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Article

Prešić Type Nonself Operators and Related Best Proximity Results

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Abstract: The purpose of this article is to discuss the existence of best proximity points for Prešić-type nonself operators, say $T\colon A^k \to B$. We also give several examples to support our results. As a consequence of our results, we have provided some interesting formulations of Prešić fixed point results.

Keywords: best proximity point; equilibrium point; Prešić operators

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1. Introduction and Preliminaries

The k^{th} order nonlinear difference equation is of the form:

$$x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}) \tag{1}$$

where T is a continuous function from $I^k \subset \mathbb{R}^k$ into $I \subset \mathbb{R}$. A point $x \in I$ is an equilibrium point of (1) if $x = T(x, x, \ldots, x)$. The existence of the equilibrium point of a certain difference equation is of interest and has been extensively discussed in the literature; see for example Prešić [1]. On the other hand, Equation (1) appears in many iteration methods, for example the variational iteration method and the homotopy perturbation method [2,3].

In the literature of fixed point theory, the result of Prešić [1] is considered as one of the most important extensions of the Banach contraction principle for the operators defined on product spaces. This famous extension [1] was stated as: Let (X, d) be a complete metric space, k be a positive integer, and $T\colon X^k \to X$ be a mapping such that:

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{k} a_i d(x_i, x_{i+1}) \tag{2}$$

for every $x_1, x_2, \ldots, x_k, x_{k+1} \in X$, where $a_1, a_2, \ldots, a_k$ are nonnegative constants such that $\sum_{i=1}^{k} a_i < 1$. Then, there exists a unique point $x \in X$ such that $T(x, x, \ldots, x) = x$. Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for each $n \in \mathbb{N}$, we have:

$$x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}) \tag{3}$$

then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)$.

Later on, this result was further extended by Ćirić and Prešić [4] as: Let (X, d) be a complete metric space, k be a positive integer, and $T\colon X^k \to X$ be a mapping such that:

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}$$

for every $x_1, x_2, \ldots, x_k, x_{k+1} \in X$, where $\lambda \in (0,1)$. Then, there exists a point $x \in X$ such that $T(x, x, \ldots, x) = x$. Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for each $n \in \mathbb{N}$, we have:

$$x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}),$$

then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)$.

Note that, if the operator $T\colon X^k \to X$ satisfies (2), then it is considered as a Prešić-type operator [4], and the fixed point of such operators is an element x of X such that $T(x, x, \ldots, x) = x$. Furthermore, Equation (3) can be considered as the representation of the k^{th} order nonlinear difference equation. Thus, the fixed points of T are the equilibrium points of difference Equation (3). Therefore, the above stated results are taken as tools to ensure the existence and uniqueness of an equilibrium point of a k^{th} order nonlinear difference equation. To study some other forms of Prešić's result, we refer to the works of: Berinde and Păcurar [5], Khan et al. [6], Păcurar [7], and Shukla et al. [8,9].

The purpose of this paper is to study the existence of an approximate solution of the equation $x = T(x, x, \ldots, x)$, where $T\colon A^k \to B$. This equation may have a solution if A and B have some common elements, but when A and B have no common element, then the above equation has no solution; hence, in this case, we can only discuss the approximate solution of the equation. The approximate solution of the equation $x = T(x, x, \ldots, x)$ with the error term equal to $d(A, B)$ is called the best proximity point of $T\colon A^k \to B$.

The study of approximate solutions of $x = Tx$ was inspired by the classical result of approximation theory given by Fan [10] as: Let A be a nonempty compact convex subset of normed linear space X and $T\colon A \to X$ be a continuous function. Then, there exists $x \in A$ such that:

$$\|x - Tx\| = \inf_{a \in A}\{\|Tx - a\|\}.$$

In the literature, we have seen that the existence of the best proximity points has been investigated by several researchers by using different techniques, for example: Jleli and Samet [11] used α-ψ-proximal contraction to studied the best proximity points of single-valued mappings; Abkar and Gbeleh [12] used asymptotic cyclic contraction in their results; Abkar and Gbeleh [13] also proved the existence of best proximity points for multivalued nonself mappings satisfying contraction and nonexpansive condition along with P-property; Alghamdi et al. [14] studied the best proximity point theorems in geodesic metric spaces; Choudhury et al. [15] used the structure of partially-ordered metric spaces to discuss best proximity and couple best proximity points; Bari et al. [16] used cyclic Meir-Keeler contraction in their discussion; Eldred and Veeramani [17] used cyclic proximal contraction to discuss the existence of best proximity point in metric space, and they further provided an algorithm to calculate a best proximity point over the structure of a uniformly-convex Banach space; Jacob et al. [18] gave hybrid algorithms for nonself nonexpansive mappings and provided an iterative sequence of the algorithm, which converges to the proximity point of the mapping; Markin and Shahzad [19] studied the best proximity points of relatively u-continuous mappings; Sadiq Basha et al. [20] discussed the existence of best proximity points of two mappings satisfying the min-max condition; Shatanawi and Pitea [21] used the notions of P-property and weak P-property in their best proximity theorems; Vetro [22] gave the existence and convergence theorems for best proximity points of the mappings satisfying the p-cyclic ϕ-contraction.

We will use the following notions and definition in this article: Let (X, d) be a metric space and A, B be nonempty subsets of X, then $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$, $d(x, B) =$

$\inf\{d(x,b) : b \in B\}$, $A_0 = \{a \in A : d(a,b) = d(A,B)$ for some $b \in B\}$, and $B_0 = \{b \in B : d(a,b) = d(A,B)$ for some $a \in A\}$.

The following definition was introduced by Basha and Shahzad [23].

