mathematics

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Solution of Ambartsumian Delay Differential Equation with Conformable Derivative

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Received: 30 March 2019; Accepted: 6 May 2019; Published: 13 May 2019

Abstract: This paper addresses the modelling of Ambartsumian equation using the conformable derivative as an application of the theory of surface brightness in astronomy. The homotopy perturbationmethod is applied to solve this model, where the approximate solution is given in terms of the conformable derivative order and the exponential functions. The present solution reduces to the corresponding one in the relevant literature as a special case. Moreover, a rapid rate of convergence has been achieved for the obtained approximate solutions. Furthermore, the accuracy of the obtained numerical results is validated via calculating the residual against the impeded parameters. It is shown graphically that the obtained residual approaches zero in various cases, which proves the efficiency of the current analysis.

Keywords: Ambartsumian equation; conformable derivative; homotopy perturbation method; series solution

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1. Introduction

In this paper, we consider a generalized model of Ambartsumian equation (AE), a delay differential equation, in the form:

$$D_t^\alpha z(t) = -z(t) + \frac{1}{q}\, z\left(\frac{t}{q}\right), \quad 0 < \alpha \le 1, \quad q > 1, \tag{1}$$

where q is a constant for the given model and α is the order of conformable derivative (CD). Equation (1) is subjectedto the initial condition:

$$z(0) = \lambda, \tag{2}$$

where λ is also a constant. When $\alpha \to 1$, Equations (1) and (2) describe the surface brightness in the Milky Way, given by [1]:

$$z'(t) = -z(t) + \frac{1}{q}\, z\left(\frac{t}{q}\right). \tag{3}$$

Moreover, Equation (3) was derived more than 25 years earlier by Ambartsumian [1] to describe the absorption of light by the interstellar matter. The existence and uniqueness of the standard model (Equations (2) and (3)) was analyzed by Kato and McLeod [2]. In the literature [3–6], the standard model Equations (2) and (3)) was solved via several analytical methods. Patade and Bhalekar [3] applied the Daftardar–Gejji and Jafari Method to obtain a power series solution and provided a theoretical analysis for convergence. Later, Bakodah and Ebaid [4] obtained the exact solution of the standard model (Equations (2) and (3)), i.e., as $\alpha \to 1$, by means of Laplace transform. In addition, Ebaid et al. [5] implemented the Adomian decomposition method (ADM) to derive an accurate approximate solution. Very recently, Alharbi and Ebaid [6] introduced a closed-form analytical solution in terms of exponential functions for the standard model (Equations (2) and (3)). They demonstrated that their new closed-form solution has many advantages over the published one by Patade and Bhalekar [3].

However, the present model (Equations (1) and (2)) is a generalized form of the standard AE (Equations (2) and (3)). In fractional calculus, Kumar et al. [7] solved the fractional model (Equations (1) and (2)) by means of the homotopy transform analysis method considering the Caputo's fractional derivative definition.

However, the conformable derivative (CD) is one of the most prominent operators in this context. To solve the generalized model (Equations (1) and (2)) using the CD, several analytical approaches can be implemented such as the Adomian decomposition method (ADM) [8–20], the homotopy perturbation method (HPM) [21–23], the differential transform method (DTM)/Taylor expansion [24,25], and the the homotopy analysis method (HAM) [6]. In addition, many applications of the CD have been recently discussed by several authors [26–29]. Recent applications and advances of fractional operators in various fields can be also found in Refs. [30–33]. Therefore, the present paper extends the application of the CD to analyze the fractional model (Equations (1) and (2)). The objective of this paper is to investigate the system (Equations (1) and (2)) in view of the CD using the HPM. It will be declared that the HPM is an effective tool to deal with the current model. Moreover, it is shown that a few terms of the series is sufficient to obtain accurate numerical results, where a very small residual error is obtained.

