

Article

# Similarity Measures of q-Rung Orthopair Fuzzy Sets Based on Cosine Function and Their Applications

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**Abstract:** In this article, we propose another form of ten similarity measures by considering the function of membership degree, non-membership degree, and indeterminacy membership degree between the q-ROFSs on the basis of the traditional cosine similarity measures and cotangent similarity measures. Then, we utilize our presented ten similarity measures and ten weighted similarity measures between q-ROFSs to deal with multiple attribute decision-making (MADM) problems including pattern recognition and scheme selection. Finally, two numerical examples are provided to illustrate the scientific and effective of the similarity measures for pattern recognition and scheme selection.

**Keywords:** multiple attribute decision-making (MADM); q-rung orthopair fuzzy sets (q-ROFSs); cosine function; cosine similarity measure; pattern recognition; scheme selection

## 1. Introduction

As an important branch of multiple attribute decision-making (MADM) domains, the similarity measures have been regarded as very useful tools to determine the degree of similarity between two objects. In the previous research literature, an increasing number of researchers pay attention to similarity measures between fuzzy sets (FSs) due to their broad applications in a variety of fields, for instance, pattern recognition, scheme selection, machine learning, image processing, and decision-making, many theories and applications of similarity measures between fuzzy sets (FSs) have been presented and investigated for the past few years. Atanassov [1,2] presented the definition of intuitionistic fuzzy set (IFS), which is an extension form of fuzzy set (FS). Each element contained in IFS was depicted by an ordered pair including the degree of membership  $\mu$  and non-membership  $\nu$ , and the sum of them is limited to 1. Since IFS theory was proposed, a variety of similarity measures between intuitionistic fuzzy sets (IFSs) have been studied in the document [3–6]. Based on IFS and theories of similarity measures, Li and Cheng [7] presented appropriate similarity measure and gave a numerical example of pattern recognition problems to illustrate the effective of this method. Besides, Mitchell [8] improved Li and Cheng's similarity measures to deal with MADM. According to the extension of the Hamming distance (HD) of fuzzy sets (FSs), Park et al. [9] computed the distance between IFSs based on Hamming distance (HD) and proposed some similarity measures to solve MADM problems [10]. According to the Hausdorff distance, Torra and Narukawa [11] defined some new similarity measures between IFSs. Based on geometric aggregation operators, Xia and Xu [12] proposed the intuitionistic fuzzy geometric distance and intuitionistic fuzzy similarity measures to deal with MADM problems. Ye [13] initially developed the intuitionistic fuzzy cosine similarity measure based on cosine function. Kuo-ChenHung [14] defined the likelihood-based measurement of

IFSs for the medical diagnosis and bacteria classification problems. Shi and Ye [15] further modified the cosine similarity measure of IFSs. Based on the cotangent function, Tian [16] presented the intuitionistic fuzzy cotangent similarity measure between IFSs for medical diagnosis. To contain more fuzzy information, Rajarajeswari and Uma [17] further defined the cotangent similarity measure which considered the function of membership degree, non-membership degree, and indeterminacy membership degrees described in IFSs. In addition, Szmidt [18] introduced distances between IFSs and introduced a family of similarity measures which considered the function of membership degree, non-membership degree, and indeterminacy membership degree in IFSs. Ye [19] developed two new cosine similarity measures and weighted cosine similarity measures based on cosine function and the fuzzy information denoted by the function of membership degree, non-membership degree, and indeterminacy membership degree described in intuitionistic fuzzy sets (IFSs). Wei [20] proposed some picture fuzzy similarity measures and applied them in MADM problems. Le Hoang and Pham Hong [21] defined the intuitionistic vector similarity measures for medical diagnosis. Wei and Wei [22] introduced some Pythagorean fuzzy similarity measures based on cosine function and applied them in pattern recognition and medical diagnosis.

More recently years, Pythagorean fuzzy set (PFS) [23] has emerged to describe the indeterminacy and complexity of the evaluation information. Similar to IFS, the PFS also consisted of the function of membership  $\mu$  and non-membership  $\nu$ ; the sum of squares of  $\mu$  and  $\nu$  is restricted to 1, thus it is clear that the PFS is more widespread than the IFS and can express more decision-making information. For instance, the membership is given as 0.6 and the non-membership is given as 0.8, therefore it is obvious that this problem is only valid for PFS. In other words, all the intuitionistic fuzzy decision-making problems are the special case of Pythagorean fuzzy decision-making problems, which means that PFS can more efficiently deal with MADM problems. In previous literatures, some researching works have been studied by a large amount of investigators [24–28]. Zhang and Xu [29] defined the Pythagorean fuzzy TOPSIS model to deal with the MADM problems. Peng and Yang [30] primarily proposed two Pythagorean fuzzy operations including the division and subtraction operations to better understand PFS. Reformat and Yager [31] handled the collaborative-based recommender system with Pythagorean fuzzy information. Garg [32] defined some new Pythagorean fuzzy aggregation operators including Pythagorean fuzzy Einstein weighted averaging (PFEWA) operator, Pythagorean fuzzy Einstein ordered weighted averaging (PFEOWA) operator, generalized Pythagorean fuzzy Einstein weighted averaging (GPFEWA) operator, and generalized Pythagorean fuzzy Einstein ordered weighted averaging (GPFEOWA) operator. Zeng, et al. [33] utilized the Pythagorean fuzzy ordered weighted averaging weighted average distance (PFOAWAD) operator to study Pythagorean fuzzy MADM issues. Ren, et al. [34] built the Pythagorean fuzzy TODIM model. Wei and Lu [35] developed some new Maclaurin symmetric mean (MSM) [36] operator based on Pythagorean fuzzy environment. Wei and Wei [22] defined ten cosine similarity measures under Pythagorean fuzzy environment. Liang, et al. [37] investigated some Bonferroni mean operators with Pythagorean fuzzy information. Liang, et al. [38] presented Pythagorean fuzzy Bonferroni mean aggregation operators based on geometric averaging (GA) operations. Combined the PFSs [39–41] and dual hesitant fuzzy sets (DHFSs) [42], Zhao et al. [43] introduced the definition of the dual hesitant Pythagorean fuzzy sets (DHPFSs) and proposed some dual hesitant Pythagorean fuzzy Hamacher aggregation operators.

In spite of this, to express more decision information, Yager [44] initially defined the  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs), in which the sum of the  $q$ th power of the membership and non-membership is less or equal to 1, that is to say,  $\mu^q + \nu^q \leq 1$ . Obviously,  $q$ -ROFS are more general for the IFS, and PFSs are special issues of it. Liu and Wang [45] developed the  $q$ -rung orthopair fuzzy weighted averaging ( $q$ -ROFWA) operator and the  $q$ -rung orthopair fuzzy weighted geometric ( $q$ -ROFWG) operator. Wei, et al. [46] proposed some  $q$ -rung orthopair fuzzy MSM operators, including  $q$ -rung orthopair fuzzy MSM ( $q$ -ROFMSM) operator,  $q$ -rung orthopair fuzzy weighted MSM ( $q$ -ROFWMSM) operator,  $q$ -rung orthopair fuzzy dual MSM ( $q$ -ROFDMSM) operator, and

q-rung orthopair fuzzy weighted DMSM (q-ROFWDMSM) operator. Wei, et al. [47] defined some q-rung orthopair fuzzy Heronian mean operators. Yang and Pang [48] presented some new partitioned Bonferroni mean operators under q-rung orthopair fuzzy environment. Liu and Liu [49] provided some power Bonferroni mean operators with linguistic q-rung orthopair fuzzy information. Xu, et al. [50], given the concept of q-rung dual hesitant orthopair fuzzy set (q-RDHOFs), proposed some q-rung dual hesitant orthopair fuzzy Heronian mean operators. Lei and Xu [51] gave some methods for MAGDM with q-rung interval-valued orthopair fuzzy information for green supplier selection.

Although the intuitionistic fuzzy set (IFS) [1,2] and Pythagorean fuzzy set (PFS) [23,39] have been applied in some decision-making areas, for some special cases, such as when the membership degree and non-membership degree are given as 0.7 and 0.8, it is clear that both IFS and PFS theories cannot satisfy this situation. The q-rung orthopair fuzzy set (q-ROFS) is also denoted by the degree of membership and non-membership whose *q*-th power sum of them is restricted to 1. Obviously, the q-ROFS is more general than the q-ROFS and can express more fuzzy information. In other words, the q-ROFS can deal with the MADM problems which IFS cannot, and it is clear that IFS is a part of the q-ROFS, which indicates q-ROFS can be more effective and powerful to deal with fuzzy and uncertain decision-making problems. Thus, to solve such issues, based on the cosine functions and cotangent functions, we shall propose the concept of q-rung orthopair fuzzy cosine similarity measures and q-rung orthopair fuzzy cotangent similarity measures under q-rung orthopair fuzzy environment in this paper, which is a new extension of the similarity measure of IFSs.

To do this, the rest of this article is structured as follows. In the next section, we briefly review some fundamental theories of intuitionistic fuzzy set (IFS) and some intuitionistic fuzzy similarity measures. Some q-rung orthopair fuzzy cosine similarity measures, q-rung orthopair fuzzy weighted cosine similarity measures, q-rung orthopair fuzzy cotangent similarity measures, and q-rung orthopair fuzzy weighted cotangent similarity measures are developed in Section 3. All the above-mentioned similarity measures for q-ROFSs are used to pattern recognition and scheme selection in Section 4. Section 5 concludes the paper with some remarks.

## 2. Preliminaries

In this part, we shall briefly introduce some basic theories of intuitionistic fuzzy sets (IFSs) and review some similarity measures based on cosine functions and cotangent functions between IFSs.

**Definition 1.** Suppose that  $X$  is a fixed set, then an intuitionistic fuzzy set (IFS)  $Q$  in  $X$  [1,2] can be denoted as

$$Q = \{ \langle x, \alpha_Q(x), \beta_Q(x) \rangle | x \in X \} \tag{1}$$

where  $\alpha_Q : X \rightarrow [0, 1]$  means the degree of membership and  $\beta_Q(x) : X \rightarrow [0, 1]$  means the degree of non-membership which satisfies the condition of  $0 \leq \alpha_Q(x) \leq 1, 0 \leq \beta_Q(x) \leq 1, 0 \leq \alpha_Q(x) + \beta_Q(x) \leq 1, \forall x \in X$ .

**Definition 2.** For each intuitionistic fuzzy set (IFS)  $Q$  in  $X$  [1,2], the degree of indeterminacy membership  $\pi_Q(x)$  can be expressed as

$$\pi_Q(x) = 1 - \alpha_Q(x) - \beta_Q(x), \forall x \in X. \tag{2}$$

The cosine similarity measures and cotangent similarity measures, which can calculate the degree of proximity between any two schemes, have been applied in many practical MADM problems. As we all know, the cosine and cotangent functions are monotone decreasing functions, thus, by considering the distance measures between any two alternatives, the bigger the distance values are, the smaller the calculating results by cosine and cotangent functions are and the lower similarity measures are. Therefore, to select best alternatives in decision-making problems, we always utilize cosine and cotangent similarity measures to obtain the similarity degree between each alternative and the ideal

alternative. In what follows, we will briefly review some intuitionistic fuzzy cosine and cotangent similarity measures.