Definition 1 ([23])**.** *Let A and B be nonempty subsets of metric space (X,d). Then, B is said to be approximatively compact with respect to A, if each sequence $\{v_n\}$ in B with $d(x,v_n) \to d(x,B)$, for some x in A, has a convergent subsequence.*

2. Main Results

Throughout the article, we assume that $G = (V,E)$ is a directed graph defined on a metric space (X,d) such that the set of its vertices $V = X$ and the set of its edges contain all loops, but it has no parallel edge. Further, we say that for $x, y \in V$, we have a path from x to y, denoted by xPy, if we have $\{x_i : i = 1,2,\ldots,N\} \subseteq V$ with $x_1 = x$ and $x_N = y$ satisfying $(x_i, x_{i+1}) \in E$ for each $i = 1,2,\ldots N-1$.

Definition 2. *Let A and B be nonempty subsets of metric space (X,d) endowed with the above mentioned graph G. A mapping $T\colon A \times A \to B$ is said to be path admissible, if:*

$$\begin{cases} d(u_1, T(a_1,a_2)) = d(A,B) \\ d(u_2, T(a_2,a_3)) = d(A,B) \\ a_1 P a_3 \end{cases} \Rightarrow (u_1,u_2) \in E$$

where $a_1, a_2, a_3, u_1, u_2 \in A$. Here, by a_1Pa_3, we mean, for the above-mentioned $a_1, a_2, a_3 \in V$, we have $(a_1,a_2) \in E$ and $(a_2,a_3) \in E$.

We now state and prove the first result of the article:

Theorem 1. *Let A and B be nonempty closed subsets of a complete metric space (X,d) endowed with the graph G. Let $T\colon A \times A \to B$ be a mapping such that for each a_1, a_2, a_3, u_1, u_2 in A with a_1Pa_3, that is, $(a_1,a_2), (a_2,a_3) \in E$, and $d(u_1, T(a_1,a_2)) = d(A,B) = d(u_2, T(a_2,a_3))$, we have:*

$$d(u_1,u_2) \leq \gamma \max\{d(a_1,a_2), d(a_2,a_3)\} \tag{4}$$

where $\gamma \in [0,1)$. Further, assume that the following conditions hold:

(i) *T is path admissible;*
(ii) *there exist $a_0, a_1, a_2 \in A$ satisfying $d(a_2, T(a_0,a_1)) = d(A,B)$ and a_0Pa_2;*
(iii) *A_0 is nonempty;*
(iv) *$T(A \times A_0) \subseteq B_0$;*
(v) *B is approximately compact with respect to A;*
(vi) *if $\{a_n\}$ is a sequence in X such that a_nPa_{n+2} for each $n \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$, then $(a_n, x) \in E$ for each $n \in \mathbb{N}$ and $(x,x) \in E$.*

Then, T has a best proximity point, that is there exists $u^* \in A$ satisfying $d(u^*, T(u^*,u^*)) = d(A,B)$.

Proof. Hypothesis (ii) implies that we have $a_0, a_1, a_2 \in A$ satisfying $d(a_2, T(a_0,a_1)) = d(A,B)$ and a_0Pa_2, that is, $(a_0,a_1), (a_1,a_2) \in E$. By using Hypothesis (iv), we have $T(a_1,a_2) \in B_0$, and by the definition of B_0, we have $a_3 \in A_0$ satisfying $d(a_3, T(a_1,a_2)) = d(A,B)$. Since the mapping T is path admissible, hence we have $(a_2,a_3) \in E$. Thus, a_1Pa_3. By considering the same arguments further, we construct a sequence $\{a_n\}_{n\geq 2}$ in A_0 satisfying:

$$d(a_{n+1}, T(a_{n-1},a_n)) = d(A,B) \text{ for each } n \in \mathbb{N}$$

and:

$$a_{n-1}Pa_{n+1}, \text{ that is } (a_{n-1}, a_n), (a_n, a_{n+1}) \in E, \text{ for each } n \in \mathbb{N}.$$

From (4), we have:

$$d(a_n, a_{n+1}) \leq \gamma \max\{d(a_{n-2}, a_{n-1}), d(a_{n-1}, a_n)\} \text{ for each } n = 2, 3, 4, \ldots. \tag{5}$$

For convenience, we take $d_n = d(a_n, a_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By using induction, we can get:

$$d_{n-1} \leq Z\psi^n \text{ for each } n \in \mathbb{N} \tag{6}$$

where $\psi = \gamma^{1/2}$ and $Z = \max\{d_0/\psi, d_1/\psi^2\}$. Clearly, $d_0 \leq Z\psi$ and $d_1 \leq Z\psi^2$. We obtain:

$$\begin{aligned} d_2 &\leq \gamma \max\{d_0, d_1\} \leq \gamma \max\{Z\psi, Z\psi^2\} \leq \gamma Z\psi = Z\psi^3. \\ &\vdots \\ d_m &\leq \gamma \max\{d_{m-1}, d_{m-2}\} \leq \gamma \max\{Z\psi^m, Z\psi^{m-1}\} \leq \gamma Z\psi^{m-1} = Z\psi^{m+1}. \end{aligned}$$

Thus, $d_{n-1} \leq Z\psi^n$ for each $n \in \mathbb{N}$. By using the triangle inequality, for each $m, q \in N$, we have:

$$\begin{aligned} d(a_m, a_{m+q}) &\leq d(a_m, a_{m+1}) + d(a_{m+1}, a_{m+2}) + \cdots + d(a_{m+q-1}, a_{m+q}) \\ &\leq Z\psi^{m+1} + Z\psi^{m+2} + \cdots + Z\psi^{m+q} \\ &< \frac{\psi^{m+1}}{1-\psi} Z. \end{aligned}$$