2. Analysis of the HPM

The HPM assumes the solution as an infinite series using an artificial parameter p. To implement this method to deal with the system (Equations (1) and (2)), we rewrite Equation (1) in the form:

$$D_t^\alpha z(t) = -z(t) + p\left(\frac{1}{q}\, z\left(\frac{t}{q}\right)\right), \tag{4}$$

where p is an embedding parameter, which is used to construct the homotopy series solution as

$$z(t) = \sum_{n=0}^{\infty} p^n z_n(t). \tag{5}$$

Inserting Equation (5) into Equation (4), we have

$$D_t^\alpha z_0(t) + z_0(t) + \sum_{n=0}^{\infty} p^{n+1}\left(D_t^\alpha z_{n+1}(t) + z_{n+1}(t) - \frac{1}{q}\, z_n\left(\frac{t}{q}\right)\right) = 0, \tag{6}$$

which leads to the following systems of initial value problems:

$$D_t^\alpha z_0(t) + z_0(t) = 0, \quad z_0(0) = \lambda, \tag{7}$$

$$D_t^\alpha z_{n+1}(t) + z_{n+1}(t) = \frac{1}{q}\, z_n\left(\frac{t}{q}\right), \quad z_{n+1}(0) = 0, \quad n \geq 0. \tag{8}$$

The conformable derivative of arbitrary order α, $0 < \alpha \leq 1$, of a function $z(t) : [0, \infty) \to \mathbb{R}$ is defined by [26–29]

$$D_t^\alpha z(t) = t^{1-\alpha}\left(\frac{dz}{dt}\right). \tag{9}$$

Accordingly, the systems in Equations (7) and (8) become

$$t^{1-\alpha} z_0'(t) + z_0(t) = 0, \quad z_0(0) = \lambda, \tag{10}$$

$$t^{1-\alpha} z_{n+1}'(t) + z_{n+1}(t) = \frac{1}{q}\, z_n\left(\frac{t}{q}\right), \quad z_{n+1}(0) = 0, \quad n \geq 0. \tag{11}$$

Equation (11) is defined as nth-order systems. The solution of the zeroth-order system (i.e., the system of order zero) in Equation (10) is obtained as

$$z_0(t) = \lambda \operatorname{Exp}\left(-\frac{t^\alpha}{\alpha}\right). \tag{12}$$

From Equations (11) and (12), the first-order system becomes

$$t^{1-\alpha} z_1'(t) + z_1(t) = \frac{\lambda}{q} \operatorname{Exp}\left(-\frac{q^{-\alpha} t^\alpha}{\alpha}\right), \quad z_1(0) = 0. \tag{13}$$

To solve the first-order ordinary differential equation in Equation (13), we rewrite such equation in the form:

$$z_1'(t) + t^{\alpha-1} z_1(t) = \frac{\lambda}{q}\, t^{\alpha-1} \operatorname{Exp}\left(-\frac{q^{-\alpha} t^\alpha}{\alpha}\right). \tag{14}$$

The solution of Equation (14) subject to the initial condition $z_1(0) = 0$ is given in terms of the integrating factor $\operatorname{Exp}\left(\frac{t^\alpha}{\alpha}\right)$ as

$$\left[\operatorname{Exp}\left(\frac{\tau^\alpha}{\alpha}\right) z_1(\tau)\right]_{\tau=0}^{t} = \frac{\lambda}{q} \int_0^t \tau^{\alpha-1} \operatorname{Exp}\left(\frac{\tau^\alpha}{\alpha}\right) \operatorname{Exp}\left(-\frac{q^{-\alpha} \tau^\alpha}{\alpha}\right) d\tau, \tag{15}$$

or

$$\operatorname{Exp}\left(\frac{t^\alpha}{\alpha}\right) z_1(t) = \frac{\lambda}{q\,(1-q^{-\alpha})} \left(\operatorname{Exp}\left(\frac{t^\alpha}{\alpha}\,(1-q^{-\alpha})\right) - 1\right). \tag{16}$$

Therefore, $z_1(t)$ is finally given by

$$z_1(t) = \frac{\lambda}{q\,(1-q^{-\alpha})} \left(\operatorname{Exp}\left(-\frac{q^{-\alpha} t^\alpha}{\alpha}\right) - \operatorname{Exp}\left(-\frac{t^\alpha}{\alpha}\right)\right). \tag{17}$$

