\|),
\end{aligned} \tag{16}$$

where

$$M_2 = \sup_{n\geq 1}\{\frac{1}{a} + \frac{\|x_{n-1} - S_{n-1}u_{n-1}\|}{a} + M_1 + 1\}.$$

From (15), the assumption $\sum_{n=0}^{\infty} c_n < \infty$ and conditions (i) and (ii), it can be readily seen that $\sum_{n=0}^{\infty} \alpha_n(\bar{\gamma}\theta - \gamma) = \infty$ and

$$\sum_{n=1}^{\infty} M_2(\|S_n u_n - S_{n-1}u_n\| + c_n + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + \|T^n y_{n-1} - T^{n-1}y_{n-1}\|) < \infty.$$

So, it follows from Lemma 7 and (16) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{17}$$

Again from (9) and (14) we conclude that

$$\begin{aligned} &a\|y_n - y_{n-1}\| = a\|Gu_n - Gu_{n-1}\| \leq a\|u_n - u_{n-1}\| \\ &\leq a\|x_n - x_{n-1}\| + \|S_n u_n - S_{n-1}u_n\| + |\beta_n - \beta_{n-1}|\|x_{n-1} - S_{n-1}u_{n-1}\| \to 0 \quad (n \to \infty). \end{aligned}$$

That is,

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{18}$$

Step 3. We claim that $\|x_n - Gx_n\| \to 0$ as $n \to \infty$. Indeed, note that $q = \Pi_C(p - \mu_2 B_2 p)$, $z_n = \Pi_C(u_n - \mu_2 B_2 u_n)$ and $y_n = \Pi_C(z_n - \mu_1 B_1 z_n)$. Then $y_n = Gu_n$. By Lemma 4 we have

$$\begin{aligned} \|z_n - q\|^2 &\leq \|u_n - p - \mu_2(B_2 u_n - B_2 p)\|^2 \\ &\leq 2\mu_2(\kappa^2\mu_2 - \beta)\|B_2 u_n - B_2 p\|^2 + \|u_n - p\|^2 \end{aligned} \tag{19}$$

and

$$\begin{aligned} \|y_n - p\|^2 &\leq \|z_n - q - \mu_1(B_1 z_n - B_1 q)\|^2 \\ &\leq 2\mu_1(\kappa^2\mu_1 - \alpha)\|B_1 z_n - B_1 q\|^2 + \|z_n - q\|^2. \end{aligned} \tag{20}$$

Substituting (19) for (20), we obtain

$$\|y_n - p\|^2 + 2\mu_2(\beta - \kappa^2\mu_2)\|B_2 u_n - B_2 p\|^2 + 2\mu_1(\alpha - \kappa^2\mu_1)\|B_1 z_n - B_1 q\|^2 \leq \|u_n - p\|^2. \tag{21}$$

Let $v_n := (T^n y_n - \alpha_n \theta F T^n y_n) + \alpha_n f(x_n)$. Then, from (7) and Lemma 2 we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|v_n - p\|^2 \\ &\leq 2\alpha_n\langle f(x_n) - \theta F T^n y_n, j(v_n - p)\rangle + \|T^n y_n - p\|^2 \\ &\leq 2\alpha_n\|f(x_n) - \theta F T^n y_n\|\|v_n - p\| + (\|y_n - p\| + c_n)^2 \\ &\leq \|y_n - p\|^2 + c_n M_2 + \alpha_n M_3, \end{aligned} \tag{22}$$

where $M_2 = \sup_{n\geq 0}\{2\|y_n - p\| + c_n\}$ and

$$M_3 = \sup_{n\geq 0}\{2\|f(x_n) - \theta F T^n y_n\|\|v_n - p\|\}.$$

Substituting (21) to (22), we deduce from (10) that

$$\begin{aligned} &\|x_{n+1} - p\|^2 + 2\mu_2(\beta - \kappa^2\mu_2)\|B_2 u_n - B_2 p\|^2 + 2\mu_1(\alpha - \kappa^2\mu_1)\|B_1 z_n - B_1 q\|^2 \\ &\leq \|x_n - p\|^2 + c_n M_2 + \alpha_n M_3, \end{aligned}$$

which immediately yields

$$\begin{aligned}&2\mu_2(\beta-\kappa^2\mu_2)\|B_2u_n-B_2p\|^2+2\mu_1(\alpha-\kappa^2\mu_1)\|B_1z_n-B_1q\|^2\\&\le\alpha_nM_3-\|x_{n+1}-p\|^2+c_nM_2+\|x_n-p\|^2\\&\le c_nM_2+\alpha_nM_3+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.\end{aligned}$$

