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Generalized Nonsmooth Exponential-Type Vector Variational-Like Inequalities and Nonsmooth Vector Optimization Problems in Asplund Spaces

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Abstract: The aim of this article is to study new types of generalized nonsmooth exponential type vector variational-like inequality problems involving Mordukhovich limiting subdifferential operator. We establish some relationships between generalized nonsmooth exponential type vector variational-like inequality problems and vector optimization problems under some invexity assumptions. The celebrated Fan-KKM theorem is used to obtain the existence of solution of generalized nonsmooth exponential-type vector variational like inequality problems. In support of our main result, some examples are given. Our results presented in this article improve, extend, and generalize some known results offer in the literature.

Keywords: vector variational-like inequalities; vector optimization problems; limiting (p,r)-α-(η,θ)-invexity; Lipschitz continuity; Fan-KKM theorem

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1. Introduction

The vector variational inequality has been introduced and studied in [1] in finite-dimensional Euclidean spaces. Vector variational inequalities have emerged as an efficient tool to provide imperative requirements for the solution of vector optimization problems. Vector variational-like inequalities for nonsmooth mappings are useful generalizations of vector variational inequalities. For more details on vector variational inequalities and their generalizations, see the references [2–8]. In 1998, Giannessi [9] proved a necessary and sufficient condition for the existence of an efficient solution of a vector optimization problem for differentiable and convex mappings by using a Minty type vector variational inequality problem. Under different assumptions, many researchers have studied vector optimization problems by using different types of Minty type vector variational inequality problems. Yang et al. [8] generalized the result of Giannessi [9] for differentiable but pseudoconvex mappings.

On the other hand, Yang and Yang [10] considered vector variational-like inequality problem and showed relationships between vector variational-like inequality and vector optimization problem under the assumptions of pseudoinvexity or invariant pseudomonotonicity. Later, some researchers extended above problems in the direction of nonsmooth mappings. Rezaie and Zafarani [11] established a correspondence between a solution of the generalized vector variational-like inequality problem and the nonsmooth vector optimization problem under the same assumptions of Yang and Yang [10] in the setting of Clarke's subdifferentiability. Due to the fact that Clarke's subdifferentiability is bigger class than Mordukhovich limiting subdifferentiability, many authors studied the vector variational-like inequality problems and vector optimization problems by means of Mordukhovich limiting subdifferential. Later, Long et al. [12] and Oveisiha and Zafarani [13] studied generalized vector variational-like inequality problem and discussed the relationships between generalized vector variational-like inequality problem and nonsmooth vector optimization problem for pseudoinvex mappings, whereas Chen and Huang [14] obtained similar results for invex mappings by means of Mordukhovich limiting subdifferential.

Due to several applications of invex sets and exponential mappings in engineering, economics, population growth, mathematical modelling problems, Antczak [15] introduced exponential (p,r)-invex sets and mappings. After that, Mandal and Nahak [16] introduced (p,r)-ρ-(η,θ)-invexity mapping which is the generalization of the result of Antczak [15]. By using (p,r)-invexity, Jayaswal and Choudhury [17] introduced exponential type vector variational-like inequality problem involving locally Lipschitz mappings.

In this paper, we introduce generalized nonsmooth exponential-type vector variational like inequality problems involving Mordukhovich limiting subdifferential in Asplund spaces. We obtain some relationships between an efficient solution of nonsmooth vector optimization problems and this generalized nonsmooth exponential-type vector variational like inequality problems using limiting (p,r)-α-(η,θ)-invexity mapping. Employing the Fan-KKM theorem, we establish an existence result for our problem in Asplund spaces.

2. Preliminaries

Suppose that X is a real Banach space with dual space X^* and $\langle \cdot,\cdot \rangle$ is duality pairing between them. Assume that $K \subseteq X$ is a nonempty subset, $C \subset \mathbb{R}^n$ is a pointed, closed, convex cone with nonempty interior, i.e., $intC \neq \emptyset$ and $f : K \longrightarrow \mathbb{R}$ is a non-differentiable mapping. When the mappings are non-differentiable, many authors used the concept of subdifferential such as Fréchet subdifferential, Mordukhovich limiting subdifferential, and Clarke subdifferential operators. Now, we mention some notions and results already known in the literature.