Note that $\psi = \gamma^{1/2} < 1$. Therefore, $\{a_n\}$ is a Cauchy sequence in a closed subset A of the complete metric space X. Then, there is a point a^* in A such that $a_n \to a^*$. Furthermore,

$$\begin{aligned} d(a^*, B) &\leq d(a^*, T(a_{n-1}, a_n)) \\ &\leq d(a^*, a_{n+1}) + d(a_{n+1}, T(a_{n-1}, a_n)) \\ &= d(a^*, a_{n+1}) + d(A, B) \\ &\leq d(a^*, a_{n+1}) + d(a^*, B). \end{aligned} \tag{7}$$

Therefore, $d(a^*, T(a_{n-1}, a_n)) \to d(x^*, B)$ as $n \to \infty$. Since B is approximatively compact with respect to A, the sequence $\{T(a_{n-1}, a_n)\}$ has a subsequence $\{T(a_{n_k-1}, a_{n_k})\}$, which converges to a point b^* in B. This implies that:

$$d(a^*, b^*) = \lim_{k\to\infty} d(a_{n_{k+1}}, T(a_{n_k-1}, a_{n_k})) = d(A, B).$$

Hence, $a^* \in A_0$. As we know $T(a_n, a^*) \in B_0$, we have $u \in A$ satisfying $d(u, T(a_n, a^*)) = d(A, B)$. By Hypothesis (vi), we have $(a_n, a^*) \in E$ for each $n \in \mathbb{N}$. Thus, we get $a_{n-1}Pa^*$, that is $(a_{n-1}, a_n), (a_n, a^*) \in E$, for each $n \in \mathbb{N}$. Hence, from (4), we get:

$$d(a_{n+1}, u) \leq \gamma \max\{d(a_{n-1}, a_n), d(a_n, a^*)\} \text{ for each } n \in \mathbb{N}.$$

Applying the limit when n tends to infinity in the above inequality, we get $d(a^*, u) = 0$, that is $u = a^*$. Furthermore, note that $T(a^*, a^*) \in B_0$, and there is $s \in A$ satisfying $d(s, T(a^*, a^*)) = d(A, B)$. By Hypothesis (vi), we further have $(a^*, a^*) \in E$. Hence, we have $d(a^*, T(a_n, a^*)) = d(A, B)$, $d(s, T(a^*, a^*)) = d(A, B)$, and a_nPa^*, that is, $(a_n, a^*) \in E$ and $(a^*, a^*) \in E$. Thus, from (4), we get:

$$d(a^*, s) \leq \gamma \max\{d(a_n, a^*), d(a^*, a^*)\} \text{ for each } n \in \mathbb{N}.$$

By taking limit as n tends to infinity in the above inequality, we get $d(a^*, s) = 0$, that is $s = a^*$. Thus, we have $d(a^*, T(a^*, a^*)) = d(A, B)$. □

Example 1. *Let $X = \mathbb{R} \times \mathbb{R}$ be endowed with a metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for each $x, y \in X$ and a graph G be defined as $V = X$ and $E = \{((x_1, x_2), (y_1, y_2)) : x_1, x_2, y_1, y_2 \in [0, 1]\} \cup \{(x, x) : x \in X\}$. Take $A = \{(0, x) : x \in [-2, 2]\}$ and $B = \{(1, x) : x \in [-2, 2]\}$. Define:*

$$T \colon A \times A \to B, \quad T((0,x),(0,y)) = \begin{cases} (1, \frac{x+y+2}{4}) & \text{if } x, y \geq 0 \\ (1, |x+y| - 2) & \text{otherwise.} \end{cases}$$

Then, for each $\overline{a_1} = (0, a_1), \overline{a_2} = (0, a_2), \overline{a_3} = (0, a_3), \overline{u_1} = (0, u_1) = (0, \frac{a_1+a_2+2}{4}), \overline{u_2} = (0, u_2) = (0, \frac{a_2+a_3+2}{4}) \in A$ with $\overline{a_1} P \overline{a_3}$ and $d(\overline{u_1}, T(\overline{a_1}, \overline{a_2})) = d(A, B) = d(\overline{u_2}, T(\overline{a_2}, \overline{a_3}))$, we have:

$$d(\overline{u_1}, \overline{u_2}) = \frac{1}{4}|a_1 - a_3| = \gamma \max\{d(\overline{a_1}, \overline{a_2}), d(\overline{a_2}, \overline{a_3})\}$$

where $\gamma = \frac{1}{2}$. Consider $\overline{a_1} = (0, a_1), \overline{a_2} = (0, a_2), \overline{a_3} = (0, a_3) \in A$ such that $\overline{a_1} P \overline{a_3}$ and $d((0, u_1), T((0, a_1), (0, a_2))) = d(A, B) = d((0, u_2), T((0, a_2), (0, a_3)))$, then $((0, u_1), (0, u_2)) \in E$, since $(0, u_1) = (0, \frac{a_1+a_2+2}{4})$ and $(0, u_2) = (0, \frac{a_2+a_3+2}{4})$. Thus, T is path admissible. We also have $\overline{a_1} = (0, 0)$, $\overline{a_2} = (0, 1/2)$, $\overline{a_3} = (0, 5/8) \in A$ such that $d((0, 5/8), T((0, 0), (0, 1/2))) = d(A, B)$ and $\overline{a_1} P \overline{a_3}$. Moreover, B is approximately compact with respect to A and for each sequence $\{a_n\}$ in X such that $a_n P a_{n+2}$ for each $n \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$, then $(a_n, x) \in E$ for each $n \in \mathbb{N}$ and $(x, x) \in E$. Hence, all the conditions of Theorem 1, are satisfied. Therefore, T has a best proximity point.