Since $\mu_1\kappa^2\in(0,\alpha),\ \mu_2\kappa^2\in(0,\beta),\ \lim_{n\to\infty}c_n=0$ and $\lim_{n\to\infty}\alpha_n=0$, we obtain from (17) that

$$\lim_{n\to\infty}\|B_2u_n-B_2p\|=0\quad\text{and}\quad\lim_{n\to\infty}\|B_1z_n-B_1q\|=0.\tag{23}$$

On the other hand, from Proposition 3 and Lemma 5 we have

$$\begin{aligned}2\|z_n-q\|^2&\le2\langle u_n-\mu_2B_2u_n-(p-\mu_2B_2p),j(z_n-q)\rangle\\&\le[\|u_n-p\|^2+\|z_n-q\|^2-g_1(\|u_n-z_n-(p-q)\|)]+2\mu_2\|B_2p-B_2u_n\|\|z_n-q\|,\end{aligned}$$

which implies that

$$\|z_n-q\|^2+g_1(\|u_n-z_n-(p-q)\|)\le\|u_n-p\|^2+2\mu_2\|B_2p-B_2u_n\|\|z_n-q\|.\tag{24}$$

In the same way, we derive

$$\begin{aligned}2\|y_n-p\|^2&\le2\langle z_n-\mu_1B_1z_n-(q-\mu_1B_1q),j(y_n-p)\rangle\\&\le[\|z_n-q\|^2+\|y_n-p\|^2-g_2(\|z_n-y_n+(p-q)\|)]+2\mu_1\|B_1q-B_1z_n\|\|y_n-p\|,\end{aligned}$$

which implies that

$$\|y_n-p\|^2+g_2(\|z_n-y_n+(p-q)\|)\le\|z_n-q\|^2+2\mu_1\|B_1q-B_1z_n\|\|y_n-p\|.\tag{25}$$

Substituting (24) for (25), we deduce from (10) that

$$\begin{aligned}\|y_n-p\|^2&\le\|u_n-p\|^2-g_1(\|u_n-z_n-(p-q)\|)-g_2(\|z_n-y_n+(p-q)\|)\\&\quad+2\mu_2\|B_2p-B_2u_n\|\|z_n-q\|+2\mu_1\|B_1q-B_1z_n\|\|y_n-p\|\\&\le\|x_n-p\|^2-g_1(\|u_n-z_n-(p-q)\|)-g_2(\|z_n-y_n+(p-q)\|)\\&\quad+2\mu_1\|B_1q-B_1z_n\|\|y_n-p\|+2\mu_2\|B_2p-B_2u_n\|\|z_n-q\|.\end{aligned}\tag{26}$$

Substituting (26) for (22), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2+g_1(\|u_n-z_n-(p-q)\|)+g_2(\|z_n-y_n+(p-q)\|)\\&\le\|x_n-p\|^2+2\mu_2\|B_2p-B_2u_n\|\|z_n-q\|+2\mu_1\|B_1q-B_1z_n\|\|y_n-p\|+c_nM_2+\alpha_nM_3,\end{aligned}$$

which hence yields

$$\begin{aligned}&g_1(\|u_n-z_n-(p-q)\|)+g_2(\|z_n-y_n+(p-q)\|)\\&\le(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2\mu_2\|B_2p-B_2u_n\|\|z_n-q\|\\&\quad+2\mu_1\|B_1q-B_1z_n\|\|y_n-p\|+\alpha_nM_3+c_nM_2.\end{aligned}$$

Since $\lim_{n\to\infty}c_n=0$ and $\lim_{n\to\infty}\alpha_n=0$, we conclude from (17) and (23) that

$$\lim_{n\to\infty}g_1(\|u_n-z_n-(p-q)\|)=0\quad\text{and}\quad\lim_{n\to\infty}g_2(\|z_n-y_n+(p-q)\|)=0.$$