Definition 1. *Suppose that $f : X \longrightarrow \mathbb{R}$ is a proper lower semicontinuous mapping on Banach space X. Then, the mapping f is said to be Fréchet subdifferentiable and ξ^* is Fréchet subderivative of f at x (i.e., $\xi^* \in \partial_F f(x)$) if, $x \in domf$ and*

$$\liminf_{\|h\|\to 0} \frac{f(x+h) - f(x) - \langle \xi^*, h\rangle}{\|h\|} \geq 0.$$

Definition 2 ([18])**.** *Suppose that Ω is a nonempty subset of a normed vector space X. Then, for any $x \in X$ and $\varepsilon \geq 0$, the set of ε-normals to Ω at x is defined as*

$$\widehat{N}_\varepsilon(x;\Omega) = \left\{ x^* \in X^* : \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \leq \varepsilon \right\}.$$

For $\tilde{x} \in \Omega$, the limiting normal cone to Ω at $\tilde{x}$ is

$$N(\tilde{x};\Omega) = \limsup_{x \xrightarrow{\Omega} \tilde{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x;\Omega).$$

Consider a mapping $f : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ and a finite point $\tilde{x} \in X$. Then, the limiting subdifferential of f at $\tilde{x}$ is the following set

$$\partial_L f(\tilde{x}) = \{x^* \in X^* : (x^*, -1) \in N((\tilde{x}, f(\tilde{x})); epif)\},$$

where $epif$ is defined as $epif = \{(x,a) \in X \times \mathbb{R} : f(x) \leq a\}$. If $|f(\tilde{x})| = \infty$, then we put $\partial_L f(\tilde{x}) = \emptyset$.

Remark 1 ([18]). *It is noted that the Clarke subdifferential is larger class than the Fréchet subdifferential and the limiting subdifferential with the relation* $\partial_F f(x) \subseteq \partial_L f(x) \subseteq \partial_C f(x)$.

Definition 3. *A Banach space X is said to be Asplund space if K is any open subset of X and* $f : K \longrightarrow \mathbb{R}$ *is continuous convex mapping, then f is Fréchet subdifferentiable at any point of a dense subset of K.*

Remark 2. *It is remarked that a Banach space X has the Asplundity property if every separable subspace of X has separable dual. The concept of Asplund space depicts the differentiability characteristics of continuous convex mappings on Euclidean space. All the spaces which are reflexive Banach spaces are Asplund. The space of convergent real sequences* c_0 *(whose limit is 0) is non-reflexive separable Banach space, but its is an Asplund space. For more details, we refer to [19].*

Definition 4. *A bi-mapping* $\eta : K \times K \longrightarrow K$ *is said to be affine with respect to the first argument if, for any* $\lambda \in [0, 1]$ *and* $u_1, u_2 \in K$ *with* $u = \lambda u_1 + (1 - \lambda)u_2 \in K$ *such that*

$$\eta(\lambda u_1 + (1 - \lambda)u_2, v) = \lambda\eta(u_1, v) + (1 - \lambda)\eta(u_2, v), \ \ \forall v \in K.$$

Definition 5. *A bi-mapping* $\eta : K \times K \longrightarrow X$ *is said to be continuous in the first argument if,*

$$\|\eta(u, z) - \eta(v, z)\| \to 0 \ \text{ as } \ \|u - v\| \to 0, \ \ \forall u, v \in K, \ \ z \text{ is fixed.}$$

Definition 6 ([20]). *Suppose that K is a subset of a topological vector space Y. A set-valued mapping* $T : K \longrightarrow 2^Y$ *is called a KKM-mapping if, for each nonempty finite subset* $\{y_1, y_2, \cdots, y_n\} \subset K$, *we have*

$$Co\{y_1, y_2, \cdots, y_n\} \subseteq \bigcup_{i=1}^{n} T(y_i),$$

where Co denotes the convex hull.

Theorem 1 (Fan-KKM Theorem [20]). *Suppose that K is a subset of a topological vector space Y and* $T : K \longrightarrow 2^Y$ *is a KKM-mapping. If, for each* $y \in K, T(y)$ *is closed and for at least one* $y \in K, T(y)$ *is compact, then*

$$\bigcap_{y \in K} T(y) \neq \emptyset.$$

Definition 7. *A mapping* $f : X \longrightarrow \mathbb{R}^n$ *is called locally Lipschitz continuous at* x_0 *if, there exists a* $L > 0$ *and a neighbourhood N of* x_0 *such that*

$$\|f(y) - f(z)\| \leq L\|y - z\|, \ \ \forall y, z \in N(x_0).$$

If f is locally Lipschitz continuous for each x_0 *in X, then f is locally Lipschitz continuous mapping on X.*

Slightly changing the structure of definition of (p, r)-α-(η, θ)-invexity defined in [16], we have the following definition.