Theorem 2. *Let A and B be nonempty closed subsets of a complete metric space (X, d) endowed with the graph G. Let $T \colon A \times A \to B$ be a mapping such that for each $a_1, a_2, a_3, u_1, u_2 \in A$ with $a_1 P a_3$, that is $(a_1, a_2), (a_2, a_3) \in E$, and $d(u_1, T(a_1, a_2)) = d(A, B) = d(u_2, T(a_2, a_3))$, we have:*

$$d(a_3, u_2) \leq \gamma \max\{d(a_1, a_2), d(a_2, u_1)\} \tag{8}$$

where $\gamma \in [0, 1)$. Further, assume that the following conditions hold:

- *(i)* *T is path admissible;*
- *(ii)* *there exist $a_0, a_1, a_2 \in A$ satisfying $d(a_2, T(a_0, a_1)) = d(A, B)$ and $a_0 P a_2$;*
- *(iii)* *A_0 is nonempty;*
- *(iv)* *$T(A \times A_0) \subseteq B_0$;*
- *(v)* *B is approximately compact with respect to A;*
- *(vi)* *if $\{a_n\}$ is a sequence in X such that $a_n P a_{n+2}$ for each $n \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$, then $(a_n, x) \in E$ for each $n \in \mathbb{N}$ and $(x, x) \in E$.*

Then, T has a best proximity point, that is there exists $u^* \in A$ satisfying $d(u^*, T(u^*, u^*)) = d(A, B)$.

Proof. Following the proof of the above theorem, we will construct a sequence $\{a_n : n \in \mathbb{N} \setminus \{1\}\}$ in A_0 satisfying:

$$d(a_{n+1}, T(a_{n-1}, a_n)) = d(A, B) \text{ for each } n \in \mathbb{N}$$

and:

$$a_{n-1} P a_{n+1}, \text{ that is } (a_{n-1}, a_n), (a_n, a_{n+1}) \in E, \text{ for each } n \in \mathbb{N}.$$

From (8), we have:

$$d(a_n, a_{n+1}) \leq \gamma \max\{d(a_{n-2}, a_{n-1}), d(a_{n-1}, a_n)\} \text{ for each } n = 2, 3, 4, \ldots. \tag{9}$$

Following the above inequality and the proof of Theorem 1, we conclude that $\{a_n\}$ is a Cauchy sequence in A such that $a_n \to a^*$ and $a^* \in A_0$. As $T(a_n, a^*) \in B_0$, we have $u \in A$ satisfying

$d(u, T(a_n, a^*)) = d(A, B)$. By Hypothesis (vi), we have $(a_n, a^*) \in E$ for each $n \in \mathbb{N}$. Thus, we get $a_{n-1}Pa^*$, that is $(a_{n-1}, a_n), (a_n, a^*) \in E$, for each $n \in \mathbb{N}$. Hence, from (8), we get:

$$d(a^*, u) \leq \gamma \max\{d(a_{n-1}, a_n), d(a_n, a_{n+1})\} \text{ for each } n \in \mathbb{N}.$$

Applying the limit when n tends to infinity in the above inequality, we get $d(a^*, u) = 0$, that is $u = a^*$. Furthermore, note that $T(a^*, a^*) \in B_0$, and there is $s \in A$ satisfying $d(s, T(a^*, a^*)) = d(A, B)$. By Hypothesis (vi), we further have $(a^*, a^*) \in E$. Hence, we have $d(a^*, T(a_n, a^*)) = d(A, B)$, $d(s, T(a^*, a^*)) = d(A, B)$, a_nPa^*, that is $(a_n, a^*) \in E$ and $(a^*, a^*) \in E$. Thus, from (8), we get:

$$d(a^*, s) \leq \gamma \max\{d(a_n, a^*), d(a^*, a^*)\} \text{ for each } n \in \mathbb{N}.$$

By taking the limit as n tends to infinity in the above inequality, we get $d(a^*, s) = 0$, that is $s = a^*$. Thus, we have $d(a^*, T(a^*, a^*)) = d(A, B)$. □

Example 2. *Let $X = \mathbb{R} \times \mathbb{R}$ be endowed with a metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for each $x, y \in X$ and a graph G be defined as $V = X$ and $E = \{((x_1, x_2), (y_1, y_2)) : x_1, x_2, y_1, y_2 \in [0, 1]\} \cup \{(x, x) : x \in X\}$. Take $A = \{(0, x) : x \in [-2, 2]\}$ and $B = \{(1, x) : x \in [-2, 2]\}$. Define:*

$$T\colon A \times A \to B, \quad T((0, x), (0, y)) = (1, y) \text{ for each } (0, x), (0, y) \in A.$$

Then, for each $\overline{a_1} = (0, a_1), \overline{a_2} = (0, a_2), \overline{a_3} = (0, a_3), \overline{u_1} = (0, u_1) = (0, a_2), \overline{u_2} = (0, u_2) = (0, a_3) \in A$ with $\overline{a_1}P\overline{a_3}$ and $d(\overline{u_1}, T(\overline{a_1}, \overline{a_2})) = d(A, B) = d(\overline{u_2}, T(\overline{a_2}, \overline{a_3}))$, we have:

$$d(\overline{a_3}, \overline{u_2}) = 0 \leq \gamma \max\{d(\overline{a_1}, \overline{a_2}), d(\overline{a_2}, \overline{u_1})\}$$

where $\gamma = \frac{1}{2}$. The rest of the conditions of Theorem 2 are obviously fulfilled. Thus, T has a best proximity point.