Let  $M = \{ \langle x_j, \alpha_M(x_j), \beta_M(x_j) \rangle | x_j \in X \}$  and  $N = \{ \langle x_j, \alpha_N(x_j), \beta_N(x_j) \rangle | x_j \in X \}$  be two intuitionistic fuzzy sets (IFSs), then the intuitionistic fuzzy cosine (IFC) measure between  $M$  and  $N$  proposed by Ye [13] can be shown as

$$IFC^1(M, N) = \frac{1}{n} \sum_{j=1}^n \frac{\alpha_M(x_j)\alpha_N(x_j) + \beta_M(x_j)\beta_N(x_j)}{\sqrt{\alpha_M^2(x_j) + \beta_M^2(x_j)} \sqrt{\alpha_N^2(x_j) + \beta_N^2(x_j)}} \tag{3}$$

Consider the degree of membership, non-membership and indeterminacy membership, then the intuitionistic fuzzy cosine (IFC) measure between  $M$  and  $N$  proposed by Shi and Ye [15] can be shown as

$$IFC^2(M, N) = \frac{1}{n} \sum_{j=1}^n \frac{\alpha_M(x_j)\alpha_N(x_j) + \beta_M(x_j)\beta_N(x_j) + \pi_M(x_j)\pi_N(x_j)}{\sqrt{\alpha_M^2(x_j) + \beta_M^2(x_j) + \pi_M^2(x_j)} \sqrt{\alpha_N^2(x_j) + \beta_N^2(x_j) + \pi_N^2(x_j)}} \tag{4}$$

On account of cosine function, Ye [19] developed two intuitionistic fuzzy cosine similarity (IFCS) measures between two intuitionistic fuzzy sets (IFSs)  $M$  and  $N$ .

$$IFCS^1(M, N) = \frac{1}{n} \sum_{j=1}^n \cos \left[ \frac{\pi}{2} \max \left( \left| \alpha_M(x_j) - \alpha_N(x_j) \right|, \left| \beta_M(x_j) - \beta_N(x_j) \right|, \left| \pi_M(x_j) - \pi_N(x_j) \right| \right) \right] \tag{5}$$

$$IFCS^2(M, N) = \frac{1}{n} \sum_{i=1}^n \cos \left[ \frac{\pi}{4} \left( \left| \alpha_M(x_j) - \alpha_N(x_j) \right| + \left| \beta_M(x_j) - \beta_N(x_j) \right| + \left| \pi_M(x_j) - \pi_N(x_j) \right| \right) \right] \tag{6}$$

In addition, the intuitionistic fuzzy cotangent (IFCot) similarity measure between any two intuitionistic fuzzy sets (IFSs)  $M$  and  $N$  proposed by Tian [16] is shown as

$$IFCot^1(M, N) = \frac{1}{n} \sum_{j=1}^n \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \max \left( \left| \alpha_M(x_j) - \alpha_N(x_j) \right|, \left| \beta_M(x_j) - \beta_N(x_j) \right| \right) \right] \tag{7}$$

Consider the degree of membership, non-membership, and indeterminacy membership, then the intuitionistic fuzzy cotangent (IFCot) similarity measure between any two intuitionistic fuzzy sets (IFSs)  $M$  and  $N$  proposed by Rajarajeswari and Uma [17] can be shown as

$$IFCot^2(M, N) = \frac{1}{n} \sum_{j=1}^n \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \max \left( \left| \alpha_M(x_j) - \alpha_N(x_j) \right|, \left| \beta_M(x_j) - \beta_N(x_j) \right|, \left| \pi_M(x_j) - \pi_N(x_j) \right| \right) \right] \tag{8}$$

Consider the weighting vector of the elements in IFS, the weighted intuitionistic fuzzy cosine (WIFC) measure, the weighted intuitionistic fuzzy cosine similarity (WIFCS) measure, and weighted

intuitionistic fuzzy cotangent (*WIFCot*) similarity measure between any two intuitionistic fuzzy sets (IFSs), *M* and *N* can be shown as follows [13,15–17,19]

$$WIFC^1(M, N) = \sum_{j=1}^n \omega_j \left[ \frac{\alpha_M(x_j)\alpha_N(x_j) + \beta_M(x_j)\beta_N(x_j)}{\sqrt{\alpha_M^2(x_j) + \beta_M^2(x_j)} \sqrt{\alpha_N^2(x_j) + \beta_N^2(x_j)}} \right] \tag{9}$$

$$WIFC^2(M, N) = \sum_{j=1}^n \omega_j \frac{\alpha_M(x_j)\alpha_N(x_j) + \beta_M(x_j)\beta_N(x_j) + \pi_M(x_j)\pi_N(x_j)}{\sqrt{\alpha_M^2(x_j) + \beta_M^2(x_j) + \pi_M^2(x_j)} \sqrt{\alpha_N^2(x_j) + \beta_N^2(x_j) + \pi_N^2(x_j)}} \tag{10}$$

$$WIFCS^1(M, N) = \sum_{j=1}^n \omega_j \cos \left[ \frac{\pi}{2} \max \left( \begin{matrix} |\alpha_M(x_j) - \alpha_N(x_j)|, \\ |\beta_M(x_j) - \beta_N(x_j)|, \\ |\pi_M(x_j) - \pi_N(x_j)| \end{matrix} \right) \right] \tag{11}$$

$$WIFCS^2(M, N) = \sum_{i=1}^n \omega_j \cos \left[ \frac{\pi}{4} \left( \begin{matrix} |\alpha_M(x_j) - \alpha_N(x_j)| + \\ |\beta_M(x_j) - \beta_N(x_j)| + \\ |\pi_M(x_j) - \pi_N(x_j)| \end{matrix} \right) \right] \tag{12}$$

$$WIFCot^1(M, N) = \sum_{j=1}^n \omega_j \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \max \left( \begin{matrix} |\alpha_M(x_j) - \alpha_N(x_j)|, \\ |\beta_M(x_j) - \beta_N(x_j)| \end{matrix} \right) \right] \tag{13}$$

$$WIFCot^2(M, N) = \sum_{j=1}^n \omega_j \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \max \left( \begin{matrix} |\alpha_M(x_j) - \alpha_N(x_j)|, \\ |\beta_M(x_j) - \beta_N(x_j)|, \\ |\pi_M(x_j) - \pi_N(x_j)| \end{matrix} \right) \right] \tag{14}$$

where  $\omega_j (j = 1, 2, \dots, n)$  denotes the weighting vector of elements  $x_j$ , which satisfies the condition of  $\omega_j \in [0, 1]$  and  $\sum_{j=1}^n \omega_j = 1$ .

### 3. Some Similarity Measures Based on Cosine Function for q-ROFSs

Although the intuitionistic fuzzy sets (IFSs) defined by Atanassov’s [1,2] have been broadly applied in different areas, for some special cases, such as when membership degree and non-membership degree are given as 0.7 and 0.8, it is clear that IFSs theory cannot satisfy this situation. The q-rung orthopair fuzzy set (q-ROFS) is also denoted by the degree of membership and non-membership, whose *q*-th power sum is restricted to 1, obviously, the q-ROFS is more general than the q-ROFS and can express more fuzzy information. In other words, the q-ROFS can deal with the MADM problems which IFS cannot and it is clear that IFS is a part of the q-ROFS, which indicates q-ROFS can be more effective and powerful to deal with fuzzy and uncertain decision-making problems.

**Definition 3.** Suppose *P* be a fix set, then a q-rung orthopair fuzzy set (q-ROFS) *P* in *X* [39,40] can be denoted as

$$P = \{ \langle x, (\alpha_P(x), \beta_P(x)) \rangle | x \in X \} \tag{15}$$

where  $\alpha_P : X \rightarrow [0, 1]$  means the degree of membership and  $\beta_P(x) : X \rightarrow [0, 1]$  means the degree of non-membership which satisfies the condition of  $0 \leq \alpha_P(x) \leq 1, 0 \leq \beta_P(x) \leq 1, 0 \leq (\alpha_P(x))^q + (\beta_P(x))^q \leq 1, q \geq 1, \forall x \in X$ .

**Definition 4.** For each  $q$ -rung orthopair fuzzy set ( $q$ -ROFS)  $P$  in  $X$  [39,40], the degree of indeterminacy membership  $\pi_P(x)$  can be expressed as

$$\pi_P(x) = \sqrt[q]{(\alpha_P(x))^q + (\beta_P(x))^q - (\alpha_P(x))^q(\beta_P(x))^q}, \forall x \in X. \tag{16}$$

**Definition 5.** Let  $p = (\alpha, \beta)$  be a  $q$ -ROFN, a score function can be represented [40] as follows

$$S(p) = \frac{1}{2}(1 + \alpha^q - \beta^q), S(p) \in [0, 1]. \tag{17}$$

**Definition 6.** Let  $r_j = (\alpha_j, \beta_j)(j = 1, 2, \dots, n)$  be a group of  $q$ -ROFNs with weighting vector  $w = (w_1, w_2, \dots, w_n)^T$ , which satisfies  $w_j > 0, i = 1, 2, \dots, n$  and  $\sum_{j=1}^n w_j = 1$  [40]. Then we can obtain the  $q$ -rung orthopair fuzzy weighted averaging ( $q$ -ROFWA) operator and the  $q$ -rung orthopair fuzzy weighted geometric ( $q$ -ROFWG) operator as follows

$$q\text{-ROFWA}(r_1, r_2, \dots, r_n) = \bigoplus_{j=1}^n w_j r_j = \left\langle \left(1 - \prod_{j=1}^n (1 - \alpha_j^q)^{w_j}\right)^{1/q}, \prod_{j=1}^n \beta_j^{w_j} \right\rangle \tag{18}$$

$$q\text{-ROFWG}(r_1, r_2, \dots, r_n) = \bigotimes_{j=1}^n (r_j)^{w_j} = \left\langle \prod_{j=1}^n \alpha_j^{w_j}, \left(1 - \prod_{j=1}^n (1 - \beta_j^q)^{w_j}\right)^{1/q} \right\rangle \tag{19}$$

### 3.1. Cosine Similarity Measure for $q$ -ROFSs

Suppose that  $P$  is a  $q$ -rung orthopair fuzzy set ( $q$ -ROFS) in a universe of discourse  $X = \{x\}$ , the elements contained in  $q$ -ROFS can be expressed as the function of membership degree  $\alpha_P(x)$ , the function of non-membership degree  $\beta_P(x)$ , and the function of indeterminacy membership degree  $\pi_P(x)$ . Thus, a cosine similarity measure and a weighted cosine similarity measure with  $q$ -rung orthopair fuzzy information are presented in an analogous manner to the cosine similarity measure based on Bhattacharyya's distance and cosine similarity measure for intuitionistic fuzzy set (IFS) [13].