Definition 8. *Suppose that $f : X \longrightarrow \mathbb{R}^n$ is a locally Lipschitz continuous mapping, $e = (1, 1, \cdots, 1) \in \mathbb{R}^n$ and p, r are arbitrary real numbers. If there exist the mappings $\eta, \theta : X \times X \longrightarrow X$ and a constant $\alpha \in \mathbb{R}$ such that one of the following relations*

$$\begin{aligned}
\frac{1}{r}\left\{\exp^{r(f(x)-f(u))} - 1\right\} &\geq \frac{1}{p}\left\langle \xi; \left(\exp^{p\eta(x,u)} - e\right)\right\rangle + \alpha\|\theta(x,u)\|^2 e \ (> \ if \ x \neq u) \ for \ p \neq 0, r \neq 0, \\
\frac{1}{r}\left\{\exp^{r(f(x)-f(u))} - 1\right\} &\geq \langle \xi; \eta(x,u)\rangle + \alpha\|\theta(x,u)\|^2 e \ (> \ if \ x \neq u) \ for \ p = 0, r \neq 0, \\
f(x) - f(u) &\geq \frac{1}{p}\left\langle \xi; \left(\exp^{p\eta(x,u)} - e\right)\right\rangle + \alpha\|\theta(x,u)\|^2 e \ (> \ if \ x \neq u) \ for \ p \neq 0, r = 0, \\
f(x) - f(u) &\geq \langle \xi; \eta(x,u)\rangle + \alpha\|\theta(x,u)\|^2 e \ (> \ if \ x \neq u) \ for \ p = 0, r = 0,
\end{aligned}$$

holds for each $\xi \in \partial_L f(u)$, then f is called limiting (p,r)-α-(η,θ)-invex (strictly limiting (p,r)-α-(η,θ)-invex) with respect to η and θ at the point u on X. If f is limiting (p,r)-α-(η,θ)-invex with respect to η and θ at each $u \in X$, then f is limiting (p,r)-α-(η,θ)-invex with respect to the same η and θ on X.

Remark 3. *We only consider the case when $p \neq 0, r \neq 0$ to prove the results. We exclude other cases as it is straightforward in terms of altering inequality. Throughout the proof of the results, we assume that $r > 0$. Under other condition $r < 0$, the direction in the proof will be reversed.*

Problem 1. *Suppose that $f = (f_1, f_2, \cdots, f_n) : K \longrightarrow \mathbb{R}^n$ is a vector-valued mapping such that each $f_i : K \longrightarrow \mathbb{R}$ $(i = 1, 2, \cdots, n)$ is locally Lipschitz continuous mapping. The nonsmooth vector optimization problem is to*

$$\underset{C}{Maximize}\, f(x) = (f_1(x), f_2(x), \cdots, f_n(x)) \qquad (P_1)$$
$$subject\ to\ x \in K,$$

where $C \in \mathbb{R}^n$ is a pointed, closed and convex cone with $intC \neq \emptyset$.

Definition 9. *Suppose that $f : K \longrightarrow \mathbb{R}^n$ is a vector-valued mapping. A point $\bar{x} \in K$ is called*

(i) *an efficient solution (Pareto solution) of (P_1) if and only if*

$$f(y) - f(\bar{x}) \notin -C \setminus \{0\}, \ \ \forall y \in K;$$

(ii) *a weak efficient solution (weak Pareto solution) of (P_1) if and only if*

$$f(y) - f(\bar{x}) \notin -intC, \ \ \forall y \in K.$$

Now, we introduce following two kinds of generalized nonsmooth exponential-type vector variational-like inequality problems. Suppose that $K \neq \emptyset$ is a subset of an Asplund space X and $C \subset \mathbb{R}^n$ is a pointed, closed and convex cone such that $intC \neq \emptyset$. Assume that $f = (f_1, f_2, \cdots, f_n) : K \longrightarrow \mathbb{R}^n$ is a non-differentiable locally Lipschitz continuous mapping, $\eta, \theta : K \times K \longrightarrow X$ are the continuous mappings, β, p is an arbitrary real number and $e = (1, 1, \cdots, 1) \in \mathbb{R}^n$.

Problem 2. *Generalized nonsmooth exponential-type strong vector variational like inequality problem is to find a vector $\bar{x} \in K$ such that*

$$\left.\begin{aligned}
\tfrac{1}{p}\left\langle \xi; \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \beta\|\theta(y,\bar{x})\|^2 e \quad &\notin -C \setminus \{0\}, \ for \ p \neq 0, \\
\langle \xi; \eta(y,\bar{x})\rangle + \beta\|\theta(y,\bar{x})\|^2 e \quad &\notin -C \setminus \{0\}, \ for \ p = 0,
\end{aligned}\right\} \forall \xi \in \partial_L f(\bar{x}), y \in K; \qquad (P_2)$$