Using the properties of g_1 and g_2, we obtain

$$\lim_{n\to\infty}\|u_n - z_n - (p-q)\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|z_n - y_n + (p-q)\| = 0. \tag{27}$$

It follows that

$$\|u_n - y_n\| \le \|u_n - z_n - (p-q)\| + \|z_n - y_n + (p-q)\| \to 0 \quad (n\to\infty).$$

That is,

$$\lim_{n\to\infty}\|u_n - Gu_n\| = \lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{28}$$

Also, according to (8) we have

$$\|u_n - p\|^2 \le \beta_n\langle x_n - p, j(u_n - p)\rangle + (1-\beta_n)\|u_n - p\|^2,$$

which together with Lemma 5, yields

$$\begin{aligned} 2\|u_n - p\|^2 &\le 2\langle x_n - p, j(u_n - p)\rangle \\ &\le [\|x_n - p\|^2 + \|u_n - p\|^2 - g(\|x_n - u_n\|)]. \end{aligned}$$

This immediately implies that

$$\|u_n - p\|^2 + g(\|x_n - u_n\|) \le \|x_n - p\|^2,$$

which together with (22), yields

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \|u_n - p\|^2 + c_n M_2 + \alpha_n M_3 \\ &\le \|x_n - p\|^2 - g(\|x_n - u_n\|) + c_n M_2 + \alpha_n M_3. \end{aligned}$$

Hence we have

$$\begin{aligned} g(\|u_n - x_n\|) &\le \|x_n - p\|^2 + c_n M_2 - \|x_{n+1} - p\|^2 + \alpha_n M_3 \\ &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n M_3 + c_n M_2. \end{aligned}$$

Since $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, we obtain from (17) that $\lim_{n\to\infty} g(\|x_n - u_n\|) = 0$. Using the properties of g, we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{29}$$

Also, observe that

$$\|x_n - u_n\| + \|u_n - Gu_n\| \ge \|x_n - y_n\|,$$

and

$$\|x_n - y_n\| + \|u_n - x_n\| \ge \|x_n - y_n\| + \|Gu_n - Gx_n\| \ge \|x_n - Gx_n\|.$$

Then from (28) and (29) it follows that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{30}$$

Step 4. We claim that $\|Tx_n - x_n\| \to 0$ and $\|S_n x_n - x_n\| \to 0$ as $n \to \infty$. Indeed, combining (8) with (29), we obtain that

$$\|S_n u_n - u_n\| = \frac{\beta_n}{1-\beta_n}\|x_n - u_n\| \le \frac{b}{1-b}\|x_n - u_n\| \to 0 \quad (n\to\infty).$$

That is,

$$\lim_{n\to\infty} \|S_n u_n - u_n\| = 0. \tag{31}$$

Since $\{S_n\}_{n=0}^{\infty}$ is ℓ-uniformly Lipschitzian on C, we deduce from (29) and (31) that

$$\begin{aligned} \|S_n x_n - x_n\| &\le \|S_n x_n - S_n u_n\| + \|S_n u_n - u_n\| + \|u_n - x_n\| \\ &\le (\ell+1)\|x_n - u_n\| + \|S_n u_n - u_n\| \to 0 \quad (n \to \infty). \end{aligned}$$

That is,

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{32}$$

We note that

$$\begin{aligned} \|x_n - T^n y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n y_n\| \\ &=\le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - \theta F T^n y_n\|. \end{aligned}$$

Then we have

$$\begin{aligned} \|y_n - T^n y_n\| &\le \|y_n - x_n\| + \|x_n - T^n y_n\| \\ &\le \|y_n - x_n\| + \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - \theta F T^n y_n\|. \end{aligned}$$

Consequently, from (17), (30) and $\lim_{n\to\infty} \alpha_n = 0$, it follows that

$$\lim_{n\to\infty} \|y_n - T^n y_n\| = 0. \tag{33}$$

We also note that

$$\|y_n - T y_n\| \le \|y_n - T^n y_n\| + \|T^n y_n - T^{n+1} y_n\| + \|T^{n+1} y_n - T y_n\|.$$