Let  $M = \{(x_j, \alpha_M(x_j), \beta_M(x_j)) | x_j \in X\}$  and  $N = \{(x_j, \alpha_N(x_j), \beta_N(x_j)) | x_j \in X\}$  be two  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs), then the  $q$ -rung orthopair fuzzy cosine ( $q$ -ROFC) measure between  $M$  and  $N$  can be shown as

$$q\text{-ROFC}^1(M, N) = \frac{1}{n} \sum_{j=1}^n \frac{\alpha_M^q(x_j)\alpha_N^q(x_j) + \beta_M^q(x_j)\beta_N^q(x_j)}{\sqrt{(\alpha_M^q(x_j))^2 + (\beta_M^q(x_j))^2} \sqrt{(\alpha_N^q(x_j))^2 + (\beta_N^q(x_j))^2}} \tag{20}$$

Especially, when we let  $n = 1$ , the cosine similarity measure between  $q$ -ROFSs  $M$  and  $N$  can be depicted as  $C_{q\text{-ROFS}}(M, N)$ , which will become the correlation coefficient between  $M$  and  $N$ , which is depicted as  $K_{q\text{-ROFS}}(M, N)$ , i.e.,  $C_{q\text{-ROFS}}(M, N) = K_{q\text{-ROFS}}(M, N)$ . In addition, the cosine similarity measure between  $q$ -ROFSs  $M$  and  $N$  also satisfies some properties as follows.

- (1)  $0 \leq q\text{-ROFC}^1(M, N) \leq 1$ ;
- (2)  $q\text{-ROFC}^1(M, N) = q\text{-ROFC}^1(N, M)$ ;
- (3)  $q\text{-ROFC}^1(M, N) = 1$ , if  $M = N, j = 1, 2, \dots, n$ .

**Proof.**

- (1) It is clear that the proposition is true based on the cosine result.

- (2) It is clear that the proposition is true.
- (3) When  $M = N$ , it means that  $\alpha_M(x_j) = \alpha_N(x_j)$  and  $\beta_M(x_j) = \beta_N(x_j)$  for  $j = 1, 2, \dots, n$ . So  $C_{q-ROFS}^1(M, N) = 1$ .

Therefore, we have finished the proofs.  $\square$

In what follows, we shall study the distance measure of the angle as  $d(M, N) = \arccos(C_{q-ROFS}^1(M, N))$ . It satisfies some properties as follows.

- (1)  $d(M, N) \geq 0$ , if  $0 \leq C_{q-ROFS}^1(M, N) \leq 1$ ;
- (2)  $d(M, N) = \arccos(1) = 0$ , if  $C_{q-ROFS}^1(M, N) = 1$ ;
- (3)  $d(M, N) = d(N, M)$ , if  $C_{q-ROFS}^1(M, N) = C_{q-ROFS}^1(N, M)$ ,
- (4)  $d(M, T) \leq d(M, N) + d(N, T)$ , if  $M \subseteq N \subseteq T$  for any q-ROFS  $T$ .

**Proof.** Clearly the distance measure  $d(M, N)$  satisfies properties (1)–(3). In what follows we shall prove that the distance measure  $d(M, N)$  satisfies property (4).

For any q-rung orthopair fuzzy set (q-ROFS)  $T = \{ \langle x_j, (\alpha_T(x_j), \beta_T(x_j)) \rangle | x_j \in X \}$ ,  $M \subseteq N \subseteq T$ , let us investigate the distance measures of the angle between the vectors:

$$\begin{aligned} d_j(M(x_j), N(x_j)) &= \arccos(q - ROFC_i^1(M(x_j), N(x_j))) (j = 1, 2, \dots, n) \\ d_j(M(x_j), T(x_j)) &= \arccos(q - ROFC_i^1(M(x_j), T(x_j))) (j = 1, 2, \dots, n) \\ d_j(N(x_j), T(x_j)) &= \arccos(q - ROFC_i^1(N(x_j), T(x_j))) (j = 1, 2, \dots, n) \end{aligned}$$

where

$$\begin{aligned} q - ROFC_j^1(M(x_j), N(x_j)) &= \frac{\alpha_M^q(x_j)\alpha_N^q(x_j) + \beta_M^q(x_j)\beta_N^q(x_j)}{\sqrt{(\alpha_M^q(x_j))^2 + (\beta_M^q(x_j))^2} \sqrt{(\alpha_N^q(x_j))^2 + (\beta_N^q(x_j))^2}} \\ q - ROFC_j^1(M(x_j), T(x_j)) &= \frac{\alpha_M^q(x_j)\alpha_T^q(x_j) + \beta_M^q(x_j)\beta_T^q(x_j)}{\sqrt{(\alpha_M^q(x_j))^2 + (\beta_M^q(x_j))^2} \sqrt{(\alpha_T^q(x_j))^2 + (\beta_T^q(x_j))^2}} \\ q - ROFC_j^1(N(x_j), T(x_j)) &= \frac{\alpha_N^q(x_j)\alpha_T^q(x_j) + \beta_N^q(x_j)\beta_T^q(x_j)}{\sqrt{(\alpha_N^q(x_j))^2 + (\beta_N^q(x_j))^2} \sqrt{(\alpha_T^q(x_j))^2 + (\beta_T^q(x_j))^2}} \end{aligned}$$

$M(x_j) = \langle \alpha_M(x_j), \beta_M(x_j) \rangle$ ,  $N(x_j) = \langle \alpha_N(x_j), \beta_N(x_j) \rangle$ ,  $T(x_j) = \langle \alpha_T(x_j), \beta_T(x_j) \rangle$  are three vectors in one plane, if  $M(x_j) \subseteq N(x_j) \subseteq T(x_j)$ ,  $j = 1, 2, \dots, n$ . Therefore, it is clear that  $d_j(M(x_j), T(x_j)) \leq d_j(M(x_j), N(x_j)) + d_j(N(x_j), T(x_j))$  based on the triangle inequality. Combining the inequality  $0 \leq (\alpha_p(x_j))^q + (\beta_p(x_j))^q \leq 1$ , we can get  $d(M, T) \leq d(M, N) + d(N, T)$ . Therefore  $d(M, N)$  meets the property (4). So we completed the process of proof.  $\square$

If we consider three terms—membership degree, non-membership degree, and indeterminacy membership—which are contained in q-ROFSs, assume that there are two q-rung orthopair fuzzy sets,  $M = \{ \langle x_j, \alpha_M(x_j), \beta_M(x_j), \pi_M(x_j) \rangle | x_j \in X \} (j = 1, 2, \dots, n)$  and  $N = \{ \langle x_j, \alpha_N(x_j), \beta_N(x_j), \pi_N(x_j) \rangle | x_j \in X \} (j = 1, 2, \dots, n)$ , then the q-rung orthopair fuzzy cosine (q-ROFC) measures between q-ROFSs can be expressed as

$$q - ROFC^2(M, N) = \frac{1}{n} \sum_{j=1}^n \left[ \frac{\left( \alpha_M^q(x_j)\alpha_N^q(x_j) + \beta_M^q(x_j)\beta_N^q(x_j) + \pi_M^q(x_j)\pi_N^q(x_j) \right)}{\left( \sqrt{(\alpha_M^q(x_j))^2 + (\beta_M^q(x_j))^2 + (\pi_M^q(x_j))^2} \right) \left( \sqrt{(\alpha_N^q(x_j))^2 + (\beta_N^q(x_j))^2 + (\pi_N^q(x_j))^2} \right)} \right] \tag{21}$$

Especially when we let  $n = 1$ , the cosine similarity measure between  $q$ -ROFSs  $M$  and  $N$  will become the correlation coefficient between  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs)  $M$  and  $N$ . Of course, the cosine similarity measure  $q - ROFC^2(M, N)$  also satisfies some properties which are listed as follows.

- (1)  $0 \leq q - ROFC^2(M, N) \leq 1$ ;
- (2)  $q - ROFC^2(M, N) = q - ROFC^2(N, M)$ ;
- (3)  $q - ROFC^2(M, N) = 1$ , if  $M = N, j = 1, 2, \dots, n$ .

Consider the weighting vector of the elements in  $q$ -ROFS, the  $q$ -rung orthopair fuzzy weighted cosine ( $q$ -ROFWC) measure between two  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs)  $M$  and  $N$  can be shown as follows.

$$q - ROFWC^1(M, N) = \sum_{j=1}^n \omega_j \frac{\alpha_M^q(x_j)\alpha_N^q(x_j) + \beta_M^q(x_j)\beta_N^q(x_j)}{\sqrt{(\alpha_M^q(x_j))^2 + (\beta_M^q(x_j))^2} \sqrt{(\alpha_N^q(x_j))^2 + (\beta_N^q(x_j))^2}} \tag{22}$$

$$q - ROFWC^2(M, N) = \sum_{j=1}^n \omega_j \left[ \frac{\left( \begin{matrix} \alpha_M^q(x_j)\alpha_N^q(x_j) + \beta_M^q(x_j)\beta_N^q(x_j) \\ + \pi_M^q(x_j)\pi_N^q(x_j) \end{matrix} \right)}{\left( \begin{matrix} \sqrt{(\alpha_M^q(x_j))^2 + (\beta_M^q(x_j))^2 + (\pi_M^q(x_j))^2} \\ \times \sqrt{(\alpha_N^q(x_j))^2 + (\beta_N^q(x_j))^2 + (\pi_N^q(x_j))^2} \end{matrix} \right)} \right] \tag{23}$$

where  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  indicates the weighting vector of the elements  $x_j (j = 1, 2, \dots, n)$  contained in  $q$ -ROFS and the weighting vector satisfies  $\omega_j \in [0, 1], j = 1, 2, \dots, n, \sum_{j=1}^n \omega_j = 1$ . Especially, when we let weighting vector be  $\omega = (1/n, 1/n, \dots, 1/n)^T$ , then the weighted cosine similarity measure will reduce to cosine similarity measure. In other words, when  $\omega_j = \frac{1}{n}, j = 1, 2, \dots, n$ , the  $q - ROFWC^1(M, N) = q - ROFC^1(M, N)$ .