Problem 3. *Generalized nonsmooth exponential-type weak vector variational like inequality problem is to find a vector $\bar{x} \in K$ such that*

$$\left.\begin{array}{ll} \frac{1}{p}\left\langle \xi; \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \beta\|\theta(y,\bar{x})\|^2 e & \notin -intC, \ \text{for } p \neq 0, \\ \langle \xi; \eta(y,\bar{x})\rangle + \beta\|\theta(y,\bar{x})\|^2 e & \notin -intC, \ \text{for } p = 0, \end{array}\right\} \forall \xi \in \partial_L f(\bar{x}), y \in K. \qquad (P_3)$$

Special Cases:

(i) If $\theta \equiv 0$ and $\partial_L f(\cdot) = \partial f(\cdot)$, i.e., the Clarke subdifferential operator, then (P_2) and (P_3) reduces to nonsmooth exponential-type vector variational like inequality problem and nonsmooth exponential-type weak vector variational like inequality problem considered and studied by Jayswal and Choudhury [17].

(ii) For $p = 0$, a similar analogue of problems (P_2) and (P_3) was introduced and studied by Oveisiha and Zafarani [13].

Apparently, it shows that the solution of (P_2) is also a solution of (P_3). We construct the following example in support of (P_2).

Example 1. *Let us consider $X = \mathbb{R}$, $K = [-1, 1]$, $C = \mathbb{R}^2_+$, $p = 1$ and the mapping f be defined as $f = (f_1, f_2)$ by*

$$f_1(x) = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} x^2 + 2x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Now, the limiting subdifferential of f is

$$\partial_L f(x) = \begin{cases} (1, 2x+2), & \text{if } x > 0, \\ \{(s,t) : s \in [0,1], t \in [0,2]\}, & \text{if } x = 0, \\ (0,0), & \text{if } x < 0. \end{cases}$$

Define the mappings $\eta, \theta : K \times K \longrightarrow X$ by

$$\eta(y,x) = \ln(|y-x|+1) \ \text{ and } \ \theta(y,x) = \frac{y-x}{2}, \ \forall y, x \in K.$$

Then, the problem (P_2) is to find a point $\bar{x} \in K$ such that

$$\left\langle \xi; \left(\exp^{\eta(y,\bar{x})} - e\right)\right\rangle + \beta\|\theta(y,\bar{x})\|^2 e \notin -C \setminus \{0\}, \ \forall \xi \in \partial_L f(x), y \in K,$$

which is equivalent to say that

$$\left\langle \partial_L f(\bar{x}); \left(\exp^{\eta(y,\bar{x})} - e\right)\right\rangle + \beta\|\theta(y,\bar{x})\|^2 e \nsubseteq -C \setminus \{0\}, \ \forall \xi \in \partial_L f(x), y \in K.$$

For $\bar{x} = 0$ and $\beta \geq 4$, we can see that

$$\begin{aligned} & \left\langle \partial_L f(\bar{x}); \left(\exp^{\eta(y,\bar{x})} - e\right)\right\rangle + \beta\|\theta(y,\bar{x})\|^2 e \\ = & \left\{\left(s\left(\exp^{\ln(|y-x|+1)} - e\right), t\left(\exp^{\ln(|y-x|+1)} - e\right)\right) : s \in [0,1], t \in [0,2]\right\} + \beta\left\|\frac{y-\bar{x}}{2}\right\|^2 e \\ = & \{(s(|y-x|), t(|y-x|)) : s \in [0,1], t \in [0,2]\} + \frac{\beta}{4}\|y-\bar{x}\|^2 e \\ = & \{(s|y|, t|y|) : s \in [0,1], t \in [0,2]\} + \frac{\beta}{4}|y|^2 e \\ \nsubseteq & -C \setminus \{0\}. \end{aligned}$$

Hence, $\bar{x} = 0$ is the solution of the problem (P_2).

3. Main Results

Now, we prove a result which ensures that the solution of (P_2) is an efficient solution of (P_1).

Theorem 2. *Suppose that $K \neq \emptyset$ is a subset of Asplund space X, $C = \mathbb{R}^n_+$ and $f = (f_1, f_2, \cdots, f_n) : K \longrightarrow \mathbb{R}^n$ is a locally Lipschitz continous mapping on K. Let $\eta, \theta : K \times K \longrightarrow X$ be the mappings such that each f_i $(i = 1, 2, \cdots, n)$ is limiting (p, r)-α_i-(η, θ)-invex mapping with respect to η and θ. If $\bar{x} \in K$ is a solution of (P_2), then $\bar{x}$ is an efficient solution of (P_1).*

Proof. Assume that $\bar{x} \in K$ is a solution of (P_2). We will prove that $\bar{x} \in K$ is an efficient solution of (P_1). Indeed, let us assume that $\bar{x} \in K$ is not an efficient solution of (P_1). Then, $\exists y \in K$ such that

$$(f_1(y) - f_1(\bar{x}), f_2(y) - f_2(\bar{x}), \cdots, f_n(y) - f_n(\bar{x})) = f_i(y) - f_i(\bar{x}) \in -C \setminus \{0\},$$

which implies that

$$f_i(y) - f_i(\bar{x}) \leq 0, \ \ \forall i = 1, 2, \cdots, n, \tag{1}$$

and strict inequality holds for some $1 \leq k \leq n$.