By the assumption $\sum_{n=0}^{\infty} \|T^{n+1} y_n - T^n y_n\| < \infty$, (33) and the condition that $T : C \to C$ is uniformly continuous, we get

$$\lim_{n\to\infty} \|y_n - T y_n\| = 0. \tag{34}$$

In addition, noticing that

$$\|x_n - T x_n\| \le \|x_n - y_n\| + \|y_n - T y_n\| + \|T y_n - T x_n\|,$$

we deduce from (30), (34) and the uniform continuity of T that

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0. \tag{35}$$

Step 5. We claim that $\|x_n - \overline{S} x_n\| \to 0$ as $n \to \infty$ where $\overline{S} := (2I - S)^{-1}$. Indeed, first, let us show that $S : C \to C$ is pseudocontractive and ℓ-Lipschitzian such that $\lim_{n\to\infty} \|S x_n - x_n\| = 0$ where $Sx = \lim_{n\to\infty} S_n x \; \forall x \in C$. Observe that for all $x, y \in C$, $\lim_{n\to\infty} \|S_n x - S x\| = 0$ and $\lim_{n\to\infty} \|S_n y - S y\| = 0$. Since S_n is pseudocontractive operator, we get

$$\langle Sx - Sy, j(x-y) \rangle \le \|x - y\|^2.$$

This means that S is pseudocontractive. Noting that $\{S_n\}_{n=0}^{\infty}$ is ℓ-uniformly Lipschitzian on C, we have

$$\|Sx - Sy\| = \lim_{n\to\infty} \|S_n x - S_n y\| \le \ell \|x - y\|, \quad \forall x, y \in C.$$

This means that S is ℓ-Lipschitzian. Taking into account the boundedness of $\{x_n\}$ and putting $D = \overline{\text{conv}}\{x_n : n \geq 0\}$ (the closed convex hull of the set $\{x_n : n \geq 0\}$), by the assumption we have $\sum_{n=1}^{\infty} \sup_{x \in D} \|S_{n-1}x - S_n x\| < \infty$. Hence, by Proposition 1 we get

$$\lim_{n \to \infty} \sup_{x \in D} \|S_n x - Sx\| = 0,$$

which immediately yields

$$\lim_{n \to \infty} \|S_n x_n - Sx_n\| = 0. \tag{36}$$

Thus, combining (32) with (36) we have

$$\|x_n - Sx_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0 \quad (n \to \infty).$$

That is,

$$\lim_{n \to \infty} \|x_n - Sx_n\| = 0. \tag{37}$$

Now, let us show that if we define $\overline{S} := (2I - S)^{-1}$, then $\overline{S} : C \to C$ is nonexpansive, $\text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$ and $\lim_{n \to \infty} \|x_n - \overline{S}x_n\| = 0$. Indeed, put $\overline{S} := (2I - S)^{-1}$, where I is the identity mapping on E. Then it is known that $\overline{S}$ is nonexpansive and the fixed-point set $\text{Fix}(S) = \text{Fix}(\overline{S}) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$. From (37) it follows that

$$\begin{aligned} \|x_n - \overline{S}x_n\| &= \|\overline{S}\overline{S}^{-1}x_n - \overline{S}x_n\| \\ &\leq \|\overline{S}^{-1}x_n - x_n\| \\ &= \|(2I - S)x_n - x_n\| = \|x_n - Sx_n\| \to 0 \quad (n \to \infty). \end{aligned}$$

That is,

$$\lim_{n \to \infty} \|x_n - \overline{S}x_n\| = 0. \tag{38}$$

Step 6. We claim that

$$\limsup_{n \to \infty} \langle f(x^*) - \theta F(x^*), j(x_n - x^*) \rangle \leq 0, \tag{39}$$

where $x^* = \Pi_\Omega(f + I - \theta F)(x^*)$. Indeed, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \to \infty} \langle f(x^*) - \theta F(x^*), j(x_n - x^*) \rangle = \lim_{i \to \infty} \langle f(x^*) - \theta F(x^*), j(x_{n_i} - x^*) \rangle.$$