The second-order system takes the from of Equation (11):

$$\begin{aligned} t^{1-\alpha} z_2'(t) + z_2(t) &= \frac{1}{q}\, z_1\left(\frac{t}{q}\right), \\ &= \frac{\lambda}{q^2\,(1-q^{-\alpha})} \left(\operatorname{Exp}\left(-\frac{q^{-2\alpha} t^\alpha}{\alpha}\right) - \operatorname{Exp}\left(-\frac{q^{-\alpha} t^\alpha}{\alpha}\right)\right),\ z_2(0) = 0. \end{aligned} \tag{18}$$

Proceeding as above, the solution of the system in Equation (18) can be solved as follows:

$$\begin{aligned} \left[\operatorname{Exp}\left(\frac{\tau^\alpha}{\alpha}\right) z_2(\tau)\right]_{\tau=0}^{t} &= \frac{\lambda}{q^2(1-q^{-\alpha})} \int_0^t \tau^{\alpha-1} [\left(\operatorname{Exp}\left(-\frac{q^{-\alpha}\tau^\alpha}{\alpha}\right) - \operatorname{Exp}\left(-\frac{\tau^\alpha}{\alpha}\right)\right) \operatorname{Exp}\left(\frac{\tau^\alpha}{\alpha}\right)]\, d\tau, \\ &= \frac{\lambda}{q^2(1-q^{-\alpha})} \int_0^t \tau^{\alpha-1} \left(\operatorname{Exp}\left(\frac{\tau^\alpha}{\alpha}(1-q^{-2\alpha})\right) - \operatorname{Exp}\left(\frac{\tau^\alpha}{\alpha}(1-q^{-\alpha})\right)\right) d\tau, \\ &= \frac{\lambda}{q^2(1-q^{-\alpha})} \left(I_1 - I_2\right), \end{aligned} \tag{19}$$

where I_1 and I_2 are defined, respectively, by

$$I_1 = \int_0^t \tau^{\alpha-1} \operatorname{Exp}\left(\frac{\tau^\alpha}{\alpha}(1-q^{-2\alpha})\right) d\tau, \tag{20}$$

and

$$I_2 = \int_0^t \tau^{\alpha-1} \mathrm{Exp}\left(\frac{\tau^\alpha}{\alpha}(1-q^{-\alpha})\right)\, d\tau. \tag{21}$$

The two integrals above can be explicitly calculated as

$$I_1 = \frac{1}{1-q^{-2\alpha}}\left(\mathrm{Exp}\left(\frac{t^\alpha}{\alpha}(1-q^{-2\alpha})\right)-1\right), \tag{22}$$

and

$$I_2 = \frac{1}{1-q^{-\alpha}}\left(\mathrm{Exp}\left(\frac{t^\alpha}{\alpha}(1-q^{-\alpha})\right)-1\right). \tag{23}$$

From Equation (19), we obtain $z_2(t)$ in the form:

$$z_2(t) = \frac{\lambda}{q^2\left(1-q^{-\alpha}\right)}\mathrm{Exp}\left(-\frac{t^\alpha}{\alpha}\right)\left(I_1 - I_2\right). \tag{24}$$

Implementing Equations (22) and (23), the expression $\left(\mathrm{Exp}\left(-\frac{t^\alpha}{\alpha}\right)(I_1-I_2)\right)$ in Equation (24) reads

