mathematics

Article

Generalized Implicit Set-Valued Variational Inclusion Problem with $\oplus$ Operation

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Received: 6 March 2019; Accepted: 8 May 2019; Published: 10 May 2019

Abstract: In this paper, we consider a resolvent operator which depends on the composition of two mappings with $\oplus$ operation. We prove some of the properties of the resolvent operator, that is, that it is single-valued as well as Lipschitz-type-continuous. An existence and convergence result is proven for a generalized implicit set-valued variational inclusion problem with $\oplus$ operation. Some special cases of a generalized implicit set-valued variational inclusion problem with $\oplus$ operation are discussed. An example is constructed to illustrate some of the concepts used in this paper.

Keywords: algorithm; implicit; inclusion; set-valued mapping; $\oplus$ operation

MSC: 47H09; 49J40

1. Introduction

Because of applications in optimization problems, mathematical programming, equilibrium problems, engineering, economics and operation research etc., suitable progress has been achieved in both theory and application of various types of variational inequalities (inclusions) and their generalizations. After careful observation, it was noticed that the projection method and its variant forms cannot be applied for solving variational inclusions. This fact motivated researchers to use techniques based on resolvent operators. The resolvent operator and its variant forms represent an important tool for finding the approximate solutions of variational inclusions. The main idea in this technique is to establish the equivalence between the variational inclusions and the fixed point problems using the concept of resolvent. For more details, we refer to [1–20] and the references therein.

$\oplus$ operation, that is, XOR-operation is a binary operation and behaves like the ADD operation: It takes two arguments and produces one result. This operation is commutative, associative, and self-inverse. In Boolean algebra, it is the same as addition modulo(2). XOR represents the inequality function, i.e., the output is true if the inputs are not alike; otherwise, the output is false. It is interesting to note that if we take the XOR of any number with 1, then we get the complement of the number, and if we take XOR with 0, then we get the same number. XOR terminology is used to generate pseudo-random numbers, to detect error in digital communication, inside CPU it helps in addition operation, etc.

Li and his co-authors [21–23] first used the $\oplus$ operation for solving some classes of variational inclusions and after that, Ahmad and his co-authors [24–26] also solved some generalized variational inclusions with $\oplus$ operation.

In this paper, we consider a resolvent operator with $\oplus$ operation involving composition of two mappings. We proved some properties of the resolvent operator. An iterative algorithm was constructed to solved a generalized implicit set-valued variational inclusion problem with $\oplus$ operation in real ordered positive Hilbert spaces. An existence and convergence result was proven for a generalized implicit set-valued inclusion problem with $\oplus$ operation. Some special cases are discussed and an example is given in support of some of the concepts used in this work.

2. Preliminaries

Let C be a cone with partial ordering "$\leq$". An ordered Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ is called positive if $0 \leq x$ and $0 \leq y$, then $0 \leq \langle x, y\rangle$ holds. Throughout the paper, $\mathcal{H}_p$ is assumed to be a real ordered positive Hilbert space. We denote $2^{\mathcal{H}_p}$ (respectively, $C^*(\mathcal{H}_p)$) as the family of nonempty (respectively, compact) subsets of $\mathcal{H}_p$, and d is the metric induced by the norm and $\mathcal{D}(.,.)$ is the Hausdörff metric on $C^*(\mathcal{H}_p)$.

Now, we illustrate some known concepts and results which are needed to prove the main result. The following concepts and results can be found in [20–27].

Definition 1. *A nonempty closed convex subset C of $\mathcal{H}_p$ is said to be a cone if:*

(i) *for any $x \in C$ and any $\lambda > 0$, $\lambda x \in C$;*
(ii) *if $x \in C$ and $-x \in C$, then $x = 0$.*

Definition 2. *Let C be the cone, then:*

(i) *C is called a normal cone if there exists a constant $\lambda_N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \lambda_N \|y\|$, for all $x, y \in \mathcal{H}_p$;*
(ii) *for any $x, y \in \mathcal{H}_p$, $x \leq y$ if and only if $y - x \in C$;*
(iii) *x and y are said to be comparative to each other if either $x \leq y$ or $y \leq x$ holds and is denoted by $x \propto y$.*

Definition 3. *For any $x, y \in \mathcal{H}_p$, $lub\{x, y\}$ denotes the least upper bound and $glb\{x, y\}$ denotes the greatest lower bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist, then some binary operations are given below:*

(i) $x \vee y = lub\{x, y\}$;
(ii) $x \wedge y = glb\{x, y\}$;
(iii) $x \oplus y = (x - y) \vee (y - x)$;
(iv) $x \odot y = (x - y) \wedge (y - x)$.

The operations $\vee, \wedge, \oplus$, and $\odot$ are called OR, AND, XOR, and XNOR operations, respectively.

Lemma 1. *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist such that $x - y \propto y - x$ and $0 \leq (x - y) \vee (y - x)$.*

Lemma 2. *For any natural number n, $x \propto y_n$ and $y_n \to y^*$ as $n \to \infty$, then $x \propto y^*$.*

Proposition 1. *Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then the following relations hold for all $x, y, u, v, w \in \mathcal{H}_p$ and $\alpha, \beta, \lambda \in \mathbb{R}$:*

(i) *$x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;*
(ii) *if $x \propto 0$, then $-x \oplus 0 \leq x \leq x \oplus 0$;*

(*iii*) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
(*iv*) $0 \leq x \oplus y$, *if* $x \propto y$;
(*v*) *if* $x \propto y$, *then* $x \oplus y = 0$ *if and only if* $x = y$;
(*vi*) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$;
(*vii*) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$;
(*viii*) *if* x, y *and* w *are comparative to each other, then* $(x \oplus y) \leq (x \oplus w) + (w \oplus y)$;
(*ix*) $\alpha x \oplus \beta x = |\alpha - \beta|x = (\alpha \oplus \beta)x$, *if* $x \propto 0$.