Since $C = \mathbb{R}^n_+$, exponential mapping is monotonic and $r > 0$, then from (1), we have

$$\frac{1}{r}\left(\exp^{r(f_i(y) - f_i(\bar{x}))} - 1\right) \leq 0, \ \ \forall i = 1, 2, \cdots, n. \tag{2}$$

Since each f_i is limiting (p, r)-α_i-(η, θ)-invex mapping with respect to η and θ at $\bar{x}$, therefore for all $\xi_i \in \partial_L f_i(\bar{x})$, we have

$$\frac{1}{r}\left(\exp^{r(f_i(y) - f_i(\bar{x}))} - 1\right) \geq \frac{1}{p}\left\langle \xi_i; \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \alpha_i \|\theta(y, \bar{x})\|^2 e. \tag{3}$$

Set $\beta = \min\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, therefore from (3), we have

$$\frac{1}{r}\left(\exp^{r(f_i(y) - f_i(\bar{x}))} - 1\right) \geq \frac{1}{p}\left\langle \xi_i; \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \beta \|\theta(y, \bar{x})\|^2 e. \tag{4}$$

Now by using (2) and (4), we get

$$\frac{1}{p}\left\langle \xi_i; \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \beta \|\theta(y, \bar{x})\|^2 e \leq 0,$$

which implies that for all $\xi_i \in \partial_L f_i(\bar{x})$

$$\frac{1}{p}\left\langle \xi_i; \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \beta \|\theta(y, \bar{x})\|^2 e \in -C \setminus \{0\},$$

which counteracts the hypothesis that $\bar{x}$ is a solution of (P_2). Hence, $\bar{x}$ is an efficient solution of (P_1). This completes the proof. □

Next, we show the converse of the above conclusion.

Theorem 3. *Suppose that $f = (f_1, f_2, \cdots, f_n) : K \longrightarrow \mathbb{R}^n$ is a locally Lipschitz continuous mapping on K. If each $-f_i$ is limiting (p, r)-α_i-(η, θ)-invex mapping with respect to η and θ, and $\bar{x}$ is an efficient solution of (P_1), then $\bar{x}$ is a solution of (P_2).*

Proof. Assume that $\bar{x}$ is an efficient solution of (P_1). On contrary suppose that $\bar{x}$ is not a solution of (P_2). Then, each β ensures the existence of x_β satisfying

$$\frac{1}{p}\left\langle \xi_i; \left(\exp^{p\eta(x_\beta,\bar{x})} - e\right)\right\rangle + \beta\|\theta(x_\beta,\bar{x})\|^2 e \in -C \setminus \{0\},$$

for all $\xi_i \in \partial_L f_i(x_\beta)$. Since $C = \mathbb{R}^n_+$, from above relation, we have

$$\frac{1}{p}\left\langle \xi_i; \left(\exp^{p\eta(x_\beta,\bar{x})} - e\right)\right\rangle + \beta\|\theta(x_\beta,\bar{x})\|^2 e \leq 0, \tag{5}$$

and strict inequality holds for some $1 \leq k \leq n$.

As each $-f_i$ is limiting (p,r)-α_i-(η,θ)-invex mapping with respect to η and θ with constants α_i, therefore for any $y \in K$, $\exists \xi_i \in \partial_L f_i(y)$ such that

$$\frac{1}{r}\left(\exp^{r(-f_i(y)+f_i(\bar{x}))} - 1\right) \geq \frac{1}{p}\left\langle (-\xi_i); \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \alpha_i\|\theta(y,\bar{x})\|^2 e,$$

which implies that

$$\frac{1}{r}\left(\exp^{r(-f_i(y)+f_i(\bar{x}))} - 1\right) \geq \frac{1}{p}\left\langle (-\xi_i); \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \beta\|\theta(y,\bar{x})\|^2 e, \tag{6}$$

where $\beta = \min\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.

Using (5), (6) and monotonic property of exponential mapping, it is easy to deduce that $\exists y \in K$ such that

$$f_i(\bar{x}) - f_i(y) \geq 0,$$

and strict inequality holds for $i = k$ and equivalently

$$f_i(\bar{x}) - f_i(y) \in C \setminus \{0\},$$

which counteracts the hypothesis that $\bar{x}$ is an efficient solution of (P_1). Therefore, $\bar{x}$ is a solution of (P_2). This completes the proof. □

Based on equivalent arguments as used in Theorems 2 and 3, we have the following theorem which associates the problems (P_1) and (P_3).