Remark 1. *Note that Theorem 1 is not applicable on the above example. To see this, use $\overline{a_1} = (0, \frac{5}{8})$, $\overline{a_2} = (0, \frac{1}{2})$ and $\overline{a_3} = (0, 0)$ in (4).*

Theorem 3. *Let A and B be nonempty closed subsets of a complete metric space (X, d) endowed with the graph G. Let $T\colon A \times A \to B$ be a mapping such that for each a_1, a_2, a_3, u_1, u_2 in A with a_1Pa_3, that is $(a_1, a_2), (a_2, a_3) \in E$, and $d(u_1, T(a_1, a_2)) = d(A, B) = d(u_2, T(a_2, a_3))$, we have:*

$$d(T(a_2, u_1), T(a_3, u_2)) \leq \gamma d(T(a_1, a_2), T(a_2, a_3)) \tag{10}$$

where $\gamma \in [0, 1)$. Further, assume that the following conditions hold:

(i) *T is path admissible;*
(ii) *there exist $a_0, a_1, a_2 \in A$ satisfying $d(a_2, T(a_0, a_1)) = d(A, B)$ and a_0Pa_2;*
(iii) *A_0 is nonempty;*
(iv) *$T(A \times A_0) \subseteq B_0$;*
(v) *A is approximately compact with respect to B;*
(vi) *if $\{a_n\}$ and $\{\overline{a}_n\}$ are sequences in X such that $a_n \to a$ and $\overline{a}_n \to \overline{a}$, then $T(a_n, \overline{a}_n) \to T(a, \overline{a})$.*

Then, T has a best proximity point, that is there exists $u^* \in A$ satisfying $d(u^*, T(u^*, u^*)) = d(A, B)$.

Proof. Based on a similar argument to the one used in the proof of Theorem 1, we will construct a sequence $\{a_n : n \in \mathbb{N} \setminus \{1\}\}$ in A_0 satisfying:

$$d(a_{n+1}, T(a_{n-1}, a_n)) = d(A, B) \text{ for each } n \in \mathbb{N}$$

and:

$$a_{n-1}Pa_{n+1}, \text{ that is } (a_{n-1}, a_n), (a_n, a_{n+1}) \in E, \text{ for each } n \in \mathbb{N}.$$

From (10), we have:

$$d(T(a_{n-1}, a_n), T(a_n, a_{n+1})) \leq \gamma d(T(a_{n-2}, a_{n-1}), T(a_{n-1}, a_n)) \text{ for each } n = 2, 3, 4, \ldots.$$

Inductively, we get:

$$d(T(a_{n-1}, a_n), T(a_n, a_{n+1})) \leq \gamma^{n-1} d(T(a_0, a_1), T(a_1, a_2)) \text{ for each } n = 2, 3, 4, \ldots.$$

Based on the triangle inequality, from the above inequality, for each $m, p \in \mathbb{N}$, we get:

$$d(T(a_m, a_{m+1}), T(a_{m+p}, a_{m+p+1})) \leq \sum_{i=m}^{m+p-1} d(T(a_i, a_{i+1}), T(a_{i+1}, a_{i+2}))$$

This proves that $\{T(a_{n-1}, a_n)\}$ is a Cauchy sequence in the closed subset B of a complete space X. Then, there is a point b^* in B such that $T(a_{n-1}, a_n) \to b^*$. Furthermore, we have:

$$\begin{aligned} d(b^*, A) &\leq d(b^*, a_{n+1}) \\ &\leq d(b^*, T(a_{n-1}, a_n)) + d(T(a_{n-1}, a_n), a_{n+1}) \\ &= d(b^*, T(a_{n-1}, a_n)) + d(A, B) \\ &\leq d(b^*, T(a_{n-1}, a_n)) + d(b^*, A). \end{aligned} \tag{11}$$

Therefore, $d(b^*, a_{n+1}) \to d(b^*, A)$ as $n \to \infty$. Since A is approximatively compact with respect to B, the sequence $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to a point a^* in A. This implies that:

$$d(a^*, T(a^*, a^*)) = \lim_{k\to\infty} d(a_{n_{k+1}}, T(a_{n_k - 1}, a_{n_k})) = d(A, B),$$

and the proof is complete. □

Example 3. *Let $X = \mathbb{R} \times \mathbb{R}$ be endowed with a metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for each $x, y \in X$ and a graph G be defined as $V = X$ and $E = X \times X$. Take $A = \{(0, x) : x \in [-2, 2]\}$ and $B = \{(1, x) : x \in [-2, 2]\}$. Define:*

$$T\colon A \times A \to B, \quad T((0, x), (0, y)) = \left(1, \frac{y}{2}\right) \text{ for each } (0, x), (0, y) \in A.$$

Then, for each $\overline{a_1} = (0, a_1), \overline{a_2} = (0, a_2), \overline{a_3} = (0, a_3), \overline{u_1} = (0, u_1) = (0, \frac{a_2}{2}), \overline{u_2} = (0, u_2) = (0, \frac{a_3}{2}) \in A$ with $d(\overline{u_1}, T(\overline{a_1}, \overline{a_2})) = d(A, B) = d(\overline{u_2}, T(\overline{a_2}, \overline{a_3}))$, we have:

$$\begin{aligned} d(T(\overline{a_2}, \overline{u_1}), T(\overline{a_3}, \overline{u_2})) &= d\left(\left(1, \frac{a_2}{4}\right), \left(1, \frac{a_3}{4}\right)\right) \\ &= \frac{1}{4}|a_2 - a_3| \\ &= \frac{1}{2} d\left(\left(1, \frac{a_2}{2}\right), \left(1, \frac{a_3}{2}\right)\right) \\ &= \gamma d(T(a_1, a_2), T(a_2, a_3)). \end{aligned}$$

where $\gamma = \frac{1}{2}$. One can easily check that the remaining conditions of Theorem 3 are also satisfied. Thus, T has a best proximity point.