Now we show that $\Pi_\Omega(f + I - \theta F)$ is a contraction mapping. Since F is bounded linear strongly positive, for all $x, y \in C$, we have

$$\begin{aligned} &\|\Pi_\Omega(f + I - \theta F)(x) - \Pi_\Omega(f + I - \theta F)(y)\| \\ &\leq \|f(x) - f(y)\| + \|(I - \theta F)(x) - (I - \theta F)(y)\| \\ &\leq [1 + (\gamma - \bar{\gamma}\theta)]\|x - y\|, \end{aligned}$$

which implies that $\Pi_\Omega(f + I - \theta F)$ is a contraction mapping. Banach's contraction mapping principle guarantees that $\Pi_\Omega(f + I - \theta F)$ has a unique fixed point. Say $x^* \in C$, that is, $x^* = \Pi_\Omega(f + I - \theta F)(x^*)$. Since $\{x_n\}$ is a bounded sequence in C, we may assume that $x_{n_i} \rightharpoonup \bar{x} \in C$. Please note that G and $\overline{S}$ are nonexpansive and that T is asymptotically nonexpansive in the intermediate sense. Since $(I - G)x_n \to 0$ and $(I - \overline{S})x_n \to 0$ (due to (30) and (37)), by Lemma 6 we have that $\bar{x} \in \text{Fix}(G) = \text{GSVI}(C, B_1, B_2)$ and $\bar{x} \in \text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$. From (35), we have that $\lim_{i \to \infty} \|x_{n_i} - Tx_{n_i}\| = 0$ for the subsequence $\{x_{n_i}\}$ of $\{x_n\}$. It follows that $\bar{x} \in \text{Fix}(T)$. Then, $\bar{x} \in \Omega = \bigcap_{n=0}^{\infty} \text{Fix}(S_n) \cap$

$\mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T)$. Since E admits a weakly sequentially continuous duality mapping $j(\cdot)$ and $x_{n_i} \rightharpoonup \bar{x}$, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - \theta F(x^*), j(x_n - x^*)\rangle &= \lim_{i\to\infty}\langle f(x^*) - \theta F(x^*), j(x_{n_i} - x^*)\rangle \\ &= \langle f(x^*) - \theta F(x^*), j(\bar{x} - x^*)\rangle \le 0,\end{aligned}$$

which implies that (39) holds. Noticing that $j(\cdot)$ is also norm-to-norm uniformly continuous on bounded subsets of E, we obtain from (17) that

$$\limsup_{n\to\infty}\langle f(x^*) - \theta F(x^*), j(x_{n+1} - x^*)\rangle \le 0. \tag{40}$$

Step 7. We claim that $x_n \to x^*$ as $n \to \infty$. Indeed, it follows from $x_{n+1} = \Pi_C v_n$ and Proposition 3 (iii) that

$$\langle \Pi_C v_n - v_n, j(\Pi_C v_n - x^*)\rangle \le 0,$$

which leads to

$$\langle x_{n+1} - v_n, j(x_{n+1} - x^*)\rangle \le 0.$$

Then, from (11) we have

$$\begin{aligned}&2\|x_{n+1} - x^*\|^2 \\ &\le 2\langle x^* - v_n, j(x^* - x_{n+1})\rangle \\ &\le 2\alpha_n\gamma\|x_n - x^*\|\|x_{n+1} - x^*\| + 2(1 - \alpha_n\bar{\gamma}\theta)(\|y_n - x^*\| + c_n)\|x_{n+1} - x^*\| \\ &\quad + 2\alpha_n\langle f(x^*) - \theta F(x^*), j(x_{n+1} - x^*)\rangle \\ &\le 2[1 - \alpha_n(\bar{\gamma}\theta - \gamma)]\|x_n - x^*\|\|x_{n+1} - x^*\| + 2c_nM_4 + 2\alpha_n\langle f(x^*) - \theta F(x^*), j(x_{n+1} - x^*)\rangle \\ &\le \|x_{n+1} - x^*\|^2 + (1 - \alpha_n(\bar{\gamma}\theta - \gamma))\|x_n - x^*\|^2 + 2c_nM_4 + 2\alpha_n\langle f(x^*) - \theta F(x^*), j(x_{n+1} - x^*)\rangle,\end{aligned}$$

where $M_4 = \sup_{n\ge 0}\|x_{n+1} - x^*\|$. This immediately implies that

$$\|x_{n+1} - x^*\|^2 \le [1 - \alpha_n(\bar{\gamma}\theta - \gamma)]\|x_n - x^*\|^2 + \alpha_n(\bar{\gamma}\theta - \gamma)\frac{2\langle f(x^*) - \theta F(x^*), j(x_{n+1} - x^*)\rangle}{\bar{\gamma}\theta - \gamma} + 2c_nM_4. \tag{41}$$