Theorem 4. *Suppose that $K \neq \emptyset$ is a subset of Asplund space X, $C = \mathbb{R}^n_+$ and $f = (f_1, f_2, \cdots, f_n) : K \longrightarrow \mathbb{R}^n$ a locally Lipschitz continuous mapping on K. If each $-f_i$ $(1 \leq i \leq n)$ is strictly limiting (p,r)-α_i-(η,θ)-invex mapping with respect to η and θ and $\bar{x} \in K$ is a weak efficient solution of (P_1), then $\bar{x} \in K$ is also a solution of (P_3). Conversely, if each f_i $(1 \leq i \leq n)$ is limiting (p,r)-α_i-(η,θ)-invex mapping with respect to η and θ and $\bar{x} \in K$ is the solution of (P_3), then $\bar{x} \in K$ is also a weak efficient solution of (P_1).*

We contrive the following example in support of Theorem 4.

Example 2. *Let us consider $X = \mathbb{R}$, $K = [0,1]$, $C = \mathbb{R}^2_+$ and $p = 1$. Define the nonsmooth vector optimization problem*

$$\begin{aligned} &\min_C f(x) = (f_1(x), f_2(x)) \\ &\textit{subject to } x \in K, \end{aligned} \tag{7}$$

where $f_1(x) = \ln\left(x^2 + \sqrt{x} + 1\right)$ *and* $f_2(x) = \ln\left(x^2 + \frac{\sqrt{x}}{2}\right)$. *Clearly, f is locally Lipschitz mapping at* $x = 0$. *Now, the limiting subdifferential of f is as follows:*

$$\partial_L f(x) = \begin{cases} \left(\frac{2x+\frac{1}{2\sqrt{x}}}{x^2+\sqrt{x}+1}, \frac{4x+\frac{1}{2\sqrt{x}}}{2x^2+\sqrt{x}}\right), & \text{if } x > 0, \\ \{(s,t) : s,t \in [0,\infty)\}, & \text{if } x = 0. \end{cases}$$

Define the mappings $\theta, \eta : K \times K \longrightarrow X$ *by*

$$\eta(y,x) = \ln\left(-\frac{\sqrt{y}}{2} + x + 1\right) \;\; \text{and} \;\; \theta(y,x) = y - x, \;\; \forall y,x \in K.$$

For $r = 1$, *we can see that for* $\alpha = 1$ *at* $\bar{x} = 0$

$$\begin{aligned} & \left(\exp^{f_1(y)-f_1(\bar{x})} - 1\right) - \left\langle \xi_1; \left(\exp^{\eta(y,\bar{x})} - e\right)\right\rangle - \alpha\|\theta(y,\bar{x})\|^2 \\ = & \left(\exp^{\ln\left(\frac{y^2+\sqrt{y}+1}{\bar{x}^2+\sqrt{\bar{x}}+1}\right)} - 1\right) - \left\langle \xi_1; \left(\exp^{\ln\left(-\frac{\sqrt{y}}{2}+\bar{x}+1\right)} - e\right)\right\rangle - \|y - \bar{x}\|^2 \\ = & \left(\frac{y^2+\sqrt{y}+1}{\bar{x}^2+\sqrt{\bar{x}}+1} - 1\right) - \left\langle \xi_1; \left(-\frac{\sqrt{y}}{2} + \bar{x} + 1\right) - e\right\rangle - \|y - \bar{x}\|^2 \\ = & \left(y^2 + \sqrt{y}\right) + \xi_1\left(\frac{\sqrt{y}}{2}\right) - |y|^2 \\ = & y^2 + \sqrt{y}\left(1 + \frac{\xi_1}{2}\right) - |y|^2 \geq 0. \end{aligned}$$

Similarly, we can show that

$$\left(\exp^{f_2(y)-f_2(\bar{x})} - 1\right) - \left\langle \xi_2; \left(\exp^{\eta(y,\bar{x})} - e\right)\right\rangle - \alpha\|\theta(y,\bar{x})\|^2 \geq 0.$$

Therefore, f is $(1,1)$-1-(η,θ)*-invex mapping at* $\bar{x} = 0$.