**Example 1.** Suppose there are two  $q$ -ROFSs  $M = \{(x_1, 0.7, 0.4), (x_2, 0.5, 0.6), (x_3, 0.3, 0.8)\}$  and  $N = \{(x_1, 0.9, 0.2), (x_2, 0.4, 0.3), (x_3, 0.7, 0.6)\}$ , assume  $q = 3, \omega_j = (0.2, 0.3, 0.5)$  then according to Equation (19), the weighted cosine similarity measure between  $M$  and  $N$  can be calculated as

$$\begin{aligned} q - ROFWC^1(M, N) &= \sum_{j=1}^n \omega_j \frac{\alpha_M^q(x_j)\alpha_N^q(x_j) + \beta_M^q(x_j)\beta_N^q(x_j)}{\sqrt{(\alpha_M^q(x_j))^2 + (\beta_M^q(x_j))^2} \sqrt{(\alpha_N^q(x_j))^2 + (\beta_N^q(x_j))^2}} \\ &= \left( \begin{matrix} \frac{0.2 \times (0.7^3 \times 0.9^3 + 0.4^3 \times 0.2^3)}{\sqrt{(0.7^3)^2 + (0.4^3)^2} \times \sqrt{(0.9^3)^2 + (0.2^3)^2}} + \frac{0.3 \times (0.5^3 \times 0.4^3 + 0.6^3 \times 0.3^3)}{\sqrt{(0.5^3)^2 + (0.6^3)^2} \times \sqrt{(0.4^3)^2 + (0.3^3)^2}} \\ + \frac{0.5 \times (0.3^3 \times 0.7^3 + 0.8^3 \times 0.6^3)}{\sqrt{(0.3^3)^2 + (0.8^3)^2} \times \sqrt{(0.7^3)^2 + (0.6^3)^2}} \end{matrix} \right) \\ &= 0.7247 \end{aligned}$$



**Example 2.** Suppose there are two  $q$ -ROFSs  $M = \{(x_1, 0.7, 0.4), (x_2, 0.5, 0.6), (x_3, 0.3, 0.8)\}$  and  $N = \{(x_1, 0.9, 0.2), (x_2, 0.4, 0.3), (x_3, 0.7, 0.6)\}$ , assume  $q = 3, \omega_j = (0.2, 0.3, 0.5)$  then according to Equation (3) and Equation (20), the weighted cosine similarity measure between  $M$  and  $N$  can be calculated as

$$\begin{aligned}
 q - ROFWC^2(M, N) &= \sum_{j=1}^n \omega_j \left[ \frac{\left( \frac{\alpha_M^q(x_j)\alpha_N^q(x_j) + \beta_M^q(x_j)\beta_N^q(x_j)}{\pi_M^q(x_j)\pi_N^q(x_j)} \right)}{\left( \frac{\sqrt{(\alpha_M^q(x_j))^2 + (\beta_M^q(x_j))^2 + (\pi_M^q(x_j))^2}}{\sqrt{(\alpha_N^q(x_j))^2 + (\beta_N^q(x_j))^2 + (\pi_N^q(x_j))^2}} \right)} \right] \\
 &= \left[ \begin{aligned}
 &0.2 \times \left( \frac{0.7^3 \times 0.9^3 + 0.4^3 \times 0.2^3 + 0.7^3 \times 0.9^3}{\sqrt{(0.7^3)^2 + (0.4^3)^2 + (0.7^3)^2} \times \sqrt{(0.9^3)^2 + (0.2^3)^2 + (0.9^3)^2}} \right) \\
 &+ 0.3 \times \left( \frac{0.5^3 \times 0.4^3 + 0.6^3 \times 0.3^3 + 0.7^3 \times 0.4^3}{\sqrt{(0.5^3)^2 + (0.6^3)^2 + (0.7^3)^2} \times \sqrt{(0.4^3)^2 + (0.3^3)^2 + (0.4^3)^2}} \right) \\
 &+ 0.5 \times \left( \frac{0.3^3 \times 0.7^3 + 0.8^3 \times 0.6^3 + 0.8^3 \times 0.8^3}{\sqrt{(0.3^3)^2 + (0.8^3)^2 + (0.8^3)^2} \times \sqrt{(0.7^3)^2 + (0.6^3)^2 + (0.8^3)^2}} \right)
 \end{aligned} \right] \\
 &= 0.8789
 \end{aligned}$$

Evidently, similar to cosine similarity measure  $q - ROFC^1(M, N)$ , the weighted cosine similarity measure  $q - ROFWC^1(M, N)$  also meets three properties as follows.

- (1)  $0 \leq q - ROFWC^1(M, N) \leq 1$ ,
- (2)  $q - ROFWC^1(M, N) = q - ROFWC^1(N, M)$ ,
- (3)  $q - ROFWC^1(M, N) = 1$ , if  $M = N, i = 1, 2, \dots, n$ .

### 3.2. Similarity Measures of $q$ -ROFSs Based on Cosine Function

In this section, according to the cosine function, we will present some  $q$ -rung orthopair fuzzy cosine similarity measures ( $q$ -ROFCS) between  $q$ -ROFSs and discuss their properties.

**Definition 7.** Assume that there are any two  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs)  $M = \{(x_j, (\alpha_M(x_j), \beta_M(x_j))) | x_j \in X\}$  and  $N = \{(x_j, (\alpha_N(x_j), \beta_N(x_j))) | x_j \in X\}$ . Then, we shall propose two  $q$ -rung orthopair fuzzy cosine similarity ( $q$ -ROFCS) measures between  $q$ -ROFSs  $M$  and  $N$  as follows

$$q - ROFCS^1(M, N) = \frac{1}{n} \sum_{j=1}^n \cos \left[ \frac{\pi}{2} \left( \max \left( \left| \frac{\alpha_M^q(x_j) - \alpha_N^q(x_j)}{\beta_M^q(x_j) - \beta_N^q(x_j)} \right|, \dots \right) \right) \right] \tag{24}$$

$$q - ROFCS^2(M, N) = \frac{1}{n} \sum_{j=1}^n \cos \left[ \frac{\pi}{4} \left( \left| \frac{\alpha_M^q(x_j) - \alpha_N^q(x_j)}{\beta_M^q(x_j) - \beta_N^q(x_j)} \right| + \dots \right) \right] \tag{25}$$

**Proposition 1.** Assume that there are any two  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs)  $M$  and  $N$  in  $X = \{x_1, x_2, \dots, x_n\}$ , the  $q$ -rung orthopair fuzzy cosine similarity measures  $q - ROFCS^k(M, N) (k = 1, 2)$  should satisfy the properties (1)–(4):

- (1)  $0 \leq q - ROFCS^k(M, N) \leq 1$ ;
- (2)  $q - ROFCS^k(M, N) = 1$  if and only if  $M = N$ ;
- (3)  $q - ROFCS^k(M, N) = q - ROFCS^k(N, M)$ ;

(4) Let  $M, N, T$  be three  $q$ -ROFSs in  $X$  and  $M \subseteq N \subseteq T$ , then  $q - ROFCS^k(M, T) \leq q - ROFCS^k(M, N)$ ,  $q - ROFCS^k(M, T) \leq q - ROFCS^k(N, T)$ .

**Proof.** (1) Since the calculated results based on the cosine function are within  $[0, 1]$ , the  $q$ -rung orthopair fuzzy cosine similarity measures based on the cosine function are also within  $[0, 1]$ . Thus  $0 \leq q - ROFCS^k(M, N) \leq 1, k = 1, 2$ .

(2) For two  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs)  $M$  and  $N$  in  $X = \{x_1, x_2, \dots, x_n\}$ , if  $M = N$ , then  $\alpha_M^q(x_j) = \alpha_N^q(x_j), \beta_M^q(x_j) = \beta_N^q(x_j), j = 1, 2, \dots, n$ . Thus,  $|\alpha_M^q(x_j) - \alpha_N^q(x_j)| = 0, |\beta_M^q(x_j) - \beta_N^q(x_j)| = 0$ . So,  $q - ROFCS^k(M, N) = 1, k = 1, 2$ . If  $q - ROFCS^k(M, N) = 1, k = 1, 2$ , it implies  $|\alpha_M^q(x_j) - \alpha_N^q(x_j)| = 0, j = 1, 2, \dots, n, |\beta_M^q(x_j) - \beta_N^q(x_j)| = 0, j = 1, 2, \dots, n$ . Since  $\cos(0) = 1$ . Then, there are  $\alpha_M^q(x_j) = \alpha_N^q(x_j), \beta_M^q(x_j) = \beta_N^q(x_j), j = 1, 2, \dots, n$ . Hence  $M = N$ .

(3) Proof is straightforward.

(4) If  $M \subseteq N \subseteq T$ , that means  $\alpha_M(x_j) \leq \alpha_N(x_j) \leq \alpha_T(x_j), \beta_M(x_j) \geq \beta_N(x_j) \geq \beta_T(x_j)$ , for  $j = 1, 2, \dots, n$ . Then  $\alpha_M^q(x_j) \leq \alpha_N^q(x_j) \leq \alpha_T^q(x_j), \beta_M^q(x_j) \geq \beta_N^q(x_j) \geq \beta_T^q(x_j)$ . Thus, we have

$$\begin{aligned} |\alpha_M^q(x_j) - \alpha_N^q(x_j)| &\leq |\alpha_M^q(x_j) - \alpha_T^q(x_j)|, |\alpha_N^q(x_j) - \alpha_T^q(x_j)| \leq |\alpha_M^q(x_j) - \alpha_T^q(x_j)|, \\ |\beta_M^q(x_j) - \beta_N^q(x_j)| &\leq |\beta_M^q(x_j) - \beta_T^q(x_j)|, |\beta_N^q(x_j) - \beta_T^q(x_j)| \leq |\beta_M^q(x_j) - \beta_T^q(x_j)|. \end{aligned}$$

Thus  $q - ROFCS^k(M, T) \leq q - ROFCS^k(M, N), q - ROFCS^k(M, T) \leq q - ROFCS^k(M, N)$ , as the cosine function is a decreasing function with the interval  $[0, \pi/2]$ . Then, we finished the process of proofs.  $\square$

If we consider three terms including membership degree, non-membership degree, and indeterminacy membership, which are contained in  $q$ -ROFSs, assume that there are two  $q$ -rung orthopair fuzzy sets  $M = \{(x_j, \alpha_M(x_j), \beta_M(x_j), \pi_M(x_j)) | x_j \in X\} (j = 1, 2, \dots, n)$  and  $N = \{(x_j, \alpha_N(x_j), \beta_N(x_j), \pi_N(x_j)) | x_j \in X\} (j = 1, 2, \dots, n)$ , then the  $q$ -rung orthopair fuzzy cosine similarity ( $q$ -ROFCS) measures between  $M$  and  $N$  can be expressed as

$$q - ROFCS^3(M, N) = \frac{1}{n} \sum_{j=1}^n \cos \left[ \frac{\pi}{2} \left( \max \left( \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right|, \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right|, \left| \pi_M^q(x_j) - \pi_N^q(x_j) \right| \right) \right) \right] \tag{26}$$

where  $q - ROFCS^3(M, N)$  means the  $q$ -rung orthopair fuzzy cosine similarity measures between  $M$  and  $N$ , which consider the maximum distance based on the membership, indeterminacy membership, and non-membership degree.

$$q - ROFCS^4(M, N) = \frac{1}{n} \sum_{j=1}^n \cos \left[ \frac{\pi}{4} \left( \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right| + \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| + \left| \pi_M^q(x_j) - \pi_N^q(x_j) \right| \right) \right] \tag{27}$$

where  $q - ROFCS^4(M, N)$  means the  $q$ -rung orthopair fuzzy cosine similarity measures between  $M$  and  $N$ , which consider the sum of distance based on the membership, indeterminacy membership, and non-membership degree.