Proposition 2. *Let C be a normal cone in $\mathcal{H}_p$ with constant λ_N, then for each $x, y \in \mathcal{H}_p$, the following relations hold:*

(*i*) $\|0 \oplus 0\| = \|0\| = 0$;
(*ii*) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$;
(*iii*) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_N \|x \oplus y\|$;
(*iv*) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 4. *Let $F : \mathcal{H}_p \to \mathcal{H}_p$ be a single-valued mapping, then:*

(*i*) *F is said to be comparison mapping, if for each $x, y \in \mathcal{H}_p$, $x \propto y$ then $F(x) \propto F(y)$, $x \propto F(x)$ and $y \propto F(y)$;*
(*ii*) *F is said to be strongly comparison mapping, if F is a comparison mapping and $F(x) \propto F(y)$ if and only if $x \propto y$, for all $x, y \in \mathcal{H}_p$.*

Definition 5. *A single-valued mapping $F : \mathcal{H}_p \to \mathcal{H}_p$ is said to be β-ordered compression mapping if F is a comparison mapping and:*

$$F(x) \oplus F(y) \leq \beta(x \oplus y), \text{ for } 0 < \beta < 1.$$

Definition 6. *Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Then:*

(*i*) *M is said to be a comparison mapping if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for $v_x \in M(x)$ and $v_y \in M(y)$, $v_x \propto v_y$, for all $x, y \in \mathcal{H}_p$.*
(*ii*) *A comparison mapping M is said to be α-non-ordinary difference mapping if:*

$$(v_x \oplus v_y) \oplus \alpha(x \oplus y) = 0 \; holds, \; for \; all \; x, y \in \mathcal{H}_p, \; v_x \in M(x) \; and v_y \in M(y);$$

(*iii*) *A comparison mapping M is said to be θ-ordered rectangular if there exists a constant $\theta > 0$ such that:*

$$\langle v_x \odot v_y, -(x \oplus y) \rangle \geq \theta \|x \oplus y\|^2 \; holds, \; for \; all \; x, y \in \mathcal{H}_p, \; there \; exists \; v_x \in M(x) and v_y \in M(y).$$

Definition 7. *A set-valued mapping $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be λ-XOR-ordered strongly monotone compression mapping if $x \propto y$, then there exists a constant $\lambda > 0$ such that:*

$$\lambda(v_x \oplus v_y) \geq x \oplus y, \text{ for all } x, y \in \mathcal{H}_p, v_x \in M(x), v_y \in M(y).$$

Definition 8. *A set-valued mapping $T : \mathcal{H}_p \to C^*(\mathcal{H}_p)$ is said to be $\mathcal{D}$-Lipschitz continuous if for all $x, y \in \mathcal{H}_p$, $x \propto y$, there exists a constant $\lambda_T > 0$ such that:*

$$\mathcal{D}(T(x), T(y)) \leq \lambda_T \|x \oplus y\|.$$

Definition 9. *A single-valued mapping $F : \mathcal{H}_p \to \mathcal{H}_p$ is said to be Lipschitz-type-continuous if there exists a constant $\delta > 0$ such that:*

$$\|F(x) \oplus F(y)\| \leq \delta \|x \oplus y\|, \text{ for all } x, y \in \mathcal{H}_p.$$

Let $H, F : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings, we consider the composition of H and F as:

$$(H \circ F)(x) = H(F(x)), \text{ for all } x \in \mathcal{H}_p.$$

Definition 10. *Let $H, F : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $H \circ F$ is strongly comparison and β-ordered compression mapping. Then, a set-valued comparison mapping $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be (α, λ)-XOR-NODSM if M is an α-non-ordinary difference mapping and λ-XOR-ordered strongly monotone compression mapping and $[(H \circ F) \oplus \lambda M](\mathcal{H}_p) = \mathcal{H}_p$, for $\alpha, \beta, \lambda > 0$.*

Definition 11. *Let $H, F : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mapping such that $H \circ F$ is strongly comparison and β-ordered compression mapping. Suppose that the set-valued mapping $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is (α, λ)-XOR-NODSM mapping. We define the resolvent operator $\mathcal{J}_{\lambda,M}^{H,F} : \mathcal{H}_p \to \mathcal{H}_p$ by:*

$$\mathcal{J}_{\lambda,M}^{H,F}(x) = [(H \circ F) \oplus \lambda M]^{-1}(x), \textit{ for all } x \in \mathcal{H}_p \textit{ and } \alpha, \lambda > 0. \tag{1}$$

Now, we present some properties of the resolvent operator defined by (1).

Proposition 3. *Let $H, F : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $(H \circ F)$ is β-ordered compression mapping and $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is the set-valued θ-ordered rectangular mapping with $\lambda\theta > \beta$. Then, the resolvent operator $\mathcal{J}_{\lambda,M}^{H,F} : \mathcal{H}_p \to \mathcal{H}_p$ is single-valued.*

Proof. For any given $u \in \mathcal{H}_p$ and $\lambda > 0$, let $x, y \in [(H \circ F) \oplus \lambda M]^{-1}(u)$. Then:

$$v_x = \frac{1}{\lambda}(u \oplus (H \circ F)(x)) = \frac{1}{\lambda}(u \oplus H(F(x))) \in M(x),$$

and:

$$v_y = \frac{1}{\lambda}(u \oplus (H \circ F)(y)) = \frac{1}{\lambda}(u \oplus (H \circ F)(y)) \in M(y).$$

Using (i) and (ii) of Proposition 1, we obtain:

$$\begin{aligned}
v_x \odot v_y &= \frac{1}{\lambda}(u \oplus H(F(x))) \odot \frac{1}{\lambda}(u \oplus H(F(y))) \\
&= \frac{1}{\lambda}[(u \oplus H(F(x))) \odot (u \oplus H(F(y)))] \\
&= -\frac{1}{\lambda}[(u \oplus H(F(x))) \oplus (u \oplus H(F(y)))] \\
&= -\frac{1}{\lambda}[(u \oplus u) \oplus (H(F(x)) \oplus H(F(y)))] \\
&= -\frac{1}{\lambda}[0 \oplus (H(F(x)) \oplus H(F(y)))] \\
&\leq -\frac{1}{\lambda}[H(F(x)) \oplus H(F(y))].
\end{aligned}$$