Applying Lemma 7 to (41), we infer that $x_n \to x^*$ as $n \to \infty$. The proof is completed. $\square$

It is remarkable that according to the proof of Theorem 1, we know that $\{y_n\}$ is bounded. We now give two examples to illustrate partial conditions of Theorem 1 to be satisfied.

Example 1. *Let $T : C \to C$ be a contraction mapping with a constant $\beta \in (0, 1)$. We take $S_n := T^n$ and obtain*

$$\sup_{x\in D}\|S_nx - S_{n-1}x\| = \sup_{x\in D}\|T^nx - T^{n-1}x\| \le \beta^{n-1}\cdot\sup_{x\in D}\|Tx - x\|, \quad \forall n \ge 1,$$

for any bounded subset D of C. Therefore, it follows that

$$+\infty > \sum_{n=1}^{\infty}\beta^{n-1}\cdot\sup_{x\in D}\|Tx - x\| \ge \sum_{n=1}^{\infty}\sup_{x\in D}\|S_nx - S_{n-1}x\|.$$

In particular, whenever D is a bounded sequence $\{x_n\}_{n=0}^{\infty}$ in C, we have

$$+\infty > \sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x - S_nx\| \ge \sum_{n=0}^{\infty}\|S_{n+1}x_n - S_nx_n\| = \sum_{n=0}^{\infty}\|T^{n+1}x_n - T^nx_n\|.$$

Since T is a contraction mapping, Banach's Contraction Mapping Principle guarantees that T has a unique fixed point. Say $p \in C$. We define $Sx := p$ for all $x \in C$. It is easy to see that $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$ and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$.

Example 2. *Let $E = \mathbf{R}$ and $C = [-1,1]$, and let $T : C \to C$ be an identity mapping, i.e., $Tx = x$ for all $x \in C$. Moreover, we define*

$$S_n x := \frac{\sin x + x}{2+n}, \quad \forall x \in C, \ \forall n \geq 0.$$

Then we obtain

$$\begin{aligned} |S_n x - S_n y| &= |\frac{\sin x +}{n+2} - \frac{\sin y + y}{n+2}| \\ &\leq \frac{|y-x| + |\sin y - \sin x|}{n+2} \leq \frac{2|x-y|}{n+2} \leq |x-y|, \quad \forall x,y \in C, \ \forall n \geq 0, \end{aligned}$$

and

$$\begin{aligned} \sup_{x\in C}|S_n x - S_{n-1}x| &= \sup_{x\in C}|\frac{x+\sin x}{n+2} - \frac{x+\sin x}{n+1}| = \sup_{x\in C}|\frac{x+\sin x}{(n+2)(n+1)}| \\ &\leq \frac{2}{(n+2)(n+1)} \leq \frac{2}{(n+1)^2}, \quad \forall n \geq 0. \end{aligned}$$

Therefore, it follows that for any bounded subset D of C,

$$\sum_{n=1}^{\infty} \sup_{x\in D} |S_n x - S_{n-1}x| < \infty.$$

In addition, whenever $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence in C, it is clear that

$$\sum_{n=0}^{\infty} |T^n x_n - T^{n+1} x_n| < \infty.$$

Also, we define $Sx := 0$ for all $x \in C$. Then, it is clear that $Sx = \lim_{n\to\infty} S_n x \ \forall x \in C$, and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$.

4. Applications

Now, we give an application to solve CFPPs of asymptotically nonexpansive and pseudocontractive mappings, and variational inequality problems for strict pseudocontractive mappings in Banach spaces.