$$\begin{aligned}
\mathrm{Exp}\left(-\frac{t^\alpha}{\alpha}\right)(I_1-I_2) \quad &= \frac{1}{1-q^{-2\alpha}}\mathrm{Exp}\left(-\frac{q^{-2\alpha}t^\alpha}{\alpha}\right) + \left(\frac{1}{1-q^{-\alpha}} - \frac{1}{1-q^{-2\alpha}}\right)\times \\
&\quad \mathrm{Exp}\left(-\frac{t^\alpha}{\alpha}\right) - \frac{1}{1-q^{-\alpha}}\mathrm{Exp}\left(-\frac{q^{-\alpha}t^\alpha}{\alpha}\right), \\
&= \frac{1}{1-q^{-2\alpha}}\mathrm{Exp}\left(-\frac{q^{-2\alpha}t^\alpha}{\alpha}\right) + \left(\frac{q^{-\alpha}}{1-q^{-2\alpha}}\right)\mathrm{Exp}\left(-\frac{t^\alpha}{\alpha}\right) - \\
&\quad \frac{1}{1-q^{-\alpha}}\mathrm{Exp}\left(-\frac{q^{-\alpha}t^\alpha}{\alpha}\right),
\end{aligned} \tag{25}$$

where the following identity:

$$\left(\frac{1}{1-q^{-\alpha}} - \frac{1}{1-q^{-2\alpha}}\right) = \frac{q^{-\alpha}}{1-q^{-2\alpha}}, \tag{26}$$

is used to obtain Equation (25). From Equations (24) and (25), we finally get $z_2(t)$ as

$$z_2(t) = \frac{\lambda}{q^2(1-q^{-\alpha})(1-q^{-2\alpha})}\left[\mathrm{Exp}\left(-\frac{q^{-2\alpha}t^\alpha}{\alpha}\right) + q^{-\alpha}\mathrm{Exp}\left(-\frac{t^\alpha}{\alpha}\right) - \left(1+q^{-\alpha}\right)\mathrm{Exp}\left(-\frac{q^{-\alpha}t^\alpha}{\alpha}\right)\right]. \tag{27}$$

In the same manner, the solutions of higher-order systems can be provided. Now, the HPM gives the n-term approximate analytic solution $\rho_n(t)$ when $p \to 1$ as

$$\rho_n(t) = \sum_{i=0}^{n-1} z_i(t). \tag{28}$$

Accordingly, we have the following three-term approximate analytic solution:

$$\begin{aligned}
\rho_3(t) \quad &= \lambda\, \mathrm{Exp}\left(-\frac{t^\alpha}{\alpha}\right) + \frac{\lambda}{q\left(1-q^{-\alpha}\right)}\left(\mathrm{Exp}\left(-\frac{q^{-\alpha}t^\alpha}{\alpha}\right) - \mathrm{Exp}\left(-\frac{t^\alpha}{\alpha}\right)\right) + \\
&\quad \left[\mathrm{Exp}\left(-\frac{q^{-2\alpha}t^\alpha}{\alpha}\right) + q^{-\alpha}\mathrm{Exp}\left(-\frac{t^\alpha}{\alpha}\right) - \left(1+q^{-\alpha}\right)\mathrm{Exp}\left(-\frac{q^{-\alpha}t^\alpha}{\alpha}\right)\right]\times \\
&\quad \frac{\lambda}{q^2\left(1-q^{-\alpha}\right)\left(1-q^{-2\alpha}\right)}.
\end{aligned} \tag{29}$$

At this point, the approximate solution in Equation (29) can be validated as $\alpha \to 1$. In that case, the approximate solution in Equation (29) reduces to

$$\begin{aligned}\rho_3(t) \quad &= \lambda\,\mathrm{Exp}\,(-t) + \frac{\lambda}{q-1}\,(\mathrm{Exp}\,(-t/q) - \mathrm{Exp}\,(-t)) + \frac{\lambda}{(q-1)\,(q^2-1)} \times \\ &\left[q\,\mathrm{Exp}\left(-t/q^2\right) + \mathrm{Exp}\,(-t) - (q+1)\,\mathrm{Exp}\,(-t/q)\right], \end{aligned} \tag{30}$$

which agrees with the third-order approximate solution in [5] for the model in Equations (1)–(3). Hence, the present results are applicable for $0 < \alpha \leq 1$ and more general than those in relevant literature [3–6] when $\alpha \to 1$. The applicability and validity of the obtained results are illustrated in the next section, where the convergence of the approximate solutions in Equation (28) is valid in the whole domain $t \geq 0$ for all values of $q > 1$. Moreover, the present approximate numerical results are validated by calculating the residual $|RE_n(t)|$ defined by

$$|RE_n(t)| = \left|t^{1-\alpha}\rho_n'(t) + \rho_n(t) - \frac{1}{q}\,\rho_n\left(\frac{t}{q}\right)\right|, \quad n \geq 1, \tag{31}$$

by using the n-term approximate solution to estimate $z(t)$.