Thus, we have:

$$v_x \odot v_y \leq -\frac{1}{\lambda}[H(F(x)) \oplus H(F(y))]. \tag{2}$$

Since M is θ-ordered rectangular mapping, $(H \circ F)$ is β-ordered compression mapping and using (2), we have:

$$\begin{aligned}
\theta\|x \oplus y\|^2 &\leq \langle v_x \odot v_y, -(x \oplus y)\rangle \\
&\leq \langle -\frac{1}{\lambda}[H(F(x)) \oplus H(F(y))], -(x \oplus y)\rangle \\
&\leq \frac{1}{\lambda}\langle (H \circ F)(x) \oplus (H \circ F)(y), x \oplus y\rangle \\
&\leq \frac{1}{\lambda}\langle \beta(x \oplus y), x \oplus y\rangle \\
&= \frac{\beta}{\lambda}\|x \oplus y\|^2,
\end{aligned}$$

i.e.,

$$\Big(\theta - \frac{\beta}{\lambda}\Big)\|x \oplus y\|^2 \leq 0, \text{ for } \lambda\theta > \beta,$$

which shows that:

$$\|x \oplus y\| = 0, \text{ which implies } x \oplus y = 0.$$

Therefore $x = y$, i.e., the resolvent operator $\mathcal{J}^{H,F}_{\lambda,M}$ is single-valued, for $\lambda\theta > \beta$. □

Proposition 4. *Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be (α, λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^{H,F}_{\lambda,M}$. Let $H, F : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $(H \circ F)$ is strongly comparison mapping with respect to $\mathcal{J}^{H,F}_{\lambda,M}$. Then the resolvent operator $\mathcal{J}^{H,F}_{\lambda,M} : \mathcal{H}_p \to \mathcal{H}_p$ is a comparison mapping.*

Proof. Since M is (α, λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^{H,F}_{\lambda,M}$, i.e., M is α-non-ordinary difference as well as λ-XOR-ordered strongly monotone compression mapping with respect to $\mathcal{J}^{H,F}_{\lambda,M}$. For any $x, y \in \mathcal{H}_p$, let $x \propto y$ and:

$$v_x^* = \frac{1}{\lambda}(x \oplus (H \circ F)(\mathcal{J}^{H,F}_{\lambda,M}(x))) = \frac{1}{\lambda}(x \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))) \in M(\mathcal{J}^{H,F}_{\lambda,M}(x)), \tag{3}$$

and:

$$v_y^* = \frac{1}{\lambda}(y \oplus (H \circ F)(\mathcal{J}^{H,F}_{\lambda,M}(y))) = \frac{1}{\lambda}(y \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))) \in M(\mathcal{J}^{H,F}_{\lambda,M}(y)). \tag{4}$$

Since M is λ-XOR-ordered strongly monotone compression mapping and using (3) and (4), we have:

$$\begin{aligned}
(x \oplus y) &\leq \lambda(v_x^* \oplus v_y^*) \\
(x \oplus y) &\leq \Big(x \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))\Big) \oplus \Big(y \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))\Big) \\
(x \oplus y) &\leq (x \oplus y) \oplus \Big(H(F(\mathcal{J}^{H,F}_{\lambda,M}(x))) \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))\Big) \\
0 &\leq H(F(\mathcal{J}^{H,F}_{\lambda,M}(x))) \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y))) \\
0 &\leq \Big[H(F(\mathcal{J}^{H,F}_{\lambda,M}(x))) - H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))\Big] \\
&\quad \vee \Big[H(F(\mathcal{J}^{H,F}_{\lambda,M}(y))) - H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))\Big],
\end{aligned}$$

which implies either:

$$\begin{aligned} 0 &\leq \Big[H(F(\mathcal{J}^{H,F}_{\lambda,M}(x))) - H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))\Big] \text{ or} \\ 0 &\leq \Big[H(F(\mathcal{J}^{H,F}_{\lambda,M}(y))) - H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))\Big]. \end{aligned}$$

Thus, in both cases, we have:

$$(H \circ F)(\mathcal{J}^{H,F}_{\lambda,M}(x)) \propto (H \circ F)(\mathcal{J}^{H,F}_{\lambda,M}(y)).$$

Since $(H \circ F)$ is strongly comparison mapping with respect to $\mathcal{J}^{H,F}_{\lambda,M}$, thus, we have $\mathcal{J}^{H,F}_{\lambda,M}(x) \propto \mathcal{J}^{H,F}_{\lambda,M}(y)$, i.e., the resolvent operator $\mathcal{J}^{H,F}_{\lambda,M}$ is a comparison mapping. □

Proposition 5. *If all the mappings and conditions are the same as those stated in Proposition 3, then the following condition holds:*

$$\Big\|\mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y)\Big\| \leq \frac{1}{(\lambda\theta - \beta)}\|x \oplus y\|, \text{ for } \lambda\theta > \beta \text{ and } \alpha, \beta, \lambda > 0,$$

i.e., the resolvent operator $\mathcal{J}^{H,F}_{\lambda,M}$ is Lipschitz-type-continuous mapping.

Proof. Let $x, y \in \mathcal{H}_p$, and:

$$v^*_x = \frac{1}{\lambda}(x \oplus (H \circ F)(\mathcal{J}^{H,F}_{\lambda,M}(x))) = \frac{1}{\lambda}(x \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))) \in M(\mathcal{J}^{H,F}_{\lambda,M}(x)), \tag{5}$$

and:

$$v^*_y = \frac{1}{\lambda}(y \oplus (H \circ F)(\mathcal{J}^{H,F}_{\lambda,M}(y)) = \frac{1}{\lambda}(y \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))) \in M(\mathcal{J}^{H,F}_{\lambda,M}(y)). \tag{6}$$

Since $(H \circ F)$ is β-ordered compression mapping and using (5) and (6), we have

$$\begin{aligned} v^*_x \oplus v^*_y &= \frac{1}{\lambda}\Big[\Big(x \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))\Big) \oplus \Big(y \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))\Big)\Big] \\ &= \frac{1}{\lambda}\Big[(x \oplus y) \oplus \Big(H(F(\mathcal{J}^{H,F}_{\lambda,M}(x))) \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))\Big)\Big] \\ &\leq \frac{1}{\lambda}[(x \oplus y) \oplus \beta(\mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y))]. \end{aligned} \tag{7}$$

Since M is θ-ordered rectangular mapping and using (7), for any:

$$\mathcal{J}^{H,F}_{\lambda,M}(x) \in M(\mathcal{J}^{H,F}_{\lambda,M}(x)) \text{ and } \mathcal{J}^{H,F}_{\lambda,M}(y) \in M(\mathcal{J}^{H,F}_{\lambda,M}(y)), \text{ we have:}$$

$$
\begin{aligned}
\theta\|\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)\|^2 &\leq \Big\langle v_x^* \odot v_y^*, -(\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y))\Big\rangle \\
&\leq \Big\langle v_x^* \oplus v_y^*, (\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y))\Big\rangle \\
&\leq \frac{1}{\lambda}\Big[\Big\langle (x \oplus y) \oplus \beta(\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)), \mathcal{J}_{\lambda,M}^{H,F}(x) \\
&\quad \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)\Big\rangle\Big] \\
&\leq \frac{1}{\lambda}\Bigg[\Big\|(x \oplus y) \oplus \beta(\mathcal{J}_{\lambda,M}^{H,F}(x) \\
&\quad \oplus \mathcal{J}_{\lambda,M}^{H,F}(y))\Big\| \, \Big\|\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)\Big\|\Bigg].
\end{aligned}
$$