Consider the weighting vector of the elements in q-ROFS, the q-rung orthopair fuzzy weighted cosine similarity (q-ROFWCS) measure between two q-rung orthopair fuzzy sets (q-ROFSs)  $M$  and  $N$  can be shown as follows.

$$q-ROFWCS^1(M, N) = \sum_{j=1}^n \omega_j \cos \left[ \frac{\pi}{2} \left( \max \left( \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right|, \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| \right) \right) \right] \tag{28}$$

where  $q-ROFWCS^1(M, N)$  means the q-rung orthopair fuzzy weighted cosine similarity measures between  $M$  and  $N$ , which consider the maximum distance based on the membership and non-membership degree.

$$q-ROFWCS^2(M, N) = \sum_{j=1}^n \omega_j \cos \left[ \frac{\pi}{4} \left( \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right| + \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| \right) \right] \tag{29}$$

where  $q-ROFWCS^2(M, N)$  means the q-rung orthopair fuzzy weighted cosine similarity measures between  $M$  and  $N$ , which consider the sum of distance based on the membership and non-membership degree.

$$q-ROFWCS^3(M, N) = \sum_{j=1}^n \omega_j \cos \left[ \frac{\pi}{2} \left( \max \left( \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right|, \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right|, \left| \pi_A^2(x_j) - \pi_B^2(x_j) \right| \right) \right) \right] \tag{30}$$

where  $q-ROFWCS^3(M, N)$  means the q-rung orthopair fuzzy weighted cosine similarity measures between  $M$  and  $N$ , which consider the maximum distance based on the membership, indeterminacy membership, and non-membership degree.

$$q-ROFWCS^4(M, N) = \sum_{j=1}^n \omega_j \cos \left[ \frac{\pi}{4} \left( \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right| + \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| + \left| \pi_A^2(x_j) - \pi_B^2(x_j) \right| \right) \right] \tag{31}$$

where  $q-ROFWCS^4(M, N)$  means the q-rung orthopair fuzzy weighted cosine similarity measures between  $M$  and  $N$ , which consider the sum of distance based on the membership, indeterminacy membership, and non-membership degree.

where  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  indicates the weighting vector of the elements  $x_j (j = 1, 2, \dots, n)$  contained in q-ROFS, and the weighting vector satisfies  $\omega_j \in [0, 1], j = 1, 2, \dots, n, \sum_{j=1}^n \omega_j = 1$ . Especially, when we let weighting vector be  $\omega = (1/n, 1/n, \dots, 1/n)^T$ , then the weighted cosine similarity measure will reduce to cosine similarity measure. In other words, when  $\omega_j = \frac{1}{n}, j = 1, 2, \dots, n$ , the  $q-ROFWCS^k(M, N) = q-ROFCS^k(M, N) (k = 1, 2, 3, 4)$ .

**Example 3.** Suppose there are two q-ROFSs,  $M = \{(x_1, 0.7, 0.4), (x_2, 0.5, 0.6), (x_3, 0.3, 0.8)\}$  and  $N = \{(x_1, 0.9, 0.2), (x_2, 0.4, 0.3), (x_3, 0.7, 0.6)\}$ , assume  $q = 3, \omega_j = (0.2, 0.3, 0.5)$ , then according to Equation (25), the weighted cosine similarity measure between  $M$  and  $N$  can be calculated as

$$\begin{aligned} q-ROFWCS^1(M, N) &= \sum_{j=1}^n \omega_j \cos \left[ \frac{\pi}{2} \left( \max \left( \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right|, \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| \right) \right) \right] \\ &= \left( 0.2 \times \cos \left[ \frac{\pi}{2} \max \left( \left| 0.7^3 - 0.9^3 \right|, \left| 0.4^3 - 0.2^3 \right| \right) \right] + 0.3 \times \cos \left[ \frac{\pi}{2} \max \left( \left| 0.5^3 - 0.4^3 \right|, \left| 0.6^3 - 0.3^3 \right| \right) \right] \right. \\ &\quad \left. + 0.5 \times \cos \left[ \frac{\pi}{2} \max \left( \left| 0.3^3 - 0.7^3 \right|, \left| 0.8^3 - 0.6^3 \right| \right) \right] \right) \\ &= 0.8909 \end{aligned}$$

Evidently, similar to cosine similarity measure  $q - ROFCS^k(M, N) (k = 1, 2, 3, 4)$ , the weighted cosine similarity measure  $q - ROFWCS^k(M, N) (k = 1, 2, 3, 4)$  also meets some properties as follows.

**Proposition 2.** Assume that there are any two  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs)  $M$  and  $N$  in  $X = \{x_1, x_2, \dots, x_n\}$ , the  $q$ -rung orthopair fuzzy weighted cosine similarity measures  $q - ROFWCS^k(M, N) (k = 1, 2, 3, 4)$  should satisfy the properties (1)–(4):

- (1)  $0 \leq q - ROFWCS^k(M, N) \leq 1$ ;
- (2)  $q - ROFWCS^k(M, N) = 1$  if and only if  $M = N$ ;
- (3)  $q - ROFWCS^k(M, N) = q - ROFWCS^k(N, M)$ ;
- (4) If  $T$  is a  $q$ -ROFS in  $X$  and  $M \subseteq N \subseteq T$ , then  $q - ROFWCS^k(M, T) \leq q - ROFWCS^k(M, N)$ ,  $q - ROFWCS^k(M, T) \leq q - ROFWCS^k(N, T)$ .

The proof is similar to Proposition 1, so it is omitted here.

### 3.3. Similarity Measures of $q$ -ROFSs Based on Cotangent Function

In this section, according to the cotangent function, we will present some  $q$ -rung orthopair fuzzy cotangent similarity measures ( $q$ -ROFCot) between  $q$ -ROFSs and discuss their properties.

**Definition 8.** Assume that there are any two  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs)  $M = \{\langle x_j, (\alpha_M(x_j), \beta_M(x_j)) \rangle | x_j \in X\}$  and  $N = \{\langle x_j, (\alpha_N(x_j), \beta_N(x_j)) \rangle | x_j \in X\}$ . Then, we shall propose two  $q$ -rung orthopair fuzzy cotangent ( $q$ -ROFCot) measures between  $q$ -ROFSs  $M$  and  $N$  as follows

$$q - ROFCot^1(M, N) = \frac{1}{n} \sum_{j=1}^n \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( \max \left( \left| \frac{\alpha_M^q(x_j) - \alpha_N^q(x_j)}{\beta_M^q(x_j) - \beta_N^q(x_j)} \right|, \right) \right) \right] \tag{32}$$

where  $q - ROFCot^1(M, N)$  means the  $q$ -rung orthopair fuzzy cotangent similarity measures between  $M$  and  $N$ , which consider the maximum distance based on the membership and non-membership degree.

$$q - ROFCot^2(M, N) = \frac{1}{n} \sum_{j=1}^n \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \left| \frac{\alpha_M^q(x_j) - \alpha_N^q(x_j)}{\beta_M^q(x_j) - \beta_N^q(x_j)} \right| + \right) \right] \tag{33}$$

where  $q - ROFCot^2(M, N)$  means the  $q$ -rung orthopair fuzzy cotangent similarity measures between  $M$  and  $N$ , which consider the sum of distance based on the membership and non-membership degree.

If we consider three terms—membership degree, non-membership degree and indeterminacy membership—which are contained in  $q$ -ROFSs, assume that there are two  $q$ -rung orthopair fuzzy sets  $M = \{\langle x_j, \alpha_M(x_j), \beta_M(x_j), \pi_M(x_j) \rangle | x_j \in X\} (j = 1, 2, \dots, n)$  and  $N = \{\langle x_j, \alpha_N(x_j), \beta_N(x_j), \pi_N(x_j) \rangle | x_j \in X\} (j = 1, 2, \dots, n)$ , then the  $q$ -rung orthopair fuzzy cotangent ( $q$ -ROFCot) similarity measures between  $M$  and  $N$  can be expressed as

$$q - ROFCot^3(M, N) = \frac{1}{n} \sum_{j=1}^n \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( \max \left( \left| \frac{\alpha_M^q(x_j) - \alpha_N^q(x_j)}{\beta_M^q(x_j) - \beta_N^q(x_j)} \right|, \left| \frac{\pi_A^2(x_j) - \pi_B^2(x_j)}{\beta_M^q(x_j) - \beta_N^q(x_j)} \right|, \right) \right) \right] \tag{34}$$

where  $q - ROFCot^3(M, N)$  means the q-rung orthopair fuzzy cotangent similarity measures between  $M$  and  $N$ , which consider the maximum distance based on the membership, indeterminacy membership, and non-membership degree.

$$q - ROFCot^4(M, N) = \frac{1}{n} \sum_{j=1}^n \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \begin{array}{l} \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right| + \\ \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| + \\ \left| \pi_A^2(x_j) - \pi_B^2(x_j) \right| \end{array} \right) \right] \quad (35)$$

where  $q - ROFCot^4(M, N)$  means the q-rung orthopair fuzzy cotangent similarity measures between  $M$  and  $N$ , which consider the sum of distance based on the membership, indeterminacy membership, and non-membership degree.

Consider the weighting vector of the elements in q-ROFS, the q-rung orthopair fuzzy weighted cotangent (q-ROFWCot) similarity measure between two q-rung orthopair fuzzy sets (q-ROFSs)  $M$  and  $N$  can be shown as follows.

$$q - ROFWCot^1(M, N) = \sum_{j=1}^n \omega_j \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( \max \left( \begin{array}{l} \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right|, \\ \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| \end{array} \right) \right) \right] \quad (36)$$

where  $q - ROFWCot^1(M, N)$  means the q-rung orthopair fuzzy weighted cotangent similarity measures between  $M$  and  $N$ , which consider the maximum distance based on the membership and non-membership degree.

$$q - ROFWCot^2(M, N) = \sum_{j=1}^n \omega_j \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \begin{array}{l} \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right| + \\ \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| \end{array} \right) \right] \quad (37)$$

where  $q - ROFWCot^2(M, N)$  means the q-rung orthopair fuzzy weighted cotangent similarity measures between  $M$  and  $N$ , which consider the sum of distance based on the membership and non-membership degree.

$$q - ROFWCot^3(M, N) = \sum_{j=1}^n \omega_j \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( \max \left( \begin{array}{l} \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right|, \\ \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right|, \\ \left| \pi_A^2(x_j) - \pi_B^2(x_j) \right| \end{array} \right) \right) \right] \quad (38)$$

where  $q - ROFWCot^3(M, N)$  means the q-rung orthopair fuzzy weighted cotangent similarity measures between  $M$  and  $N$ , which consider the maximum distance based on the membership, indeterminacy membership and non-membership degree.

$$q - ROFWCot^4(M, N) = \sum_{j=1}^n \omega_j \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \begin{array}{l} \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right| + \\ \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| + \\ \left| \pi_A^2(x_j) - \pi_B^2(x_j) \right| \end{array} \right) \right] \quad (39)$$

where  $q - ROFWCot^4(M, N)$  means the q-rung orthopair fuzzy weighted cotangent similarity measures between  $M$  and  $N$ , which consider the sum of distance based on the membership, indeterminacy membership and non-membership degree.