Let C be a nonempty, closed, and convex subset of a real Banach space E. A mapping $T : C \to C$ is said to be λ-strictly pseudocontractive if for every $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle + \lambda\|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2, \quad \text{for some } \lambda \in (0,1). \tag{42}$$

A simple computation shows that (42) is equivalent to the following inequality:

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \lambda\|(I-T)x - (I-T)y\|^2, \tag{43}$$

for every $x, y \in C$ and for some $j(x-y) \in J(x-y)$. Therefore, $I - T$ is λ-inverse-strongly accretive.

By Theorem 1, we can obtain the following results easily.

Theorem 2. *Let C be a convex closed set in a 2-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous duality mapping. Let Π_C be a sunny nonexpansive retraction from E onto C. Let the mappings $B_1, B_2 : C \to C$ be α-strictly pseudocontractive and β-strictly pseudocontractive, respectively. Let $f : C \to C$ be a contraction mapping with coefficient $\gamma \in [0,1)$ and $F : E \to E$ be a strongly* *positive linear bounded operator with the coefficient $\bar{\gamma}$ such that $0 < \gamma < \bar{\gamma}\theta$ and $0 < \theta \leq \|F\|^{-1}$. Let $T : C \to C$ be uniformly continuous and asymptotically nonexpansive mapping in the intermediate sense, and $\{S_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C such that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, I - B_1, I - B_2) \cap \mathrm{Fix}(T) \neq \emptyset$ where $\mathrm{GSVI}(C, I - B_1, I - B_2)$ is the fixed-point set of the mapping $G := [(1 - \mu_1)I + \mu_1 B_1][(1 - \mu_2)I + \mu_2 B_2]$ with $0 < \mu_1\kappa^2 < \min\{1, \alpha\}$ and $0 < \mu_2\kappa^2 < \min\{1, \beta\}$ for κ the 2-uniformly smooth constant of E. Assume that $\sum_{n=0}^{\infty} c_n < \infty$, where c_n is defined by (2). For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = (1 - \mu_1)z_n + \mu_1 B_1 z_n, \\ z_n = (1 - \mu_2)u_n + \mu_2 B_2 u_n, \\ u_n = \beta_n(x_n - S_n u_n) + S_n u_n, \\ x_{n+1} = \Pi_C[(T^n y_n - \alpha_n \theta F T^n y_n) + \alpha_n f(x_n)], \quad \forall n \geq 0, \end{cases} \tag{44}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $[0, 1]$ satisfying the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Assume that $\sum_{n=1}^{\infty} \sup_{x \in D} \|S_n x - S_{n-1}x\| < \infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$. If $\sum_{n=0}^{\infty} \|T^{n+1}y_n - T^n y_n\| < \infty$, then $\{x_n\}$ converges strongly to $x^* \in \Omega$. In this case,

(a) $x^* \in \Omega$ *solves the VI:* $\langle f(x^*) - \theta F(x^*), j(x - x^*)\rangle \leq 0$, $\forall x \in \Omega$;
(b) (x^*, y^*) *is a solution of GSVI (1.3) for two inverse-strongly accretive mappings* $I - B_i$, $i = 1, 2$, *where* $y^* = (1 - \mu_2)x^* + \mu_2 B_2 x^*$.

Proof. Since the mappings $B_1, B_2 : C \to C$ are α-strictly pseudocontractive and β-strictly pseudocontractive, respectively, it can be seen readily that $I - B_1, I - B_2 : C \to E$ are α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Please note that $0 < \mu_1\kappa^2 < \min\{1, \alpha\}$ and $0 < \mu_2\kappa^2 < \min\{1, \beta\}$ for κ the 2-uniformly smooth constant of E. Then, $\mathrm{GSVI}(C, I - B_1, I - B_2)$ is the fixed-point set of the following mapping

$$\begin{aligned} Gx: \quad &= \Pi_C[(1 - \mu_1)I + \mu_1 B_1]\Pi_C[(1 - \mu_2)x + \mu_2 B_2 x] \\ &= \Pi_C\{(1 - \mu_1)[(1 - \mu_2)x + \mu_2 B_2 x] + \mu_1 B_1[(1 - \mu_2)x + \mu_2 B_2 x]\} \\ &= [(1 - \mu_1)I + \mu_1 B_1][(1 - \mu_2)I + \mu_2 B_2]x, \quad \forall x \in C. \end{aligned}$$