Using (iii) of Proposition 2, we have:

$$
\begin{aligned}
\theta\|\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)\|^2 &\leq \frac{1}{\lambda}\Bigg[\Big\|(x \oplus y) \oplus \beta(\mathcal{J}_{\lambda,M}^{H,F}(x) \\
&\quad \oplus \mathcal{J}_{\lambda,M}^{H,F}(y))\Big\| \, \Big\|\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)\Big\|\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\lambda}\Bigg[\Big\|(x \oplus y) - \Big(\beta(\mathcal{J}_{\lambda,M}^{H,F}(x) \\
&\quad \oplus \mathcal{J}_{\lambda,M}^{H,F}(y))\Big)\Big\| \, \Big\|\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)\Big\|\Bigg] \\
&\leq \frac{1}{\lambda}\Bigg[\|x \oplus y\|\Big\|\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)\Big\|\Bigg] \\
&\quad + \frac{\beta}{\lambda}\Big\|\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)\Big\|^2.
\end{aligned}
$$

It follows that:

$$
\Big\|\mathcal{J}_{\lambda,M}^{H,F}(x) \oplus \mathcal{J}_{\lambda,M}^{H,F}(y)\Big\| \leq \frac{1}{(\lambda\theta - \beta)}\|x \oplus y\|, \text{ for } \lambda\theta > \beta.
$$

This completes the proof. □

In support of Proposition 3–5, we have the following example.

Example 1. *Let $\mathcal{H}_p = [0, \infty)$ with the usual inner product and norm, and let $C = [0, 1]$ be a normal cone in $[0, \infty)$. Let $H : \mathcal{H}_p \to \mathcal{H}_p$ and $F : \mathcal{H}_p \to \mathcal{H}_p$ be the mappings defined by:*

$$
H(x) = \frac{x}{3} + 1, \ \ \textit{and} \ \ F(x) = \frac{x}{2}, \forall\, x \in [0.\infty).
$$

Let $x, y \in \mathcal{H}_p, x \propto y$, then we calculate:

$$\begin{aligned}
(H \circ F)(x) \oplus (H \circ F)(y) &= H(F(x)) \oplus H(F(y)) \\
&= \Big(\frac{F(x)}{3} + 1\Big) \oplus \Big(\frac{F(y)}{3} + 1\Big) \\
&= \Big(\frac{x}{6} + 1\Big) \oplus \Big(\frac{y}{6} + 1\Big) \\
&= \Big(\Big(\frac{x}{6} + 1\Big) - \Big(\frac{y}{6} + 1\Big)\Big) \vee \Big(\Big(\frac{y}{6} + 1\Big) - \Big(\frac{x}{6} + 1\Big)\Big) \\
&= \Big(\frac{x}{6} - \frac{y}{6}\Big) \vee \Big(\frac{y}{6} - \frac{x}{6}\Big) \\
&= \frac{1}{6}((x - y) \vee (y - x)) \\
&= \frac{1}{6}(x \oplus y) \\
&\leq \frac{1}{5}(x \oplus y),
\end{aligned}$$

i.e.,

$$(H \circ F)(x) \oplus (H \circ F)(y) \leq \frac{1}{5}(x \oplus y), \ \forall\, x, y \in [0, \infty).$$

Hence, $H \circ F$ is $\frac{1}{5}$-ordered compression mapping.

Suppose that $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is a the set-valued mapping defined by:

$$M(x) = \{x + 1\}, \forall\, x \in [0, \infty).$$

It can be easily verified that M is a comparison mapping, 1-XOR-ordered strongly monotone comparison mapping, and 1-non-ordinary difference mapping.

Let $v_x = x + 1 \in M(x)$ and $v_y = y + 1 \in M(y)$, then we evaluate:

$$\begin{aligned}
\langle v_x \odot v_y, -(x \oplus y)\rangle &= \langle v_x \oplus v_y, x \oplus y\rangle \\
&= \langle (x + 1) \oplus (y + 1), x \oplus y\rangle \\
&= \langle x \oplus y, x \oplus y\rangle \\
&= \|x \oplus y\|^2 \\
&\geq \frac{1}{2}\|x \oplus y\|^2,
\end{aligned}$$

i.e.,

$$\langle v_x \odot v_y, -(x \oplus y)\rangle \geq \frac{1}{2}\|x \oplus y\|^2, \ \forall\, x, y \in [0, \infty).$$

Thus, M is a $\frac{1}{2}$-ordered rectangular comparison mapping. Further, it is clear that for $\lambda = 1$, $[(H \circ F) \oplus \lambda M][0, \infty) = [0, \infty)$. Hence, M is an $(1, 1)$-XOR-NODSM set-valued mapping.

The resolvent operator defined by (1) is given by:

$$\mathcal{J}^{H,F}_{\lambda,M}(x) = \frac{6x}{5}, \forall\, x \in [0, \infty). \tag{8}$$

It is easy to check that the resolvent operator defined above is a comparison and single-valued mapping.

Further:

$$\begin{aligned}\left\|\mathcal{J}^{H,F}_{\lambda,M}(x)\oplus\mathcal{J}^{H,F}_{\lambda,M}(y)\right\| &= \left\|\frac{6x}{5}\oplus\frac{6y}{5}\right\|\\ &= \frac{6}{5}\|x\oplus y\|\\ &\leq \frac{10}{3}\|x\oplus y\|,\end{aligned}$$

i.e.,

$$\|\mathcal{J}^{H,F}_{\lambda,M}(x)\oplus\mathcal{J}^{H,F}_{\lambda,M}(y)\|\leq\frac{10}{3}\|x\oplus y\|, \forall\, x,y\in[0,\infty).$$

That is, the resolvent operator $\mathcal{J}^{H,F}_{\lambda,M}$ is $\frac{10}{3}$-Lipschitz-type-continuous.