Where  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  indicates the weighting vector of the elements  $x_j (j = 1, 2, \dots, n)$  contained in q-ROFS and the weighting vector satisfies  $\omega_j \in [0, 1], j = 1, 2, \dots, n, \sum_{j=1}^n \omega_j = 1$ . Especially, when we let weighting vector be  $\omega = (1/n, 1/n, \dots, 1/n)^T$ , then the weighted cotangent similarity measure will reduce to cotangent similarity measure. In other words, when  $\omega_j = \frac{1}{n}, j = 1, 2, \dots, n$ , the  $q - ROFWCot^k(M, N) = q - ROFCot^k(M, N) (k = 1, 2, 3, 4)$ .

**Example 4.** Suppose there are two  $q$ -ROFSs  $M = \{(x_1, 0.7, 0.4), (x_2, 0.5, 0.6), (x_3, 0.3, 0.8)\}$  and  $N = \{(x_1, 0.9, 0.2), (x_2, 0.4, 0.3), (x_3, 0.7, 0.6)\}$ , assume  $q = 3, \omega_j = (0.2, 0.3, 0.5)$ , then according to Equation (33), the weighted cotangent similarity measure between  $M$  and  $N$  can be calculated as

$$\begin{aligned}
 q-ROFWCot^1(M, N) &= \sum_{j=1}^n \omega_j \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( \max \left( \left| \alpha_M^q(x_j) - \alpha_N^q(x_j) \right|, \left| \beta_M^q(x_j) - \beta_N^q(x_j) \right| \right) \right) \right] \\
 &= \left( \begin{aligned} &0.2 \times \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \max \left( \left| 0.7^3 - 0.9^3 \right|, \left| 0.4^3 - 0.2^3 \right| \right) \right] + 0.3 \times \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \max \left( \begin{aligned} &\left| 0.5^3 - 0.4^3 \right|, \\ &\left| 0.6^3 - 0.3^3 \right| \end{aligned} \right) \right] \right. \\
 &\quad \left. + 0.5 \times \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( \max \left( \left| 0.3^3 - 0.7^3 \right|, \left| 0.8^3 - 0.6^3 \right| \right) \right) \right] \right) \\
 &= 0.6245
 \end{aligned}$$

#### 4. Applications

In this section, we shall give two applications about the cosine similarity measures and cotangent similarity measures under  $q$ -rung orthopair fuzzy environment. The methods proposed in this paper are applied to pattern recognition and scheme selection to demonstrate the effectiveness of these methods.

##### 4.1. Numerical Example 1—Pattern Recognition

There is no doubt that the quantity of construction mainly depends on the quality of building materials. Therefore, building material inspection is the premise of good engineering quality. In the selection of materials must be strictly controlled. Inspection can not only enable the builders to accurately identify qualified materials, but also ensure and improve the quality of the project. Let us consider the pattern recognition problems about classification of building materials, suppose there are five known building materials  $A_i (i = 1, 2, 3, 4, 5)$ , which are depicted by the  $q$ -ROFSs  $A_i (i = 1, 2, 3, 4, 5)$  in the feature space  $X = \{x_1, x_2, x_3, x_4, x_5\}$  as

$$\begin{aligned}
 A_1 &= \{(x_1, 0.5, 0.8), (x_2, 0.6, 0.4), (x_3, 0.8, 0.3), (x_4, 0.6, 0.9), (x_5, 0.1, 0.4)\} \\
 A_2 &= \{(x_1, 0.6, 0.7), (x_2, 0.7, 0.3), (x_3, 0.6, 0.2), (x_4, 0.8, 0.6), (x_5, 0.3, 0.5)\} \\
 A_3 &= \{(x_1, 0.3, 0.4), (x_2, 0.7, 0.5), (x_3, 0.9, 0.3), (x_4, 0.4, 0.8), (x_5, 0.2, 0.3)\} \\
 A_4 &= \{(x_1, 0.5, 0.3), (x_2, 0.4, 0.4), (x_3, 0.6, 0.2), (x_4, 0.4, 0.7), (x_5, 0.2, 0.6)\} \\
 A_5 &= \{(x_1, 0.4, 0.7), (x_2, 0.2, 0.6), (x_3, 0.5, 0.4), (x_4, 0.5, 0.3), (x_5, 0.4, 0.2)\}
 \end{aligned}$$

Consider an unknown building material  $A \in q-ROFSs(X)$  that will be recognized, which is depicted as

$$A = \{(x_1, 0.7, 0.6), (x_2, 0.8, 0.2), (x_3, 0.4, 0.3), (x_4, 0.7, 0.8), (x_5, 0.4, 0.2)\}$$

The purpose of this problem is classify the pattern  $A$  in one of the following classes,  $A_1, A_2, A_3, A_4$ , or  $A_5$ . For it, the cosine similarity measures and cotangent similarity measures proposed in this paper have been utilized to compute the similarity from  $A$  to  $A_i (i = 1, 2, 3, 4, 5)$  and the results are listed as follows. (Suppose  $q = 3$ )

For q-rung orthopair fuzzy cosine ( $q$ -ROFC<sup>1</sup>) measures, we can obtain

$$\begin{aligned}
 & q - ROFC^1(A_1, A) \\
 &= \frac{1}{5} \left( \frac{(0.5^3 \times 0.7^3 + 0.8^3 \times 0.6^3)}{\sqrt{(0.5^3)^2 + (0.8^3)^2} \times \sqrt{(0.7^3)^2 + (0.6^3)^2}} + \frac{(0.6^3 \times 0.8^3 + 0.4^3 \times 0.2^3)}{\sqrt{(0.6^3)^2 + (0.4^3)^2} \times \sqrt{(0.8^3)^2 + (0.2^3)^2}} \right) \\
 & \quad + \frac{(0.8^3 \times 0.4^3 + 0.3^3 \times 0.3^3)}{\sqrt{(0.8^3)^2 + (0.3^3)^2} \times \sqrt{(0.4^3)^2 + (0.3^3)^2}} + \frac{(0.6^3 \times 0.7^3 + 0.9^3 \times 0.8^3)}{\sqrt{(0.6^3)^2 + (0.9^3)^2} \times \sqrt{(0.7^3)^2 + (0.8^3)^2}} \\
 & \quad + \frac{(0.1^3 \times 0.4^3 + 0.4^3 \times 0.2^3)}{\sqrt{(0.1^3)^2 + (0.4^3)^2} \times \sqrt{(0.4^3)^2 + (0.2^3)^2}} \\
 &= 0.7443
 \end{aligned}$$

Similarly, we can get

$$\begin{aligned}
 q - ROFC^1(A_2, A) &= 0.8033, q - ROFC^1(A_3, A) = 0.7988, \\
 q - ROFC^1(A_4, A) &= 0.7345, q - ROFC^1(A_5, A) = 0.6897.
 \end{aligned}$$

(1) For q-rung orthopair fuzzy cosine ( $q$ -ROFC<sup>2</sup>) measures we can obtain

$$\begin{aligned}
 q - ROFC^2(A_1, A) &= 0.8795, q - ROFC^2(A_2, A) = 0.9116, \\
 q - ROFC^2(A_3, A) &= 0.9124, q - ROFC^2(A_4, A) = 0.8766, \\
 q - ROFC^2(A_5, A) &= 0.8543.
 \end{aligned}$$

(2) For q-rung orthopair fuzzy cosine similarity ( $q$ -ROFCS<sup>1</sup>) measures we can obtain

$$\begin{aligned}
 q - ROFCS^1(A_1, A) &= 0.8975, q - ROFCS^1(A_2, A) = 0.9588, \\
 q - ROFCS^1(A_3, A) &= 0.8496, q - ROFCS^1(A_4, A) = 0.9057, \\
 q - ROFCS^1(A_5, A) &= 0.8654.
 \end{aligned}$$

(3) For q-rung orthopair fuzzy cosine similarity ( $q$ -ROFCS<sup>2</sup>) measures we can obtain

$$\begin{aligned}
 q - ROFCS^2(A_1, A) &= 0.9559, q - ROFCS^2(A_2, A) = 0.9774, \\
 q - ROFCS^2(A_3, A) &= 0.9498, q - ROFCS^2(A_4, A) = 0.9561, \\
 q - ROFCS^2(A_5, A) &= 0.9291.
 \end{aligned}$$

(4) For q-rung orthopair fuzzy cosine similarity ( $q$ -ROFCS<sup>3</sup>) measures we can obtain

$$\begin{aligned}
 q - ROFCS^3(A_1, A) &= 0.8975, q - ROFCS^3(A_2, A) = 0.9588, \\
 q - ROFCS^3(A_3, A) &= 0.8364, q - ROFCS^3(A_4, A) = 0.8880, \\
 q - ROFCS^3(A_5, A) &= 0.8540.
 \end{aligned}$$

(5) For q-rung orthopair fuzzy cosine similarity ( $q$ -ROFCS<sup>4</sup>) measures we can obtain

$$\begin{aligned}
 q - ROFCS^4(A_1, A) &= 0.8964, q - ROFCS^4(A_2, A) = 0.9630, \\
 q - ROFCS^4(A_3, A) &= 0.8386, q - ROFCS^4(A_4, A) = 0.8701, \\
 q - ROFCS^4(A_5, A) &= 0.8362.
 \end{aligned}$$

(6) For q-rung orthopair fuzzy cotangent similarity ( $q$ -ROFCot<sup>1</sup>) measures we can obtain

$$\begin{aligned}
 q - ROFCot^1(A_1, A) &= 0.6618, q - ROFCot^1(A_2, A) = 0.7633, \\
 q - ROFCot^1(A_3, A) &= 0.6362, q - ROFCot^1(A_4, A) = 0.6613, \\
 q - ROFCot^1(A_5, A) &= 0.6766.
 \end{aligned}$$

(7) For  $q$ -rung orthopair fuzzy cotangent similarity ( $q$ -ROFCot<sup>2</sup>) measures we can obtain

$$\begin{aligned}
 q - ROFCot^2(A_1, A) &= 0.7571, q - ROFCot^2(A_2, A) = 0.8257, \\
 q - ROFCot^2(A_3, A) &= 0.7613, q - ROFCot^2(A_4, A) = 0.7544, \\
 q - ROFCot^2(A_5, A) &= 0.7522.
 \end{aligned}$$

(8) For  $q$ -rung orthopair fuzzy cotangent similarity ( $q$ -ROFCot<sup>3</sup>) measures we can obtain

$$\begin{aligned}
 q - ROFCot^3(A_1, A) &= 0.6618, q - ROFCot^3(A_2, A) = 0.7633, \\
 q - ROFCot^3(A_3, A) &= 0.6198, q - ROFCot^3(A_4, A) = 0.6318, \\
 q - ROFCot^3(A_5, A) &= 0.6596.
 \end{aligned}$$

(9) For  $q$ -rung orthopair fuzzy cotangent similarity ( $q$ -ROFCot<sup>4</sup>) measures we can obtain

$$\begin{aligned}
 q - ROFCot^4(A_1, A) &= 0.6588, q - ROFCot^4(A_2, A) = 0.7702, \\
 q - ROFCot^4(A_3, A) &= 0.6259, q - ROFCot^4(A_4, A) = 0.6085, \\
 q - ROFCot^4(A_5, A) &= 0.6496.
 \end{aligned}$$

Thereafter, the above computing results are concluded to list in Table 1 as follows.