In this case, it is easy to see that the iterative scheme (7) reduces to (44). Therefore, by Theorem 1 we obtain the desired result. □

Theorem 3. *Let C be a bounded, convex and closed set in a 2-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous duality mapping. Let Π_C be a sunny nonexpansive retraction from E onto C. Let the mappings $B_1, B_2 : C \to C$ be α-strictly pseudocontractive and β-strictly pseudocontractive, respectively. Let $f : C \to C$ be a contraction mapping with coefficient $\gamma \in [0, 1)$ and $F : E \to E$ be a strongly positive linear bounded operator with the coefficient $\bar{\gamma}$ such that $0 < \gamma < \bar{\gamma}\theta$ and $0 < \theta \leq \|F\|^{-1}$. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_n\} \subset [0, \infty)$ satisfying $\sum_{n=0}^{\infty} \theta_n < \infty$, and $\{S_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C such that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, I - B_1, I - B_2) \cap \mathrm{Fix}(T) \neq \emptyset$ where $\mathrm{GSVI}(C, I - B_1, I - B_2)$ is the fixed-point set of the mapping $G := [(1 - \mu_1)I + \mu_1 B_1][(1 - \mu_2)I + \mu_2 B_2]$ with $0 < \mu_1\kappa^2 < \min\{1, \alpha\}$*

and $0 < \mu_2\kappa^2 < \min\{1, \beta\}$ *for* κ *the* 2*-uniformly smooth constant of E. For arbitrarily given* $x_0 \in C$*, let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} z_n = (1-\mu_2)u_n + \mu_2 B_2 u_n, \\ u_n = \beta_n(x_n - S_n u_n) + S_n u_n, \\ y_n = (1-\mu_1)z_n + \mu_1 B_1 z_n, \\ x_{n+1} = \Pi_C[\alpha_n f(x_n) + (I - \alpha_n\theta F)T^n y_n], \quad \forall n \geq 0, \end{cases} \tag{45}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are the sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$ *for any bounded subset D of C, and let S be a mapping of C into itself defined by* $Sx = \lim_{n\to\infty} S_n x$ *for all* $x \in C$*, and suppose that* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$*. If* $\sum_{n=0}^{\infty} \|T^{n+1}y_n - T^n y_n\| < \infty$*, then* $\{x_n\}$ *converges strongly to* $x^* \in \Omega$*. In this case,*

(a) $x^* \in \Omega$ *solves the VI:* $\langle f(x^*) - \theta F(x^*), j(x - x^*)\rangle \leq 0$, $\forall x \in \Omega$;

(b) (x^*, y^*) *is a solution of GSVI (1.3) for two inverse-strongly accretive mappings* $I - B_i$, $i = 1, 2$*, where* $y^* = (1-\mu_2)x^* + \mu_2 B_2 x^*$.

Proof. Since set C is a bounded set, we know that $\mathrm{diam}(C) = \sup_{x,y\in C} \|x - y\| < \infty$. We have that $T : C \to C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_n\} \subset [0, \infty)$ satisfying $\sum_{n=0}^{\infty} \theta_n < \infty$. Then, we deduce that for all $x, y \in C$,

$$\|T^n x - T^n y\| \leq (1 + \theta_n)\|x - y\| = \|x - y\| + \theta_n\|x - y\| \leq \|x - y\| + \theta_n \mathrm{diam}(C).$$

Hence, we get

$$c_n := \max\{0, \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|)\} \leq \theta_n \mathrm{diam}(C),$$

which immediately attains $\sum_{n=0}^{\infty} c_n < \infty$. Therefore, by Theorem 1 we derive the desired result. □

5. Conclusions

In this work, we studied problem of solving a general system of monotone variational inequalities whose solutions are also the solutions of the CFPP of countably many nonlinear operators via a hybrid viscosity implicit iteration method. Strong convergence theorems were established in 2-uniformly smooth and uniformly convex Banach spaces. An application to CFPPs of asymptotically nonexpansive and pseudocontractive mappings, and variational inequality problems for strict pseudocontractive mappings was also given in Banach spaces. We also provided two examples to support the main results of this paper.

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