Theorem 4. *Let A and B be nonempty closed subsets of a complete metric space (X, d) endowed with the graph G. Let $T\colon A \times A \to B$ be a mapping such that for each $a_1, a_2, a_3, u_1, u_2 \in A$ with a_1Pa_3, that is $(a_1, a_2), (a_2, a_3) \in E$, and $d(u_1, T(a_1, a_2)) = d(A, B) = d(u_2, T(a_2, a_3))$, we have:*

$$d(T(a_2,u_1),T(a_3,u_2)) \leq \gamma \max\{d(T(a_1,a_2),T(a_2,a_3)), d(T(a_2,a_3),T(u_1,u_2))\} \tag{12}$$

where $\gamma \in [0,1)$. Further, assume that the following conditions hold:

- *(i)* *T is path admissible;*
- *(ii)* *there exist $a_0, a_1, a_2 \in A$ satisfying $d(a_2, T(a_0,a_1)) = d(A,B)$ and $a_0 P a_2$;*
- *(iii)* *A_0 is nonempty;*
- *(iv)* *$T(A \times A_0) \subseteq B_0$;*
- *(v)* *A is approximately compact with respect to B;*
- *(vi)* *if $\{a_n\}$ and $\{\overline{a}_n\}$ are sequences in X such that $a_n \to a$ and $\overline{a}_n \to \overline{a}$, then $T(a_n, \overline{a}_n) \to T(a, \overline{a})$.*

Then, T has a best proximity point, that is there exists $u^* \in A$ satisfying $d(u^*, T(u^*,u^*)) = d(A,B)$.

Proof. Using the hypothesis of the theorem, we will construct a sequence $\{a_n : n \in \mathbb{N} \setminus \{1\}\}$ in A_0 satisfying:

$$d(a_{n+1}, T(a_{n-1}, a_n)) = d(A,B) \text{ for each } n \in \mathbb{N}$$

and:

$$a_{n-1} P a_{n+1}, \text{ that is } (a_{n-1}, a_n), (a_n, a_{n+1}) \in E, \text{ for each } n \in \mathbb{N}.$$

From (12), we have:

$$\begin{aligned} d(T(a_{n-1},a_n),T(a_n,a_{n+1})) &\leq \gamma \max\{d(T(a_{n-2},a_{n-1}),T(a_{n-1},a_n)), \\ &\quad d(T(a_{n-1},a_n),T(a_n,a_{n+1}))\} \\ &= \gamma d(T(a_{n-2},a_{n-1}),T(a_{n-1},a_n)) \text{ for each } n = 2,3,4,\ldots \end{aligned}$$

otherwise, we have a contradiction. Iteratively, we get:

$$d(T(a_{n-1},a_n),T(a_n,a_{n+1})) \leq \gamma^{n-1} d(T(a_0,a_1),T(a_1,a_2)) \text{ for each } n = 2,3,4,\ldots.$$

The rest of the proof is similar to the proof of Theorem 3. □

Theorem 5. *Let A and B be nonempty closed subsets of a complete metric space (X,d) endowed with the graph G. Let $T\colon A \times A \to B$ be a mapping such that for each a_1, a_2, a_3, u_1, u_2 in A with $a_1 P a_3$, that is $(a_1,a_2),(a_2,a_3) \in E$, and $d(u_1, T(a_1,a_2)) = d(A,B) = d(u_2, T(a_2,a_3))$, we have:*

$$d(T(a_2,a_3),T(u_1,u_2)) \leq \gamma \max\{d(T(a_1,a_2),T(a_2,a_3)), d(T(a_2,u_1),T(a_3,u_2))\} \tag{13}$$

where $\gamma \in [0,1)$. Further, assume that the following conditions hold:

- *(i)* *T is path admissible;*
- *(ii)* *there exist $a_0, a_1, a_2 \in A$ satisfying $d(a_2, T(a_0,a_1)) = d(A,B)$ and $a_0 P a_2$;*
- *(iii)* *A_0 is nonempty;*
- *(iv)* *$T(A \times A_0) \subseteq B_0$;*
- *(v)* *A is approximately compact with respect to B;*
- *(vi)* *if $\{a_n\}$ and $\{\overline{a}_n\}$ are sequences in X such that $a_n \to a$ and $\overline{a}_n \to \overline{a}$, then $T(a_n, \overline{a}_n) \to T(a, \overline{a})$.*

Then, T has a best proximity point, that is there exists $u^* \in A$ satisfying $d(u^*, T(u^*,u^*)) = d(A,B)$.

Proof. This theorem can be proven in a similar way to the proof of Theorem 4. □

3. Further Extension of the Main Results

In this section, we will extend the above-mentioned results for the operators that map from A^k into B, where k is any natural number.

Theorem 6. *Let A and B be nonempty closed subsets of a complete metric space (X, d) endowed with the graph G. Let $T\colon A^k \to B$ be a mapping such that for each $a_1, a_2, a_3, \ldots, a_k, a_{k+1}, u_1, u_2 \in A$ with $a_1 P a_{k+1}$, that is $(a_1, a_2), (a_2, a_3), \ldots, (a_k, a_{k+1}) \in E$, and $d(u_1, T(a_1, a_2, \ldots, a_k)) = d(u_2, T(a_2, a_3, \ldots, a_{k+1})) = d(A, B)$, satisfies one of the following inequalities:*

$$d(u_1, u_2) \leq \gamma \max\{d(a_i, a_{i+1}) : 1 \leq i \leq k\};$$

$$d(a_{k+1}, u_2) \leq \gamma \max\{d(a_i, a_{i+1}) : 1 \leq i \leq k-1, d(a_k, u_1)\},$$

where $\gamma \in [0, 1)$. Further, assume that the following conditions hold:

(i) *T is path admissible;*
(ii) *there exist $a_0, a_1, a_2, \ldots, a_k \in A$ satisfying $d(a_k, T(a_0, a_1, \ldots, a_{k-1})) = d(A, B)$ and $a_0 P a_k$;*
(iii) *A_0 is nonempty;*
(iv) *$T(A^{k-1} \times A_0) \subseteq B_0$;*
(v) *B is approximately compact with respect to A;*
(vi) *if $\{a_n\}$ is a sequence in X such that $a_n P a_{n+k}$ for each $n \in \mathbb{N}$ and $a_n \to x$, then $(a_n, x) \in E$ for each $n \in \mathbb{N}$ and $(x, x) \in E$.*

Then, T has a best proximity point, that is there exists $u^* \in A$ satisfying $d(u^*, T(u^*, u^*, \ldots, u^*)) = d(A, B)$.