3. Results

This section is devoted to the domain of applicability and validity of the obtained approximate solutions in the previous section for the conformable form of the AE. The convergence of the Homotopy perturbation method (HPM) was discussed in detail for ODEs (ordinary differential equations) by Ayati and Biazar [23]. For FPDEs (fractional partial differential equations), the convergence of the HPM was recently analyzed by Touchent et al. [34] and Sene and Fall [33]. Ayati and Biazar [23] proved that the series in Equation (28) is convergent in the limit if $\exists\ (0 \leq \epsilon_i < 1)$ such that $\epsilon_i = \frac{\|z_{i+1}\|}{\|z_i\|}$, $\forall\, i \in N$, where $\|f(t)\| = \max\limits_{0\leq t\leq 1} |f(t)|$. For illustration of convergence, we consider $\alpha = 0.8$, $\lambda = 1$, and $q = 1.4$. Accordingly, we have $\epsilon_1 = 0.2988$, $\epsilon_2 = 0.3853$, $\epsilon_3 = 0.1932$, $\epsilon_4 = 0.1089$, $\epsilon_5 = 0.0657$, $\epsilon_6 = 0.0414$, $\epsilon_7 = 0.0268$, $\epsilon_8 = 0.0175$, and $\epsilon_9 = 0.0120$, which proves the convergence of the present approximate solutions, even if the domain $0 \leq t \leq 1$ is extended. If the domain is enlarged to be $0 \leq t \leq \tau,\ \ \tau > 1$, we still have $0 \leq \epsilon_i < 1$, $\forall\, i \in N$.

The convergence of the sequence $\rho_n(t)$ in Equation (28) is demonstrated in Figure 1, where the approximate solutions $\rho_6(t)$, $\rho_8(t)$ and $\rho_{10}(t)$ are depicted at $\alpha = 0.8$, $\lambda = 1$, and $q = 1.6$. This figure shows that the approximate solutions using a few terms approach to a certain curve as the number of terms increases, even for large values of the independent variable. This may explain how the convergence of approximate solution is estimated, in addition to the above analysis of convergence. Furthermore, it is shown below that the approximate solution $\rho_{10}(t)$ leads to highly accurate numerical results, especially, when $q \geq 2$.

The variation of $\rho_{10}(t)$ versus t is displayed in Figure 2 at various values of the arbitrary order α of the CD. To validate the current results, the residual $|RE_{10}|$ is plotted versus t in Figures 3 and 4 at two particular values of the delay parameter q, $q = 1.6$ and $q = 2$, respectively, for various values of α. The obtained results in these two figures reveal that $|RE_{10}|$ is small when $q = 1.6$, while it rapidly decreases as q increases as observed from Figure 4 when $q = 2$. In addition, the profile of $|RE_{10}|$ against two sub-domains of the parameter q (> 1) is introduced through Figure 5 $(1.1 \leq q \leq 1.6)$ and Figure 6 $(5 \leq q \leq 10)$ in the domain $t \in [0, 50]$ at $\lambda = 1$ and $\alpha = 0.8$. It is noticed from these figures that the maximum values of $|RE_{10}|$ in these sub-domains are, respectively, 6×10^{-2} and 2×10^{-16}. In addition, it can be concluded that the residual becomes very small and rapidly decreases (approaches zero) for relatively higher values of q, i.e., when $q \geq 2$.