**Table 1.** The similarity measures between  $A_i (i = 1, 2, 3, 4, 5)$  and  $A$ .

Similarity Measures	(A <sub>1</sub> ,A)	(A <sub>2</sub> ,A)	(A <sub>3</sub> ,A)	(A <sub>4</sub> ,A)	(A <sub>5</sub> ,A)
$q - ROFC^1(A_i, A)$	0.7433	0.8003	0.7988	0.7345	0.6897
$q - ROFC^2(A_i, A)$	0.8795	0.9116	0.9124	0.8766	0.8543
$q - ROFCS^1(A_i, A)$	0.8975	0.9588	0.8496	0.9057	0.8654
$q - ROFCS^2(A_i, A)$	0.9559	0.9774	0.9498	0.9561	0.9291
$q - ROFCS^3(A_i, A)$	0.8975	0.9588	0.8364	0.8880	0.8540
$q - ROFCS^4(A_i, A)$	0.8964	0.9630	0.8386	0.8701	0.8362
$q - ROFCot^1(A_i, A)$	0.6618	0.7633	0.6362	0.6613	0.6766
$q - ROFCot^2(A_i, A)$	0.7571	0.8257	0.7613	0.7544	0.7522
$q - ROFCot^3(A_i, A)$	0.6618	0.7633	0.6198	0.6318	0.6596
$q - ROFCot^4(A_i, A)$	0.6588	0.7702	0.6259	0.6085	0.6496

According to the above calculated results listed in Table 1, except for  $q - ROFC^2(A_i, A)$ , we can easily find that the degree of similarity between  $A_2$  and  $A$  is the largest as derived by the other nine similarity measures. This indicates the nine similarity measures allocate the unknown building material  $A$  to the known building material  $A_2$  based on the principle of maximum similarity between  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs).

In practical decision-making problems, it is important to take the weights of elements into account, if we let the weights of elements  $x_i (i = 1, 2, 3, 4, 5)$  be  $\omega_i = (0.15, 0.20, 0.25, 0.10, 0.30)$ , respectively. Then the weighted cosine similarity measures and weighted cotangent similarity measures proposed in this paper have been utilized to compute the similarity from  $A$  to  $A_i (i = 1, 2, 3, 4, 5)$  and the results are listed in Table 2 (suppose  $q = 3$ ). (The calculation process is similar to not weighted situation, so it is omitted here.)

According to the above calculated results listed in Table 2, except for  $q - ROFC^2(A_i, A)$  and  $q - ROFWC^2(A_i, A)$ , we can easily find that the degree of similarity between  $A_2$  and  $A$  is the largest one as derived by the other eight similarity measures. This indicates the eight similarity measures allocate the unknown building material  $A$  to the known building material  $A_2$  based on the principle of maximum similarity between  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs).



**Table 2.** The weighted similarity measures between  $A_i(i = 1, 2, 3, 4, 5)$  and  $A$ .

Similarity Measures	$(A_1, A)$	$(A_2, A)$	$(A_3, A)$	$(A_4, A)$	$(A_5, A)$
$q - ROFWC^1(A_i, A)$	0.6728	0.7515	0.7553	0.6584	0.7336
$q - ROFWC^2(A_i, A)$	0.8457	0.8901	0.8937	0.8406	0.8735
$q - ROFWCS^1(A_i, A)$	0.8962	0.9673	0.8398	0.9114	0.8976
$q - ROFWCS^2(A_i, A)$	0.9601	0.9838	0.9487	0.9621	0.9464
$q - ROFWCS^3(A_i, A)$	0.8962	0.9673	0.8299	0.8986	0.8910
$q - ROFWCS^4(A_i, A)$	0.8961	0.9693	0.8303	0.8883	0.8830
$q - ROFWCot^1(A_i, A)$	0.6740	0.7831	0.6478	0.6735	0.7474
$q - ROFWCot^2(A_i, A)$	0.7740	0.8482	0.7700	0.7733	0.8065
$q - ROFWCot^3(A_i, A)$	0.6740	0.7831	0.6356	0.6522	0.7324
$q - ROFWCot^4(A_i, A)$	0.6727	0.7866	0.6356	0.6389	0.7284

In order to illustrate the effective and scientific of our proposed methods, we shall compare with other decision-making methods, such as the q-rung orthopair fuzzy weighted averaging (q-ROFWA) operator and the q-rung orthopair fuzzy weighted geometric (q-ROFWG) operator proposed by Liu and Wang [40], we obtained the following results in Table 3.

**Table 3.** The fused values of  $A_i(i = 1, 2, 3, 4, 5)$  and  $A$ .

	The q-ROFWA Operator	The q-ROFWG Operator
$A_1$	(0.2387,0.4479)	(0.3665,0.2428)
$A_2$	(0.2300,0.3845)	(0.5173,0.1262)
$A_3$	(0.3453,0.3827)	(0.4263,0.1162)
$A_4$	(0.0980,0.3848)	(0.3719,0.1258)
$A_5$	(0.0751,0.3724)	(0.3766,0.1249)
$A$	(0.2479,0.2998)	(0.5285,0.1123)

Then, based on distance measure between q-rung orthopair fuzzy numbers (q-ROFNs), we can allocate the unknown building material  $A$  to the known building material  $A_i$ , the distance measure  $d(M, N)$  between q-ROFNs  $M = (\alpha_1, \beta_1)$  and  $N = (\alpha_2, \beta_2)$  can be depicted as

$$d(M, N) = \frac{|(\alpha_1)^q - (\alpha_2)^q| + |(\beta_1)^q - (\beta_2)^q|}{2} \tag{40}$$

For q-ROFWA operator, we can obtain the distance results  $d(A_i, A)$  as

$$d(A_1, A) = 0.0323, d(A_2, A) = 0.0165, d(A_3, A) = 0.0275, \\ d(A_4, A) = 0.0222, d(A_5, A) = 0.0197$$

For q-ROFWG operator, we can obtain the distance results  $d(A_i, A)$  as

$$d(A_1, A) = 0.0556, d(A_2, A) = 0.0049, d(A_3, A) = 0.0352, \\ d(A_4, A) = 0.0484, d(A_5, A) = 0.0474$$

From above analysis, for q-ROFWA and q-ROFWG operators, the distance measure between  $A_2$  and  $A$  is the minimum one. This indicates that q-ROFWA and q-ROFWG operators allocate the unknown building material  $A$  to the known building material  $A_2$ . Although, based on the two operators and our developed methods, we can derive the same results, however, the q-ROFWA and q-ROFWG operators have the limitation of considering the interrelationship between attributes; our developed methods can overcome this disadvantage and derive more accuracy and scientific decision-making results.

4.2. Numerical Example 2—Scheme Selection

In this section, we shall present a numerical example to show scheme selection of construction project with q-rung orthopair fuzzy information in order to illustrate the method proposed in this paper. There is a panel with five possible construction projects.  $Y_i (i = 1, 2, 3, 4, 5)$  to select. Experts selected five attributes to evaluate from the five possible construction projects: ①  $G_1$  is the capital and technical factors; ②  $G_2$  is the Hoisting construction operation factors; ③  $G_3$  is the PC component installation factor; ④  $G_4$  is the internal and external environmental risk factors; and ⑤  $G_5$  is the professional management level factors. The five possible construction projects  $Y_i (i = 1, 2, 3, 4, 5)$  are to be evaluated using the q-rung orthopair fuzzy information by the decision maker under the above five attributes which listed as follows.

$$\begin{aligned}
 Y_1 &= \{(G_1, 0.6, 0.7), (G_2, 0.5, 0.8), (G_3, 0.6, 0.3), (G_4, 0.7, 0.3), (G_5, 0.4, 0.6)\} \\
 Y_2 &= \{(G_1, 0.7, 0.4), (G_2, 0.8, 0.3), (G_3, 0.5, 0.6), (G_4, 0.2, 0.5), (G_5, 0.6, 0.3)\} \\
 Y_3 &= \{(G_1, 0.6, 0.3), (G_2, 0.4, 0.2), (G_3, 0.7, 0.4), (G_4, 0.5, 0.2), (G_5, 0.9, 0.4)\} \\
 Y_4 &= \{(G_1, 0.8, 0.7), (G_2, 0.5, 0.6), (G_3, 0.4, 0.6), (G_4, 0.6, 0.3), (G_5, 0.4, 0.2)\} \\
 Y_5 &= \{(G_1, 0.7, 0.2), (G_2, 0.4, 0.3), (G_3, 0.5, 0.6), (G_4, 0.3, 0.5), (G_5, 0.7, 0.2)\}
 \end{aligned}$$

Let

$$Y^+ = \left\{ \left( G_1, \max_i(\alpha_{i1}), \min_i(\beta_{i1}) \right), \left( G_2, \max_i(\alpha_{i2}), \min_i(\beta_{i2}) \right), \right. \\
 \left. \left( G_3, \max_i(\alpha_{i3}), \min_i(\beta_{i3}) \right), \left( G_4, \max_i(\alpha_{i4}), \min_i(\beta_{i4}) \right), \right. \\
 \left. \left( G_5, \max_i(\alpha_{i5}), \min_i(\beta_{i5}) \right) \right\}$$

According to the evaluation results given in  $Y_1, Y_2, Y_3, Y_4$  and  $Y_5$ , we can easily obtain

$$Y^+ = \{(G_1, 0.8, 0.2), (G_2, 0.8, 0.2), (G_3, 0.7, 0.3), (G_4, 0.7, 0.2), (G_5, 0.9, 0.2)\}$$

Then the weighted cosine similarity measures and weighted cotangent similarity measures proposed in this paper have been utilized to compute the similarity from  $Y^+$  to  $Y_i (i = 1, 2, 3, 4, 5)$  and the results are listed in Table 4 (suppose  $q = 3$ ).