Now, problem (P_3) *is to find* $\bar{x} \in [0,1]$ *such that*

$$\frac{1}{p}\left\langle \xi; \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \alpha\|\theta(y,\bar{x})\|^2 e \notin -intC, \;\; \forall \xi \in \partial_L f(x), y \in K,$$

which is analogous to the following problem

$$\frac{1}{p}\left\langle \partial_L f(\bar{x}); \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \alpha\|\theta(y,\bar{x})\|^2 e \not\subseteq -intC, \;\; \forall \xi \in \partial_L f(x), y \in K.$$

Now, for $\alpha = p = 1$, *we deduce that*

$$\begin{aligned} & \left\langle \partial_L f(\bar{x}); \left(\exp^{\eta(y,\bar{x})} - e\right)\right\rangle + \alpha\|\theta(y,\bar{x})\|^2 e \\ = & \left\{\left(s\left(\exp^{\ln(-\sqrt{y}-\bar{x}+1)} - e\right), t\left(\exp^{\ln(-\sqrt{y}-\bar{x}+1)} - e\right)\right) : s,t \in [0,\infty)\right\} + \|y - \bar{x}\|^2 e \\ = & \{(s(-\sqrt{y} - \bar{x}), t(-\sqrt{y} - \bar{x})) : s,t \in [0,\infty)\} + \|y\|^2 e \\ \not\subseteq & -intC. \end{aligned}$$

Therefore, $\bar{x} = 0$ *is the solution of the problem* (P_3). *One can easily show that* $\bar{x} = 0$ *is a weakly efficient solution of vector optimization problem* (7) *by using Theorem 4.*

Following is the existence theorem for the solution of generalized nonsmooth exponential-type weak vector variational like inequality problem (P_3) by employing the Fan-KKM Theorem.

Theorem 5. *Suppose that $K \neq \emptyset$ is a convex subset of Asplund space X, C is a pointed, closed and convex cone, and $f = (f_1, f_2, \cdots, f_n) : K \longrightarrow \mathbb{R}^n$ is a locally Lipschitz mapping such that each f_i $(1 \leq i \leq n)$ is limiting (p,r)-α_i-(η,θ)-invex mapping with respect to η and θ with constants α_i. Suppose that $\eta, \theta : K \times K \longrightarrow X$ are the continuous mappings which are affine in the first argument, respectively and $\eta(x,x) = 0 = \theta(x,x)$, for all $x \in K$. For any compact subset $B \neq \emptyset$ of K and $y_0 \in B$ with the property*

$$\frac{1}{p}\left\langle \xi; \left(\exp^{p\eta(y_0,x)} - e\right)\right\rangle + \beta\|\theta(y_0,x)\|^2 e \in -intC, \ \ \forall x \in K \setminus B, \ \xi \in \partial_L f(x), \tag{8}$$

where $\beta = \min\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, then generalized nonsmooth exponential-type weak vector variational like inequality problem (P_3) admits a solution.

Proof. For any $y \in K$, consider the mapping $F : K \longrightarrow 2^K$ define by

$$F(y) = \left\{ x \in K : \frac{1}{p}\left\langle \xi; \left(\exp^{p\eta(y,x)} - e\right)\right\rangle + \beta\|\theta(y,x)\|^2 e \notin -intC, \ \ \forall \xi \in \partial_L f(x) \right\},$$

Since $y \in F(y)$, therefore F is nonempty.

Now, we will prove that F is a KKM-mapping on K. On contrary, assume that F is not a KKM-mapping. Therefore, we can find a finite set $\{x_1, x_2, \cdots, x_n\}$ and $t_i \geq 0, i = 1,2,\cdots,n$ with $\sum_{i=1}^n t_i = 1$ such that

$$x_0 = \sum_{i=1}^n t_i x_i \notin \bigcup_{i=1}^n F(x_i),$$

which implies that $x_0 \notin F(x_i), \forall i = 1,2,\cdots,n$, i.e.,

$$\frac{1}{p}\left\langle \xi; \left(\exp^{p\eta(x_i,x_0)} - e\right)\right\rangle + \beta\|\theta(x_i,x_0)\|^2 e \in -intC, \ \ \forall i = 1,2,\cdots,n.$$

In view of convexity of $\left(\exp^{\lambda x} - e\right)$, for all $x \in \mathbb{R}$ and for any $\lambda > 0$, and affinity of η and θ in the first argument with the property $\eta(x,x) = 0 = \theta(x,x)$, we obtain