At $\alpha = 0.8$ and $q = 2$, the impact of the initial condition λ $(0 \leq \lambda \leq 5)$ on the variation of $|RE_{10}|$ is depicted in Figure 7. Figure 8 shows the profile of $|RE_{10}|$ verses the arbitrary order α $(0.5 \leq \alpha \leq 1)$.

Finally, at a particular value of the independent variable $t = 100$, the behavior of the residual $|RE_{10}|$ is presented in Figure 9 versus both λ $(0 \leq \lambda \leq 5)$ and q $(2 \leq q \leq 10)$. The obtained results in Figures 7–9 confirm that the present 10-term approximate solution $\rho_{10}(t)$ is highly accurate. Therefore, the preceding discussion shows the effectiveness of the HPM in accurately and quickly solving the generalized form of AE in view of the CD.

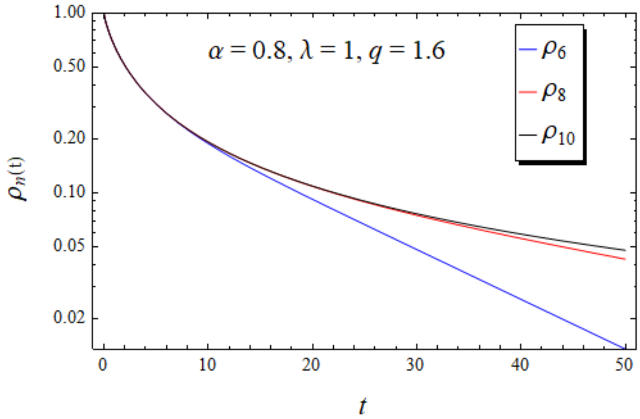

Figure 1. Logarithmic plot of the approximate solutions.

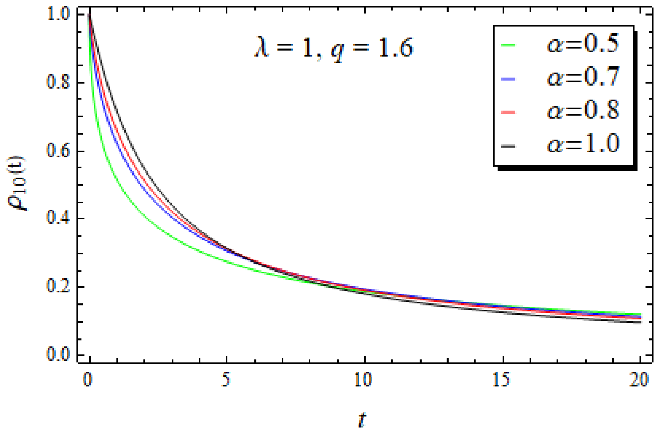

Figure 2. The variation of the approximate solution versus the conformable derivative order α.

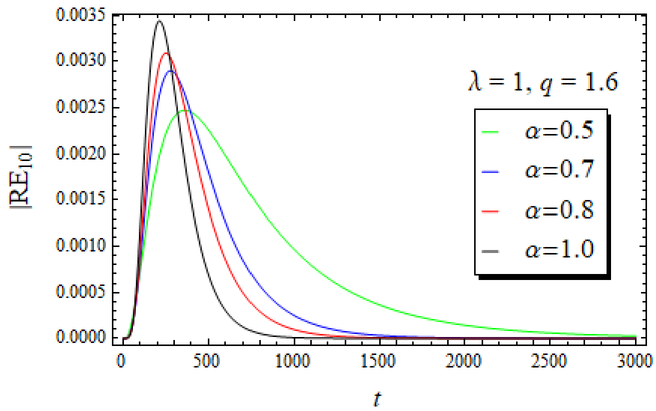

Figure 3. Effect of the conformable derivative order α on the residual at $\lambda = 1$, $q = 1.6$.