Table 4. The similarity measures between  $Y_i (i = 1, 2, 3, 4, 5)$  and  $Y^+$ .

Similarity Measures	$(Y_1, Y)$	$(Y_2, Y)$	$(Y_3, Y)$	$(Y_4, Y)$	$(Y_5, Y)$
$q - ROFC^1(Y_i, Y)$	0.6181	0.9092	0.9958	0.7401	0.7457
$q - ROFC^2(Y_i, Y)$	0.8215	0.7266	0.9975	0.8818	0.8837
$q - ROFCS^1(Y_i, Y)$	0.8099	0.8714	0.9185	0.8147	0.8741
$q - ROFCS^2(Y_i, Y)$	0.8827	0.8927	0.9785	0.9303	0.9542
$q - ROFCS^3(Y_i, Y)$	0.8099	0.9571	0.9185	0.8147	0.8741
$q - ROFCS^4(Y_i, Y)$	0.8161	0.8927	0.9194	0.8249	0.8737
$q - ROFCot^1(Y_i, Y)$	0.6088	0.8912	0.7289	0.5642	0.6103
$q - ROFCot^2(Y_i, Y)$	0.6851	0.6836	0.8494	0.7076	0.7453
$q - ROFCot^3(Y_i, Y)$	0.6088	0.7781	0.7289	0.5642	0.6103
$q - ROFCot^4(Y_i, Y)$	0.6160	0.6836	0.7371	0.5802	0.6089

According to the above calculated results listed in Table 4, we can easily find that the degree of similarity between  $Y_3$  and  $Y$  is the largest one as derived by all ten similarity measures. This indicates all ten similarity measures think the alternative  $Y_3$  is closest to be best alternative  $Y^+$  based on the principle of maximum similarity between q-rung orthopair fuzzy sets (q-ROFSs). In other words,  $Y_3$  is the best scheme selection for the construction project.

In practical decision-making problems, it is important to take the weights of elements into account, if we let the weights of elements  $x_i (i = 1, 2, 3, 4, 5)$  be  $\omega_i = (0.15, 0.20, 0.25, 0.10, 0.30)$ , respectively. Then the weighted cosine similarity measures and weighted cotangent similarity measures proposed

in this paper have been utilized to compute the similarity from  $Y$  to  $Y_i (i = 1, 2, 3, 4, 5)$  and the results are listed in Table 5 (suppose  $q = 3$ ).

**Table 5.** The weighted similarity measures between  $Y_i (i = 1, 2, 3, 4, 5)$  and  $Y^+$ .

Similarity Measures	$(Y_1, Y)$	$(Y_2, Y)$	$(Y_3, Y)$	$(Y_4, Y)$	$(Y_5, Y)$
$q - ROFWC^1(Y_i, Y)$	0.5703	0.7963	0.9956	0.7160	0.8006
$q - ROFWC^2(Y_i, Y)$	0.7994	0.9060	0.9974	0.8711	0.9095
$q - ROFWCS^1(Y_i, Y)$	0.7659	0.8744	0.9292	0.7689	0.8672
$q - ROFWCS^2(Y_i, Y)$	0.8659	0.9529	0.9813	0.9160	0.9533
$q - ROFWCS^3(Y_i, Y)$	0.7659	0.8744	0.9292	0.7689	0.8672
$q - ROFWCS^4(Y_i, Y)$	0.7714	0.8727	0.9303	0.7765	0.8667
$q - ROFWCot^1(Y_i, Y)$	0.5508	0.6633	0.7663	0.5124	0.6001
$q - ROFWCot^2(Y_i, Y)$	0.6494	0.7678	0.8706	0.6723	0.7405
$q - ROFWCot^3(Y_i, Y)$	0.5508	0.6633	0.7663	0.5124	0.6001
$q - ROFWCot^4(Y_i, Y)$	0.5571	0.6607	0.7779	0.5240	0.5983

According to the above calculated results listed in Table 5, we can easily find that the degree of similarity between  $Y_3$  and  $Y$  is the largest one as derived by other nine similarity measures. This indicates all ten similarity measures; the alternative  $Y_3$  is closest to be best alternative  $Y^+$  based on the principle of maximum similarity between  $q$ -rung orthopair fuzzy sets ( $q$ -ROFSs). In other words,  $Y_3$  is the best scheme selection for the construction project.

In order to illustrate the effective and scientific of our proposed methods, we shall compare with other decision-making methods such as the  $q$ -rung orthopair fuzzy weighted averaging ( $q$ -ROFWA) operator and the  $q$ -rung orthopair fuzzy weighted geometric ( $q$ -ROFWG) operator proposed by Liu and Wang [40], we can obtain the result which is listed in Table 6.

**Table 6.** The fused results of  $Y_i (i = 1, 2, 3, 4, 5)$ .

	The $q$ -ROFWA Operator	The $q$ -ROFWG Operator
$Y_1$	(0.1697,0.5103)	(0.5203,0.2511)
$Y_2$	(0.2693,0.3921)	(0.5568,0.0932)
$Y_3$	(0.4287,0.3112)	(0.6377,0.0420)
$Y_4$	(0.1772,0.4121)	(0.4833,0.1628)
$Y_5$	(0.2121,0.3129)	(0.5287,0.0799)

Then according to the score functions of  $q$ -rung orthopair fuzzy numbers ( $q$ -ROFNs), we can obtain the score values of  $Y_i (i = 1, 2, 3, 4, 5)$  which is listed in Table 7.

**Table 7.** The score values of  $Y_i (i = 1, 2, 3, 4, 5)$ .

	The $q$ -ROFWA Operator	The $q$ -ROFWG Operator
$s(Y_1)$	0.4360	0.5625
$s(Y_2)$	0.4796	0.5859
$s(Y_3)$	0.5243	0.6296
$s(Y_4)$	0.4678	0.5543
$s(Y_5)$	0.4895	0.5736

Then based on score values, the ordering of  $Y_i (i = 1, 2, 3, 4, 5)$  can be determined in Table 8.

From above analysis, based on the two operators and our developed methods, we can obtain that the ordering of alternatives are slightly different and the best results are same, however, the  $q$ -ROFWA and  $q$ -ROFWG operators have the limitation of considering the interrelationship between attributes,

our developed methods can overcome this disadvantage and derive more accuracy and scientific decision-making results.

**Table 8.** The ordering of  $Y_i (i = 1, 2, 3, 4, 5)$ .

	Ordering
The q – ROFWA operator	$Y_3 > Y_5 > Y_2 > Y_4 > Y_1$
The q – ROFWG operator	$Y_3 > Y_2 > Y_5 > Y_1 > Y_4$

### 4.3. Advantages of the Proposed Similarity Measures

Although, the intuitionistic fuzzy sets (IFSs), defined by Atanassov’s [1,2], have been broadly applied in different areas, for some special cases, such as when membership degree and non-membership degree are given as 0.7 and 0.8, it is clear that IFSs theory cannot satisfy this situation. The q-rung orthopair fuzzy set (q-ROFS) is also denoted by the degree of membership and non-membership, whose *q*-th power sum of them is restricted to 1; obviously, the q-ROFS is more general than the IFS and can express more fuzzy information. In other words, the q-ROFS can deal with the MADM problems which IFS cannot and it is clear that IFS is a part of the q-ROFS, which indicates q-ROFS can be more effective and powerful to deal with fuzzy and uncertain decision-making problems. Thus, the MADM problem with q-rung orthopair fuzzy information is more effective and suitable for practical scientific and engineering applications.

To date, we can get that the cosine similarity measures and cotangent similarity measures [13,15–17,19] with IFSs have been investigated by a large amount of scholars; however, as mentioned above, there are some special cases that cannot be described by IFS. Therefore, the algorithms based on similarity measures with IFS can’t deal with such problems. The cosine similarity measures and cotangent similarity measures with intuitionistic fuzzy information are special case of our proposed similarity measures with q-rung orthopair fuzzy information in this paper. Thus, our defined similarity measures are more suitable and generalized to deal with the real-life problem more accurately than the existing ones.

### 4.4. Discussion

According to above two numerical examples, we can easily find our proposed methods can express more fuzzy information and apply broadly situations in real MADM problems. Based on the q-rung orthopair fuzzy set (q-ROFS), we developed ten q-rung orthopair fuzzy similarity measures; our research results are more suitable for MADM problems than intuitionistic fuzzy similarity measures and Pythagorean fuzzy similarity measures. For pattern recognition problems, we accurately allocated the unknown building material  $A$  to the known building material  $A_2$ . For scheme selection, by utilizing our developed ten similarity measures, we obtained the best scheme selection of construction project.

Furthermore, in complicated decision-making environment, the decision-maker’s risk attitude is an important factor to think about, our methods can make this come true by altering the parameters whereas other decision-making ways such as q-ROFWA and q-ROFWG operator do not have the ability that dynamic adjust to the parameter according to the decision maker’s risk attitude, so it is difficult to solve the risk multiple attribute decision-making in real practice.

## 5. Conclusions

According to the intuitionistic fuzzy cosine similarity measures and cotangent similarity measures, based on q-rung orthopair fuzzy sets (q-ROFSs), we proposed another form of ten similarity measures by considering the function of membership degree, nonmembership degree and indeterminacy membership degree in q-ROFSs. In addition, we utilized our presented ten similarity measures and ten weighted similarity measures between q-ROFSs to deal MADM problems, including pattern recognition and scheme selection. Finally, two numerical examples and some comparative analysis

were provided to illustrate the scientific and effective of the similarity measures for pattern recognition and scheme selection. By utilizing our developed ten similarity measures, we can deal with MADM problems regarding pattern recognition and scheme selection. When comparing our developed ten similarity measures with the q-rung orthopair fuzzy weighted average (q-ROFWA) operator and q-rung orthopair fuzzy weighted geometric (q-ROFWG) operator, our proposed methods can be applied in scheme selection and pattern recognition applications as the q-ROFWA and q-ROFWG operators can be only used to select best alternatives. Moreover, q-ROFWA and q-ROFWG operators have the limitation of considering the interrelationship between fused arguments; our proposed methods can overcome this disadvantage and derive more accuracy and reasonable decision-making results. In the future, works concerning q-ROFSs could focus on dealing with other kinds of decision-making problems such as: staff selection, investment selection, machine selection, project selection, manufacturing systems, etc. [52–59].

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