3. Formulation of The Problem and Existence of Solution

Let $\mathcal{H}_p$ be a real positive Hilbert space. Let $A,B,C:\mathcal{H}_p\to C^*(\mathcal{H}_p)$ and $M:\mathcal{H}_p\to 2^{\mathcal{H}_p}$ be the set-valued mappings and let $G:\mathcal{H}_p\times\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ be a single-valued mapping. Then, we consider the following problem:

Find $x\in\mathcal{H}_p$, $w\in A(x)$, $u\in B(x)$ and $v\in C(x)$ such that:

$$0\in G(w,u,v)\oplus M(x). \tag{9}$$

We call the problem in Equation (9) a generalized implicit set-valued variational inclusion problem with $\oplus$ operation.

- (i) If $C\equiv 0$ and $G(w,u,v)=G(w,v)$, then problem (9) coincides with the problem studied by Ahmad et al. [1].
- (ii) If $B,C\equiv 0$ and A is single-valued such that $G(w,u,v)=A(x)$, then problem (9) reduces to the problem studied by Ahmad et al. [7].
- (iii) If $G\equiv 0$, then problem (9) becomes the problem studied by Li [22].

It is clear that for suitable choices of operators involved in the formulation of problem (9), one can obtain many related problems.

The following Lemma is a fixed point formulation of the problem in Equation (9).

Lemma 3. *The generalized implicit set-valued variational inclusion problem involving $\oplus$ operation (9) has a solution $x\in\mathcal{H}_p$, $w\in A(x)$, $u\in B(x)$, $v\in C(x)$ if and only if it satisfies the following equation:*

$$x=\mathcal{J}^{H,F}_{\lambda,M}[\lambda G(w,u,v)\oplus(H\circ F)(x)], \tag{10}$$

where $\lambda>0$ is a constant.

Proof. Using the definition of the resolvent operator $\mathcal{J}^{H,F}_{\lambda,M}$, and Equation (10), we get:

$$\begin{aligned} x &= \mathcal{J}^{H,F}_{\lambda,M}[\lambda G(w,u,v)\oplus(H\circ F)(x)]\\ &= [(H\circ F)\oplus\lambda M]^{-1}[\lambda G(w,u,v)\oplus(H\circ F)(x)],\\ (H\circ F)\oplus\lambda M(x) &= \lambda G(w,u,v)\oplus(H\circ F)(x),\end{aligned}$$

which implies that $0\in G(w,u,v)\oplus M(x)$, the required generalized implicit set-valued variational inclusion problem with $\oplus$ operation (9).

Conversely, suppose that generalized implicit set-valued variational inclusion problem with $\oplus$ operation (9) is satisfied, that is, $x \in \mathcal{H}_p$, $w \in A(x)$, $u \in B(x)$ and $v \in C(x)$ such that:

$$0 \in G(w,u,v) \oplus M(x),$$

which shows that:

$$\begin{aligned} G(w,u,v) &= M(x), \\ \lambda G(w,u,v) &= \lambda M(x), \\ \lambda G(w,u,v) \oplus (H \circ F)(x) &= (H \circ F)(x) \oplus \lambda M(x), \\ \lambda G(w,u,v) \oplus (H \circ F)(x) &= [(H \circ F) \oplus \lambda M](x), \\ x &= [(H \circ F) \oplus \lambda M]^{-1}[\lambda G(w,u,v) \oplus (H \circ F)(x)] \\ x &= \mathcal{J}_{\lambda,M}^{H,F}[\lambda G(w,u,v) \oplus (H \circ F)(x)] \end{aligned}$$

Thus, Equation (10) is satisfied. □

Based on Lemma 3, we establish the following iterative algorithm to obtain the solution of the problem in Equation (9).

Iterative Algorithm 1. *For any given $x_0 \in \mathcal{H}_p$, choose $w_0 \in A(x_0)$, $u_0 \in B(x_0)$, $v_0 \in C(x_0)$ and using (10), let:*

$$x_1 = (1-\alpha)x_0 + \alpha \mathcal{J}_{\lambda,M}^{H,F}[\lambda G(w_0,u_0,v_0) \oplus (H \circ F)(x_0)].$$

Since $w_0 \in A(x_0)$, $u_0 \in B(x_0)$, $v_0 \in C(x_0)$, by the Nadler's theorem [28], there exists $w_1 \in A(x_1)$, $u_1 \in B(x_1)$, $v_1 \in C(x_1)$, and using Proposition 2, we have:

$$\begin{aligned} \|w_0 \oplus w_1\| &\leq \|w_0 - w_1\| \leq (1+1)\mathcal{D}(A(x_0), A(x_1)) \\ \|u_0 \oplus u_1\| &\leq \|u_0 - u_1\| \leq (1+1)\mathcal{D}(B(x_0), B(x_1)) \\ \|v_0 \oplus v_1\| &\leq \|v_0 - v_1\| \leq (1+1)\mathcal{D}(C(x_0), C(x_1)), \end{aligned}$$

where $\mathcal{D}$ is the Hausdorff metric on $C^*(\mathcal{H}_p)$. Let:

$$x_2 = (1-\alpha)x_1 + \alpha \mathcal{J}_{\lambda,M}^{H,F}[\lambda G(w_1,u_1,v_1) \oplus (H \circ F)(x_1)].$$

Again by Nadler's theorem [28], there exist $w_2 \in F(x_2)$, $u_2 \in B(x_2)$, $v_2 \in C(x_2)$ such that:

$$\begin{aligned} \|w_1 \oplus w_2\| &\leq \|w_1 - w_2\| \leq (1+2^{-1})\mathcal{D}(A(x_1), A(x_2)), \\ \|u_1 \oplus u_2\| &\leq \|u_1 - u_2\| \leq (1+2^{-1})\mathcal{D}(B(x_1), B(x_2)), \\ \|v_1 \oplus v_2\| &\leq \|v_1 - v_2\| \leq (1+2^{-1})\mathcal{D}(C(x_1), C(x_2)). \end{aligned}$$

Continuing the above procedure inductively, we have the following scheme:

$$x_{n+1} = (1-\alpha)x_n + \alpha \mathcal{J}_{\lambda,M}^{H,F}[\lambda G(w_n,u_n,v_n) \oplus (H \circ F)(x_n)].$$

Since $w_{n+1} \in A(x_{n+1})$, $u_{n+1} \in B(x_{n+1})$, $v_{n+1} \in C(x_{n+1})$, *such that:*

$$\begin{aligned}
\|w_n \oplus w_{n+1}\| &\leq \|w_n - w_{n+1}\| \leq (1+(n+1)^{-1})\mathcal{D}(A(x_n), A(x_{n+1})), \\
\|u_n \oplus u_{n+1}\| &\leq \|u_n - u_{n+1}\| \leq (1+(n+1)^{-1})\mathcal{D}(B(x_n), B(x_{n+1})), \\
\|v_n \oplus v_{n+1}\| &\leq \|v_n - v_{n+1}\| \leq (1+(n+1)^{-1})\mathcal{D}(C(x_n), C(x_{n+1})).
\end{aligned}$$

where $\alpha \in [0,1], n = 0,1,2,\cdots$.