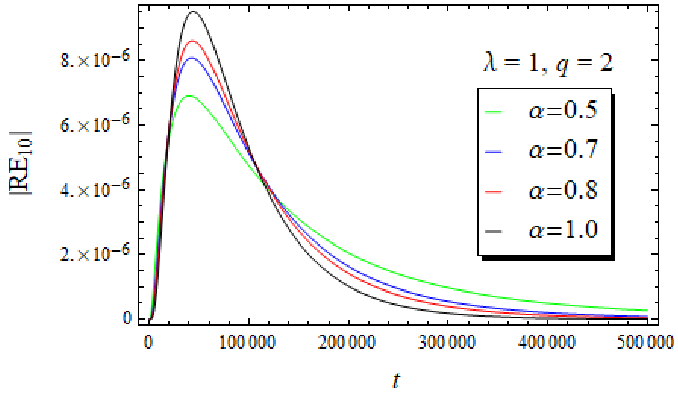

Figure 4. Effect of the conformable derivative order α on the residual at $\lambda = 1$, $q = 2$.

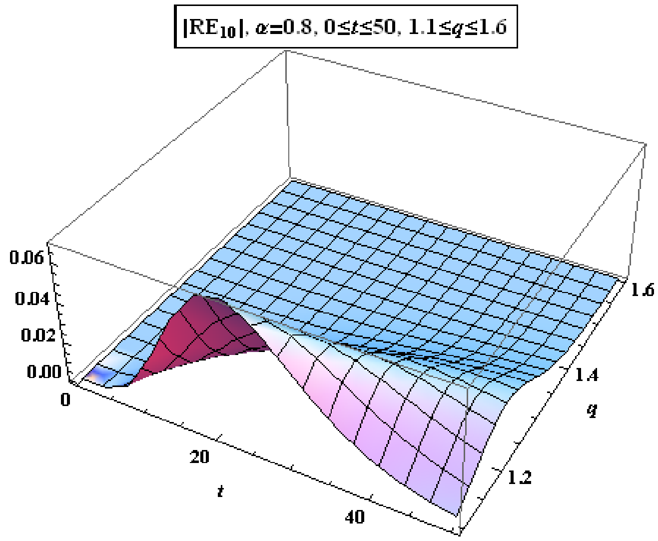

Figure 5. Residual at $\alpha = 0.8$, $\lambda = 1$, $1.1 \leq q \leq 1.6$.

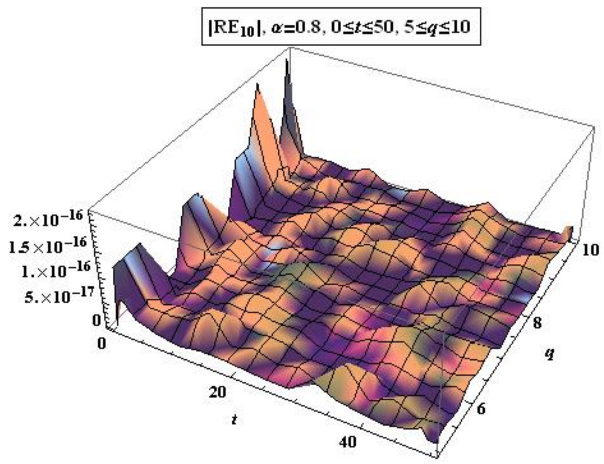

Figure 6. Residual at $\alpha = 0.8$, $\lambda = 1$, $5 \leq q \leq 10$.

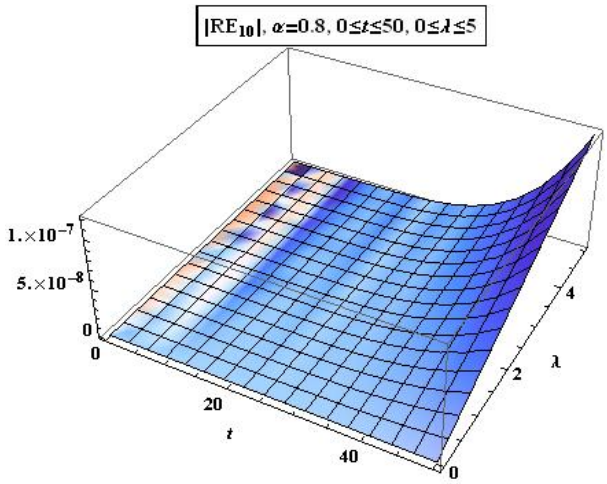

Figure 7. Residual at $\alpha = 0.8$, $q = 2$, $0 \leq \lambda \leq 5$.