$$\begin{aligned}
0 &= \frac{1}{p}\left\langle \xi; \left(\exp^{p\eta(x_0,x_0)} - e\right)\right\rangle + \left(\frac{\beta}{\sum_{i=1}^n t_i}\right)\|\theta(x_0,x_0)\|^2 e \\
&= \frac{1}{p}\left\langle \xi; \left(\exp^{p\eta\left(\sum\limits_{i=1}^n t_i x_i, x_0\right)} - e\right)\right\rangle + \left(\frac{\beta}{\sum_{i=1}^n t_i}\right)\left\|\theta\left(\sum_{i=1}^n t_i x_i, x_0\right)\right\|^2 e \\
&= \frac{1}{p}\left\langle \xi; \left(\exp^{p\sum\limits_{i=1}^n t_i \eta(x_i,x_0)} - e\right)\right\rangle + \left(\frac{\beta}{\sum_{i=1}^n t_i}\right)\left\|\sum_{i=1}^n t_i \theta\left(x_i,x_0\right)\right\|^2 e \\
&\leq_C \frac{1}{p}\left\langle \xi; \sum_{i=1}^n t_i \left(\exp^{p\eta(x_i,x_0)} - e\right)\right\rangle + \beta\left(\sum_{i=1}^n t_i\right)\|\theta\left(x_i,x_0\right)\|^2 e \\
&= \frac{1}{p}\sum_{i=1}^n t_i \left\langle \xi; \left(\exp^{p\eta(x_i,x_0)} - e\right)\right\rangle + \beta\left(\sum_{i=1}^n t_i\right)\|\theta\left(x_i,x_0\right)\|^2 e \\
&\in -intC,
\end{aligned}$$

which implies that $0 \in -intC$ and hence, a contradiction. Therefore, F is a KKM-mapping.

Next, to show that $F(y)$ is closed set, for each $y \in K$, consider any sequence $\{x_n\}$ in $F(y)$ which converges to $\bar{x}$. This implies that

$$z_n = \frac{1}{p}\left\langle \xi_n; \left(\exp^{p\eta(y,x_n)} - e\right)\right\rangle + \beta\|\theta(y,x_n)\|^2 e \notin -intC, \forall \xi_n \in \partial_L f(x_n). \tag{9}$$

Using locally Lipschitz continuity property of f, we have

$$\|f(x) - f(y)\| \leq L\|x - y\|, \ \ \forall x, y \in N(\bar{x}),$$

where $L > 0$ is a constant and $N(\bar{x})$ is the neighbourhood of $\bar{x}$. Then, we can find any $x \in N(\bar{x})$ and $\xi \in \partial_L f(x)$ such that

$$\|\xi\| \leq L.$$

Since $\partial_L f(x_n)$ is w^*-compact, then the sequence $\{\xi_n\}$ has a convergent subsequence, say $\{\xi_m\}$ in $\partial_L f(x_n)$ such that $\xi_m \to \bar{\xi} \in \partial_L f(\bar{x})$. Since η and θ are continuous mappings, we have

$$\bar{z} = \lim_m z_m = \frac{1}{p}\left\langle \bar{\xi}; \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \beta\|\theta(y,\bar{x})\|^2 e.$$

From (9), it follows that $\bar{z} \in intC$ and therefore, we have

$$\frac{1}{p}\left\langle \bar{\xi}; \left(\exp^{p\eta(y,\bar{x})} - e\right)\right\rangle + \beta\|\theta(y,\bar{x})\|^2 e \notin intC.$$

Hence $\bar{x} \in F(y)$, and thus $F(y)$ is closed set.

Using the hypothesis (8), for any compact subset $B \neq \emptyset$ of K and $y_0 \in B$, we have

$$\frac{1}{p}\left\langle \xi; \left(\exp^{p\eta(y_0,x)} - e\right)\right\rangle + \beta\|\theta(y_0,x)\|^2 e \in -intC, \ \ \forall x \in K \setminus B, \ \xi \in \partial_L f(x),$$

which shows that $F(y_0) \in B$. Due to compactness of B, we have $F(y_0)$ is also compact. Therefore, by applying the Fan-KKM Theorem 1, we obtain

$$\bigcap_{y \in K} F(y) \neq \emptyset.$$

Therefore, $\exists \tilde{x} \in K$ such that

$$\frac{1}{p}\left\langle \bar{\xi}; \left(\exp^{p\eta(y,\tilde{x})} - e\right)\right\rangle + \beta\|\theta(y,\tilde{x})\|^2 e \notin -intC, \ \ \forall \xi \in \partial_L f(\tilde{x}).$$

Thus, generalized nonsmooth exponential-type weak vector variational like inequality problem (P_3) has a solution. This completes the proof. □

4. Conclusions

We have introduced and studied a new type of generalized nonsmooth exponential type vector variational-like inequality problem involving Mordukhovich limiting subdifferential operator in Asplund spaces. We proved the relationships between our considered problems with vector optimization problems using the generalized concept of invexity, which we called limiting (p,r)-α-(η,θ)-invexity of mappings. We also derived the existence of a result for our considered problem using the Fan-KKM theorem. It is remarked that our problems and related results are more general than the previously known results.

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