Theorem 1. *Let* $C \subset \mathcal{H}_p$ *be a normal cone with constant* $\lambda_N, H, F : \mathcal{H}_p \to \mathcal{H}_p$ *and* $G : \mathcal{H}_p \times \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ *be the single-valued mappings such that* $(H \circ F)$ *be strongly comparison,* β*-ordered compression mapping,* G *is* β_1*-ordered compression mapping in the first argument,* β_2*-ordered compression mapping in the second argument, and* β_3*-ordered compression mapping in the third argument. Let* $A, B, C : \mathcal{H}_p \to C^*(\mathcal{H}_p)$ *be the set-valued mappings such that* A *is* λ_A*-*$\mathcal{D}$*-Lipschitz-continuous,* B *is* λ_B*-*$\mathcal{D}$*-Lipschitz-continuous, and* C *is* λ_C*-*$\mathcal{D}$*-Lipschitz-continuous. Suppose that* $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ *is* (α, λ)*- XOR-NODSM set-valued mapping with respect to* $\mathcal{J}^{H,F}_{\lambda,M}$ *and* θ*-ordered rectangular mapping with* $\lambda\theta > \beta$*. If* $x_{n+1} \propto x_n$*,* $n = 0,1,2,....$ *and the following condition is satisfied:*

$$|\lambda| [\beta_1\lambda_A + \beta_2\lambda_B + \beta_3\lambda_C] + \beta < \frac{1-\lambda_N(1-\alpha)}{\lambda_N\alpha\theta'}, \tag{11}$$

where $\theta' = \frac{1}{\lambda\theta-\beta}$ *and* $\lambda\theta > \beta$*;* $\beta_1, \beta_2, \beta_3, \lambda_A, \lambda_B, \lambda_C, \lambda_N, \alpha, \theta, \beta$ *all are positive constants.Then, the generalized implicit set-valued variational inclusion problem with* $\oplus$ *operation (9) has a solution* $x \in \mathcal{H}_p, w \in A(x), u \in B(x)$*, and* $v \in C(x)$*. Moreover, the iterative sequences* $\{x_n\}, \{w_n\}\{u_n\}$*, and* $\{v_n\}$ *generated by Algorithm 1 converge strongly to* $x, w, u,$ *and* v*, the solution of generalized implicit set-valued variational inclusions problem with* $\oplus$ *operation (9).*

Proof. By Algorithm 1 and Proposition 1, we have:

$$\begin{aligned}
0 &\leq x_{n+1} \oplus x_n \\
&= \Big[(1-\alpha)x_n + \alpha\Big(\mathcal{J}^{H,F}_{\lambda,M}[\lambda\, G(w_n,u_n,v_n) \oplus (H\circ F)(x_n)]\Big)\Big] \\
&\quad \oplus\Big[(1-\alpha)x_{n-1} + \alpha\Big(\mathcal{J}^{H,F}_{\lambda,M}[\lambda\, G(w_{n-1},u_{n-1},v_{n-1}) \oplus (H\circ F)(x_{n-1})]\Big)\Big] \\
&= (1-\alpha)(x_n \oplus x_{n-1}) + \alpha\Big(\mathcal{J}^{H,F}_{\lambda,M}(\lambda\, G(w_n,u_n,v_n) \oplus (H\circ F)(x_n)) \\
&\quad \oplus \mathcal{J}^{H,F}_{\lambda,M}(\lambda\, G(w_{n-1},u_{n-1},v_{n-1}) \oplus (H\circ F)(x_{n-1}))\Big).
\end{aligned} \tag{12}$$

Using Proposition 2 and Lipschitz-type-continuity of the resolvent operators (1) and (12), we have:

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &\leq \lambda_N\Big\|(1-\alpha)(x_n \oplus x_{n-1}) + \alpha\Big[\mathcal{J}^{H,F}_{\lambda,M}(\lambda\, G(w_n,u_n,v_n) \oplus (H\circ F)(x_n)) \\
&\quad \oplus \mathcal{J}^{H,F}_{\lambda,M}(\lambda\, G(w_{n-1},u_{n-1},v_{n-1}) \oplus (H\circ F)(x_{n-1}))\Big]\Big\| \\
&\leq \lambda_N\|(1-\alpha)(x_n \oplus x_{n-1})\| + \lambda_N\alpha\theta'\|(\lambda\, G(w_n,u_n,v_n) \oplus (H\circ F)(x_n)) \\
&\quad \oplus(\lambda\, G(w_n-1,u_n-1,v_n-1) \oplus (H\circ F)(x_{n-1}))\| \\
&\leq \lambda_N(1-\alpha)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\theta'\|(\lambda\, G(w_n,u_n,v_n) \oplus \lambda\, G(w_{n-1},u_{n-1},v_{n-1}) - ((H\circ F)(x_n) \\
&\quad \oplus(H\circ F)(x_{n-1}))\| \\
&\leq \lambda_N(1-\alpha)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\theta'|\lambda\,||G(w_n,u_n,v_n) \oplus \lambda\, G(w_{n-1},u_{n-1},v_{n-1})\| \\
&\quad +\lambda_N\alpha\theta'\|(H\circ F)(x_n) \oplus (H\circ F)(x_{n-1})\|,
\end{aligned} \tag{13}$$

where $\theta' = \frac{1}{\lambda\theta-\beta}$.