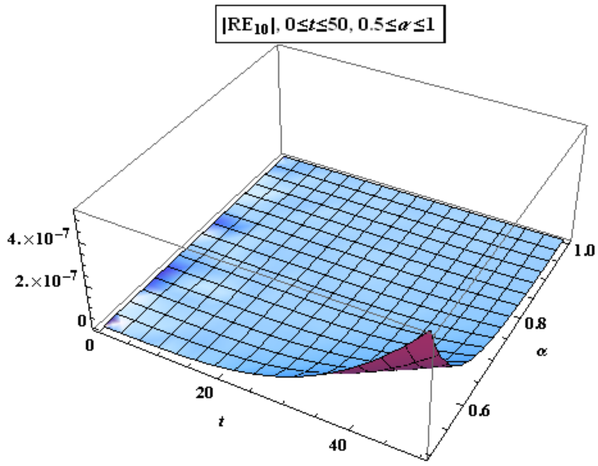

Figure 8. Residual at $\lambda = 1$, $q = 2$, $0.5 \leq \alpha \leq 1$.

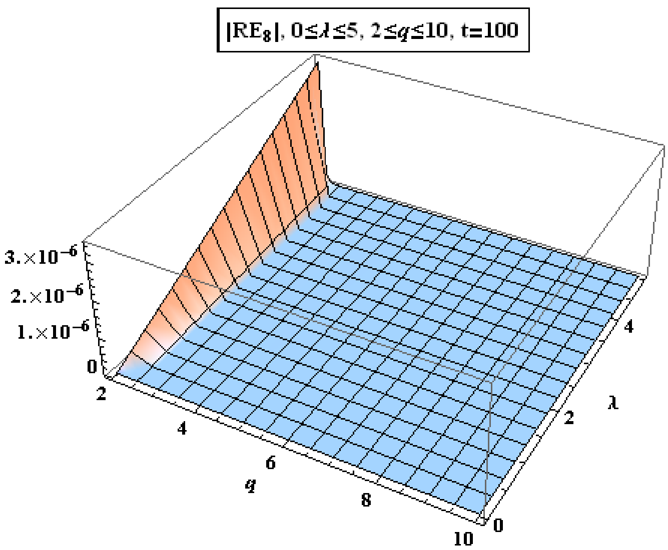

Figure 9. Residual at $\alpha = 0.8$, $2 \leq q \leq 10$, $0 \leq \lambda \leq 5$, $t = 100$.

4. Conclusions

The homotopy perturbation method is applied in this paper to solve the fractional form of the Ambartsumian equation in view of the conformable derivative. The obtained approximate solution is expressed in terms of the exponential functions and the arbitrary order of the conformable derivative. The present approximate solution reduces to the corresponding one in the literature when the order of the conformable derivative tends to unity. Moreover, the impacts of the initial condition, the delay parameter, and the arbitrary order of the conformable derivative on the residual error are discussed in detail. It is also declared that the residual achieved by using only ten terms is very small for moderate values of the delay parameter while such residual approaches zero as the the delay parameter increases.The obtained results are applicable for $0 < \alpha \leq 1$ and more general than those in relevant literature [3–6] in which $\alpha \to 1$. The obtained residual errors in Figures 7–9 are less than 3×10^{-6}, which confirms the accuracy of the approximate solution. In addition, the obtained residual error in Figure 6 is less than 2×10^{-16} which is highly accurate.

Author Contributions: The authors participated equally in this work. All the authors read and approved the final manuscript.

Funding: The authors would like to acknowledge financial support for this work from the Deanship of Scientific Research (DSR), University of Tabuk, Tabuk, Saudi Arabia, under Grant no. S-015-1439.

Conflicts of Interest: The authors declare no conflict of interest.

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