Proof. This theorem can be proven similarly to Theorems 1 and 2. □

Theorem 7. *Let A and B be nonempty closed subsets of a complete metric space (X, d) endowed with the graph G. Let $T\colon A^k \to B$ be a mapping such that for each $a_1, a_2, a_3, \ldots, a_k, a_{k+1}, u_1, u_2 \in A$ with $a_1 P a_{k+1}$, that is $(a_1, a_2), (a_2, a_3), \ldots, (a_k, a_{k+1}) \in E$, and $d(u_1, T(a_1, a_2, \ldots, a_k)) = d(u_2, T(a_2, a_3, \ldots, a_{k+1})) = d(A, B)$, satisfies one of the following inequalities:*

$$d(T(a_2, \ldots, a_k, u_1), T(a_3, \ldots, a_{k+1}, u_2)) \leq \gamma d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1}));$$

$$\begin{aligned} d(T(a_2, \ldots, a_k, u_1), T(a_3, \ldots, a_{k+1}, u_2)) \quad \leq \quad & \gamma \max\{d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})), \\ & d(T(a_2, a_3, \ldots, a_{k+1}), T(a_4, a_5, \ldots, a_{k+1}, u_1, u_2))\}; \end{aligned}$$

$$\begin{aligned} & d(T(a_2, a_3, \ldots, a_{k+1}), T(a_4, a_5, \ldots, a_{k+1}, u_1, u_2)) \\ & \quad \leq \gamma \max\{d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})), \\ & \qquad d(T(a_2, \ldots, a_k, u_1), T(a_3, \ldots, a_{k+1}, u_2))\}, \end{aligned}$$

where $\gamma \in [0, 1)$. Further, assume that the following conditions hold:

(i) *T is path admissible;*
(ii) *there exist $a_0, a_1, a_2, \ldots, a_k \in A$ satisfying $d(a_k, T(a_0, a_1, \ldots, a_{k-1})) = d(A, B)$ and $a_0 P a_k$;*
(iii) *A_0 is nonempty;*
(iv) *$T(A^{k-1} \times A_0) \subseteq B_0$;*
(v) *A is approximately compact with respect to B;*
(vi) *T is continuous with respect to each coordinate.*

Then, T has a best proximity point, that is there exists $u^* \in A$ satisfying $d(u^*, T(u^*, u^*, \ldots, u^*)) = d(A, B)$.

Proof. This theorem can be proven similarly to Theorems 3 and 4. □

Remark 2. *Note that $T\colon A^k \to B$ is path admissible, if for each $a_1, a_2, a_3, \ldots, a_k, a_{k+1}$, $u_1, u_2 \in A$ with $a_1 P a_{k+1}$, that is $(a_1, a_2), (a_2, a_3), \ldots, (a_k, a_{k+1}) \in E$ and $d(u_1, T(a_1, a_2, \ldots, a_k)) = d(u_2, T(a_2, a_3, \ldots, a_{k+1})) = d(A, B)$, we have $(u_1, u_2) \in E$*

4. Consequences

Considering $A = B = X$ in Theorems 6 and 7, then we obtain the following theorems, which ensure the existence of fixed points of the operator $T\colon X^k \to X$.

Theorem 8. *Let (X, d) be a complete metric space endowed with the graph G. Let $T\colon X^k \to X$ be a mapping such that for each $a_1, a_2, a_3, \ldots, a_k, a_{k+1} \in A$ with $a_1 P a_{k+1}$, that is (a_1, a_2), $(a_2, a_3), \ldots, (a_k, a_{k+1}) \in E$, satisfies one of the following inequalities:*

$$d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})) \leq \gamma \max\{d(a_i, a_{i+1}) : 1 \leq i \leq k\}$$

$$d(a_{k+1}, T(a_2, a_3, \ldots, a_{k+1})) \leq \gamma \max\{d(a_i, a_{i+1}) : 1 \leq i \leq k-1, d(a_k, T(a_1, a_2, \ldots, a_k))\}$$

where $\gamma \in [0, 1)$. Further, assume that the following conditions hold:

(i) *If $a_1 P a_{k+1}$, that is $(a_1, a_2), (a_2, a_3), \ldots, (a_k, a_{k+1}) \in E$, then we have:*

$$(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})) \in E;$$

(ii) *there exist $a_0, a_1, a_2, \ldots, a_k \in X$ with $a_k = T(a_0, a_1, \ldots, a_{k-1})$ and $a_0 P a_k$;*

(vi) *if $\{a_n\}$ is a sequence in X such that $a_n P a_{n+k}$ for each $n \in \mathbb{N}$ and $a_n \to x$, then $(a_n, x) \in E$ for each $n \in \mathbb{N}$ and $(x, x) \in E$.*

Then, T has a fixed point in X, that is there exists $u^* \in X$ with $u^* = T(u^*, u^*, \ldots, u^*)$.