Since G is β_1-compression mapping in the first argument, β_2-compression mapping in the second argument, and β_3-compression mapping in third argument, A is λ_A-$\mathcal{D}$-Lipschitz-continuous, B is λ_B-$\mathcal{D}$-Lipschitz-continuous, and C is λ_C-$\mathcal{D}$-Lipschitz-continuous, using Algorithm 1, we have:

$$\begin{aligned}
\|G(w_n,u_n,v_n)\oplus G(w_{n-1},u_{n-1},v_{n-1})\| &= \|G(w_n,u_n,v_n)\oplus G(w_{n-1},u_n,v_n)\oplus G(w_{n-1},u_n,v_n) \\
&\quad \oplus G(w_{n-1},u_{n-1},v_n)\oplus G(w_{n-1},u_{n-1},v_n) \\
&\quad \oplus G(w_{n-1},u_{n-1},v_{n-1})\| \\
&\leq \beta_1\|w_n\oplus w_{n-1}\|+\beta_2\|u_n\oplus u_{n-1}\|+\beta_3\|v_n\oplus v_{n-1}\| \\
&\leq \beta_1\|w_n- w_{n-1}\|+\beta_2\|u_n- u_{n-1}\|+\beta_3\|v_n- v_{n-1}\| \\
&\leq \beta_1(1+n^{-1})\mathcal{D}(A(x_n),A(x_{n-1}))+\beta_2(1+n^{-1})\mathcal{D}(B(x_n),B(x_{n-1})) \\
&\quad +\beta_3(1+n^{-1})\mathcal{D}(C(x_n),C(x_{n-1})) \\
&\leq \beta_1(1+n^{-1})\lambda_A\|x_n-x_{n-1}\|+\beta_2(1+n^{-1})\lambda_B\|x_n-x_{n-1}\| \\
&\quad +\beta_3(1+n^{-1})\lambda_C\|x_n-x_n-1\| \\
&= (\beta_1\lambda_A+\beta_2\lambda_B+\beta_3\lambda_C)(1+n^{-1})\|x_n-x_{n-1}\|.
\end{aligned} \tag{14}$$

As $(H\circ F)$ is β-ordered compression mapping, we have:

$$\|(H\circ F)(x_n)\oplus (H\circ F)(x_{n-1})\| \leq \beta\|x_n\oplus x_{n-1}\|. \tag{15}$$

Using Equations (14) and (15), (13) becomes:

$$\begin{aligned}
\|x_{n+1}\oplus x_n\| &\leq \lambda_N(1-\alpha)\|x_n\oplus x_{n-1}\|+\lambda_N\alpha\theta'|\lambda|[(\beta_1\lambda_A+\beta_2\lambda_B+\beta_3\lambda_C)(1+1/n)]\|x_n-x_{n-1}\| \\
&\quad +\lambda_N\alpha\theta'\beta\|x_n\oplus x_{n-1}\| \\
&= \lambda_N(1-\alpha)\|x_n- x_{n-1}\|+\lambda_N\alpha\theta'|\lambda|[(\beta_1\lambda_A+\beta_2\lambda_B+\beta_3\lambda_C)(1+1/n)]\|x_n-x_{n-1}\| \\
&\quad +\lambda_N\alpha\theta'\beta\|x_n- x_{n-1}\| \\
&= \partial(P_n)\|x_n-x_{n-1}\|.
\end{aligned} \tag{16}$$

As $x_{n+1}\propto x_n$, $n=0,1,2,\cdots$, we have:

$$\|x_{n+1}-x_n\| \leq \partial(P_n)\|x_n-x_{n-1}\|,$$

where $\partial(P_n)=\lambda_N(1-\alpha)+\lambda_N\alpha\theta'|\lambda|[(\beta_1\lambda_A+\beta_2\lambda_B+\beta_3\lambda_C)(1+1/n)]+\lambda_N\alpha\theta'\beta$.

Let $\partial(P)=\lambda_N(1-\alpha)+\lambda_N\alpha\theta'|\lambda|[\beta_1\lambda_A+\beta_2\lambda_B+\beta_3\lambda_C]+\lambda_N\alpha\theta'\beta$.

We know that $\partial(P_n)\to\partial(P)$ as $n\to\infty$. It follows from condition (11) that $0<\partial(p)<1$, and consequently, $\{x_n\}$ is a cauchy sequence in $\mathcal{H}_p$ and since $\mathcal{H}_p$ is complete, there exists an $x\in\mathcal{H}_p$ such that $x_n\to x$, as $n\to\infty$. From Algorithm 1, we have:

$$\begin{aligned}
\|w_n\oplus w_{n+1}\| &\leq \|w_n-w_{n+1}\| \\
&\leq (1+(n+1)^{-1})\mathcal{D}(A(x_n),A(x_{n+1})) \\
&\leq (1+(n+1)^{-1})\lambda_F\|x_n-x_{n-1}\|.
\end{aligned} \tag{17}$$

$$\begin{aligned}
\|u_n\oplus u_{n+1}\| &\leq \|u_n-u_{n+1}\| \\
&\leq (1+(n+1)^{-1})\mathcal{D}(B(x_n),B(x_{n+1})) \\
&\leq (1+(n+1)^{-1})\lambda_B\|x_n-x_{n-1}\|.
\end{aligned} \tag{18}$$

$$\begin{aligned}
\textit{and}\ \|v_n\oplus v_{n+1}\| &\leq \|v_n-v_{n+1}\| \\
&\leq (1+(n+1)^{-1})\mathcal{D}(C(x_n),C(x_{n+1})) \\
&\leq (1+(n+1)^{-1})\lambda_C\|x_n-x_{n-1}\|.
\end{aligned} \tag{19}$$

It is clear from Euqations (17)–(19) that $\{w_n\}$, $\{u_n\}$, and $\{v_n\}$ are also cauchy sequences in $\mathcal{H}_p$. Let $w_n \to w$, $u_n \to u$ and $v_n \to v$, as $n \to \infty$. In view of Lemma 3, we conclude that (x, w, u, v), such that $x \in \mathcal{H}_p$, $w \in A(x)$, $u \in B(x)$ and $v \in C(x)$ is a solution of a generalized implicit set-valued variational inclusion problem with $\oplus$ operation (9). Now, we show that with $w \in A(x)$, we have:

$$\begin{aligned} d(w, A(x)) &\leq \|w \oplus w_n\| + d(w_n, A(x)) \\ &\leq \|w - w_n\| + \|w_n \oplus A(x)\| \\ &\leq \|w - w_n\| + \mathcal{D}(A(x_n), A(x)) \\ &\leq \|w - w_n\| + \lambda_A \|x_n - x\| \to 0, \ as\ n \to \infty, \end{aligned}$$

which implies that $d(w, A(x)) = 0$, and since $A(x) \in C^*(\mathcal{H}_p)$, it follows that $w \in A(x)$. Similarly, we can show that $u \in B(x)$ and $v \in C(x)$, respectively. This complete the proof. □

4. Conclusions

In this paper, we considered a generalized implicit set-valued variational inclusion problem with $\oplus$ operation, which includes many previously studied problems in ordered spaces as special cases. A resolvent operator which involves composition of two mappings was considered, and we proved some properties of it. An existence and convergence result was proven for our problem in real ordered positive Hilbert spaces.

We remark that our results may be generalized further in higher dimensional spaces.

Author Contributions: The authors made equal contributions to this paper.

Funding: This research was supported by the Ministry of Science and Technology, Taiwan [grant number: 107-2115-M-037-001].

Acknowledgments: The authors of this paper are grateful to the referees for their valuable comments which improve the paper a lot.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ahmad, R.; Ahmad, I.; Ali, I.; Al-Homidan, S.; Wang, Y.H. H(.,)-orderd-compression mapping for solving XOR-variational inclusion problem. *J. Nonlinear Convex Anal.* **2018**, *19*, 2189–2201.
2. Ahmad, R.; Ansari, Q.H. An iterative algorithm for generalized nonlinear variational inclusions. *Appl. Math. Lett.* **2000**, *13*, 23–26. [CrossRef]
3. Qin, X.; Yao, J.C. Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators. *J. Inequal. Appl.* **2016,** *2016*, 232. [CrossRef]
4. Ansari, Q.H.; Babu, F.; Regularization of proximal point algorithms in Hadamard manifolds. *J. Fixed Point Theory Appl.* **2019**, *21*, 25. [CrossRef]
5. Qin, X.; Petrusel, A.; Yao, J.C. CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces. *J. Nonlinear Convex Anal.* **2018**, *19*, 157–165.
6. Dehaish, B.A.B. A regularization projection algorithm for various problems with nonlinear mappings in Hilbert spaces. *J. Inequal. Appl.* **2015**, *2015*, 51. [CrossRef]
7. Cottle, R.W.; Giannessi, F.; Lions, J. *Variational Inequality: Theory and Applications*; John Wiley & Sons: New York, NY, USA, 1980.
8. Fang, N. Some results on split variational inclusion and fixed point problems in Hilbert spaces. *Commun. Optim. Theory* **2017**, *2017*, 5.
9. Ceng, L.C. Approximation of common solutions of a split inclusion problem and a fixed-point problem. *J. Appl. Numer. Optim.* **2019**, *1*, 1–12.

10. Nguyen, L.V.; Qin, X. Some results on strongly pseudomonotone quasi-variational inequalities. *Set-Valued Var. Anal.* **2019**. [CrossRef]
11. Alsulami, S.M.; Latif, A.; Takahashi, W. The split common fixed point problem and strong convergence theorems by hybrid methods for new demimetric mappings in Hilbert spaces. *Appl. Anal. Optim.* **2018**, *2*, 11–26.
12. Qin, X.; Cho, S.Y.; Wang, L. Strong convergence of an iterative algorithm involving nonlinear mappings of nonexpansive and accretive type. *Optimization* **2018**, *67*, 1377–1388. [CrossRef]
13. Ceng, L.C. Convergence analysis of a Mann-like iterative algorithm in reflexive Banach spaces. *Appl. Set-Valued Anal. Optim.* **2019**, *1*, 1–32.
14. Romanus, O.M.; Nnakwe, M.O.; Nnyaba, U.V. A Krasnoselskii-type algorithm for approximating zeros of monotone maps in Banach spaces with applications. *J. Nonlinear Var. Anal.* **2018**, *2*, 305–315.
15. Akram, M.; Chen, J.W.; Dilshad, M. Generalized Yosida approximation operator with an application to a system of Yosida inclusions. *J. Nonlinear Funct. Anal.* **2018**, *2018*, 17.
16. Cho, S.Y. Viscosity approximation splitting methods for monotone and nonexpansive operators in Hilbert spaces. *J. Nonlinear Convex Anal.* **2018**, *19*, 251–264.
17. Qin, X.; Yao, J.C. Projection splitting algorithms for nonself operators. *J. Nonlinear Convex Anal.* **2017**, *18*, 925–935.
18. Chang, S.S.; Wen, C.F.; Yao, J.C. Common zero point for a finite family of inclusion problems of accretive mappings in Banach spaces. *Optimization* **2018,** *67,* 1183–1196. [CrossRef]
19. Zhao, X.; Ng, K.F.; Li, C.; Yao, J.C. Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems. *Appl. Math. Optim.* **2018**, *78*, 613–641. [CrossRef]
20. Yao, Y.; Qin, X.; Yao, J.C. Projection methods for firmly type nonexpansive operators. *J. Nonlinear Convex Anal.* **2018**, *19*, 407–415.
21. Li, H.G.; Pan, X.; Deng, Z.; Wang, C. Solving GNOVI frameworks involving (γ_G, λ)-weak-GRD set-valued mappings in positive Hilbert spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 146. [CrossRef]
22. Li, H.G. A nonlinear inclusion problem involving (α, λ)-NODM set-valued mappings in ordered Hilbert space, *Appl. Math. Lett.* **2012**, *25*, 1384–1388.
23. Li, H.G. Approximation solution for general nonlinear ordered variatinal inequalities and ordered equations in ordered Banach space. *Nonlinear Anal. Forum* **2008**, *13*, 205–214.
24. Ahmad, I.; Pang, C.T.; Ahmad, R.; Ishtyak, M. System of Yosida inclusions involving XOR operator. *J. Nonlinear Convex Anal.* **2017**, *18*, 831–845.
25. Ahmad, I.; Ahmad, R.; Iqbal, J. A Resolvent approach for solving a set-valued variational inclusion problem using weak-RRD set-valued mapping. *Korean J. Math.* **2016**, *24*, 199–213. [CrossRef]
26. Ahmad, I.; Pang, C.T.; Ahmad, R.;. Ali, I. A new resolvent operator approach for solving a general variational inclusion problem involving XOR operation with convergence and stability analysis. *J. Linear Nonlinear Anal.* **2018**, *4*, 413–430.
27. Du, Y.H. Fixed points of increasing operators in ordered Banach spaces and applications. *Appl. Anal.* **1990**, *38*, 1–20. [CrossRef]
28. Nadler, S.B. Multivalued contraction mapping. *Pac. J. Math.* **1969**, *30*, 475–488. [CrossRef]