Theorem 9. *Let (X, d) be a complete metric space endowed with the graph G. Let $T\colon X^k \to X$ be a mapping such that for each $a_1, a_2, a_3, \ldots, a_k, a_{k+1} \in A$ with $a_1 P a_{k+1}$, that is (a_1, a_2), $(a_2, a_3), \ldots, (a_k, a_{k+1}) \in E$, satisfies one of the following inequalities:*

$$\begin{aligned} &d(T(a_2, \ldots, a_k, T(a_1, a_2, \ldots, a_k)), T(a_3, \ldots, a_{k+1}, T(a_2, a_3, \ldots, a_{k+1}))) \\ &\quad \leq \gamma d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})) \end{aligned}$$

$$\begin{aligned} &d(T(a_2, \ldots, a_k, T(a_1, a_2, \ldots, a_k)), T(a_3, \ldots, a_{k+1}, T(a_2, a_3, \ldots, a_{k+1}))) \\ &\quad \leq \gamma \max\{d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})), \\ &\qquad d(T(a_2, a_3, \ldots, a_{k+1}), T(a_4, a_5, \ldots, a_{k+1}, T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})))\} \end{aligned}$$

$$\begin{aligned} &d(T(a_2, a_3, \ldots, a_{k+1}), T(a_4, a_5, \ldots, a_{k+1}, T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1}))) \\ &\quad \leq \gamma \max\{d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})), \\ &\qquad d(T(a_2, \ldots, a_k, T(a_1, a_2, \ldots, a_k)), T(a_3, \ldots, a_{k+1}, T(a_2, a_3, \ldots, a_{k+1}))\} \end{aligned}$$

where $\gamma \in [0, 1)$. Further, assume that the following conditions hold:

(i) *If $a_1 P a_{k+1}$, that is $(a_1, a_2), (a_2, a_3), \ldots, (a_k, a_{k+1}) \in E$, then we have:*

$$(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})) \in E;$$

(ii) *there exist $a_0, a_1, a_2, \ldots, a_k \in X$ with $a_k = T(a_0, a_1, \ldots, a_{k-1})$ and $a_0 P a_k$;*

(iii) *T is continuous with respect to each coordinate.*

Then, T has a fixed point in X, that is there exists $u^* \in X$ with $u^* = T(u^*, u^*, \ldots, u^*)$.

Remark 3. *Note that if $T\colon X^k \to X$ is an operator satisfying Theorem 8 or Theorem 9 and $\{a_n\}$ is a sequence in X such that $a_n P a_m$ for each $m > n \in \mathbb{N}$ and $a_{n+k+1} = T(a_{1+n}, a_{2+n}, \ldots, a_{k+n})$ for each $n \in \mathbb{N}$, then the sequence $\{a_n\}$ converges to fixed point of T.*

Considering that the graph $G = (V, E)$ is defined as $V = X$ and $E = X \times X$, then Theorems 8 and 9 reduce to the following corollaries, respectively.

Corollary 1. *Let (X, d) be a complete metric space, and let $T : X^k \to X$ be a mapping such that for each $a_1, a_2, a_3, \ldots, a_k, a_{k+1} \in X$, one of the following inequalities is satisfied:*

$$d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})) \leq \gamma \max\{d(a_i, a_{i+1}) : 1 \leq i \leq k\}$$

$$d(a_{k+1}, T(a_2, a_3, \ldots, a_{k+1})) \leq \gamma \max\{d(a_i, a_{i+1}) : 1 \leq i \leq k-1, d(a_k, T(a_1, a_2, \ldots, a_k))\}$$

where $\gamma \in [0, 1)$. Then, T has a fixed point in X, that is there exists $u^* \in X$ with $u^* = T(u^*, u^*, \ldots, u^*)$.

Corollary 2. *Let (X, d) be a complete metric space, and let $T : X^k \to X$ be a mapping such that for each $a_1, a_2, a_3, \ldots, a_k, a_{k+1} \in X$, one of the following inequalities is satisfied:*

$$\begin{aligned} &d(T(a_2, \ldots, a_k, T(a_1, a_2, \ldots, a_k)), T(a_3, \ldots, a_{k+1}, T(a_2, a_3, \ldots, a_{k+1}))) \\ &\quad \leq \gamma d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})) \end{aligned}$$

$$\begin{aligned} &d(T(a_2, \ldots, a_k, T(a_1, a_2, \ldots, a_k)), T(a_3, \ldots, a_{k+1}, T(a_2, a_3, \ldots, a_{k+1}))) \\ &\quad \leq \gamma \max\{d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})), \\ &\qquad d(T(a_2, a_3, \ldots, a_{k+1}), T(a_4, a_5, \ldots, a_{k+1}, T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})))\} \end{aligned}$$

$$\begin{aligned} &d(T(a_2, a_3, \ldots, a_{k+1}), T(a_4, a_5, \ldots, a_{k+1}, T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1}))) \\ &\quad \leq \gamma \max\{d(T(a_1, a_2, \ldots, a_k), T(a_2, a_3, \ldots, a_{k+1})), \\ &\qquad d(T(a_2, \ldots, a_k, T(a_1, a_2, \ldots, a_k)), T(a_3, \ldots, a_{k+1}, T(a_2, a_3, \ldots, a_{k+1})))\} \end{aligned}$$

where $\gamma \in [0, 1)$. Further, assume that T is continuous with respect to each coordinate. Then, T has a fixed point in X, that is there exists $u^* \in X$ with $u^* = T(u^*, u^*, \ldots, u^*)$.

5. Conclusions

In this article, we discussed several forms of Prešić-type nonself operators and studied the existence of best proximity points for such operators on a metric space equipped with a graph. In order to illustrate these results, we provided some examples. We also gave some new fixed point theorems for Prešić type operators on a metric space endowed with a graph; these fixed point theorems were obtained from our best proximity point results. This article invites researchers to work further on the development of best proximity point results for generalized forms of Prešić-type nonself operators.

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