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On $(\Lambda, \Upsilon, \Re)$-Contractions and Applications to Nonlinear Matrix Equations

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Abstract: In this paper, we study the behavior of $(\Lambda, \Upsilon, \Re)$-contraction mappings under the effect of comparison functions and an arbitrary binary relation. We establish related common fixed point theorems. We explain the significance of our main theorem through examples and an application to a solution for the following nonlinear matrix equations:
$X = D + \sum_{i=1}^{n} A_i^* X A_i - \sum_{i=1}^{n} B_i^* X B_i$
$X = D + \sum_{i=1}^{n} A_i^* \gamma(X) A_i$,
where D is an Hermitian positive definite matrix, A_i, B_i are arbitrary $p \times p$ matrices and $\gamma : H(p) \to P(p)$ is an order preserving continuous map such that $\gamma(0) = 0$. A numerical example is also presented to illustrate the theoretical findings.

Keywords: fixed point; binary relation; Λ-contraction; comparison function; nonlinear matrix equation

1. Introduction

The process of generalizations and improvements of the Banach Contraction Principle [1] (1922) geared up after the result of Kannan [2] in 1968, where he showed that discontinuous self-mapping has a unique fixed point, see Reference [3–21]. Recently, Wardowski [22] introduced a nonlinear function F under the assumptions $(F_1) - (F_3)$ (defined below) and presented a fixed point theorem for F-contractions. Piri et al. [23] replaced assumption (F_3) by continuity of function F and proved a fixed point theorem, and in this way, presented the Wardowski theorem under weak conditions. In 2014, Jleli et al. [24,25] presented another important generalization of the Banach Contraction Principle, known as θ-contractions. Liu et al. [26] discussed some important aspects of both F-contractions and θ-contractions. Liu et al. [26] also introduced weak θ-contractions ($\tilde{\theta}$-contractions) and proved that "an F-contraction and a $\tilde{\theta}$-contraction are equivalent". In the same paper, Liu et al. introduced a (Λ, Υ)-contraction, which contained both F-contractions and θ-contractions and established an important fixed point theorem extending corresponding theorems in References [22–25].

In this paper, we present a generalized $(\Lambda, \Upsilon, \Re)$-contraction containing (Λ, Υ)-contractions and, in particular, both F-contractions and θ-contractions. We prove a fixed point theorem, which generalizes the results of Liu et al. [26]. Our presented results are subject to a binary relation $\Re$, a comparison function, a generalized $(\Lambda, \Upsilon, \Re)$-contraction and two self-mappings, which are assumed to be closed and continuous.

2. Preliminaries

Let $F : (0, \infty) \to (-\infty, \infty)$ be such that

(F_1): F is strictly increasing;

(F_2): for each sequence $\{\gamma_n\}_{n=1}^{\infty} \subset (0, \infty)$;

$$\lim_{n\to\infty} F(\gamma_n) = -\infty \Leftrightarrow \lim_{n\to\infty} \gamma_n = 0;$$

(F_3): there is $\kappa \in (0, 1)$ such that $\lim_{t\to 0^+} t^{\kappa} F(t) = 0$.

Define $\mathcal{F} = \{F : \ F \text{ satisfies } (F_1) - (F_3)\}$.

Definition 1. *[22] The self-mapping $T : \Im \to \Im$ defined on a metric space $(\Im, d)$ is called an F-contraction, if there are $\tau > 0$ and $F \in \mathcal{F}$ so that*

$$\forall \zeta, \eta \in \Im, \ d(T(\zeta), T(\eta)) > 0 \Rightarrow \tau + F(d(T(\zeta), T(\eta))) \leq F(d(\zeta, \eta)),$$

Wardowski [22] proved the following remarkable result.

Theorem 1. *[22] Let $(\Im, d)$ be a complete metric space. Then every F-contraction has a unique fixed point.*

For other results dealing with F-contractions, see Reference [27–29]. Piri et al. [23] modified assumption (F_3) of F as follows:
$(\acute{F}_3)$: F is continuous.

Consider $\mathcal{F}^* = \{F : \ F \text{ satisfies } (F_1), (F_2) \text{ and } (\acute{F}_3)\}$.

Definition 2. *[23] The self-mapping $T : \Im \to \Im$ defined on a metric space $(\Im, d)$ is an $\acute{F}$-contraction, if there are $\tau > 0$ and $F \in \mathcal{F}^*$ so that*

$$\forall \zeta, \eta \in \Im, \ d(T(\zeta), T(\eta)) > 0 \Rightarrow \tau + F(d(T(\zeta), T(\eta))) \leq F(d(\zeta, \eta)).$$

Piri et al. [23] proved the following result.

Theorem 2. *[23] Let $(\Im, d)$ be a complete metric space. Then every $\acute{F}$-contraction has a unique fixed point.*

Remark 1. *It is interesting to note that there exists at least one function F defined by $F(\varsigma) = -\frac{1}{\varsigma^q}$; $q \geq 1$ which belongs to the set $\mathcal{F}^*$, but not a member of $\mathcal{F}$. Similarly, there is at least one function F defined by*

$$F(\varsigma) = -\frac{1}{(\varsigma + [\varsigma])^k}; k \in \left(0, \frac{1}{l}\right), \ l > 1$$

which belongs to $\mathcal{F}$, but not to $\mathcal{F}^*$. However, there is also at least one function F defined by $F(\varsigma) = \ln(\varsigma)$ which belongs to both $\mathcal{F}^*$ and $\mathcal{F}$. Thus, we conclude that the sets $\mathcal{F}^*$ and $\mathcal{F}$ are overlapping.

Recently, Jleli et al. [24] introduced θ-contractions as follows:

The self-mapping $T : \Im \to \Im$ defined on $(\Im, d)$ is a θ-contraction, if there are $\theta \in \Theta$ and $k \in (0,1)$ so that

$$\zeta, \eta \in \Im, d(T(\zeta), T(\eta)) \neq 0 \Longrightarrow \theta\left(d(T(\zeta), T(\eta))\right) \leq \left[\theta\left(d(\zeta,\eta)\right)\right]^k,$$

where $\Theta = \{\theta : (0,\infty) \to (1,\infty) \mid \theta \text{ satisfies } (\Theta_1) - (\Theta_3)\}$
(Θ_1) θ is non-decreasing;
(Θ_2) for each positive sequence $\{t_n\}$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ iff } \lim_{n\to\infty} t_n = 0^+;$$

(Θ_3) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\longrightarrow 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Theorem 3. *[24] Let $(\Im, d)$ be a complete metric space. Then every θ-contraction mapping has a unique fixed point.*

Liu et al. [26] observed that the condition (Θ_3) can be relaxed to (Θ_3'):

(Θ_3'): θ is continuous.

Let $\tilde{\Theta} = \{\theta : (0,\infty) \to (1,\infty) \mid \theta \text{ satisfies } (\Theta_1), (\Theta_2) \text{ and } (\Theta_3')\}$.

Liu et al. ([26] Theorem 1.7) showed that θ-contractions $(\theta \in \tilde{\Theta})$ and $\acute{F}$-contractions are equivalent. In the same paper, let $\Lambda : (0,\infty) \to (0,\infty)$ be such that
(C_1) Λ is non-decreasing;
(C_2) for each positive sequence $\{t_n\}$,

$$\lim_{n\to\infty} \Lambda(t_n) = 0 \text{ if and only if } \lim_{n\to\infty} t_n = 0;$$

(C_3) Λ is continuous.

Let $\mathcal{D} = \{\Lambda : (0,\infty) \to (0,\infty) \mid \Lambda \text{ satisfies } (C_1) - (C_3)\}$.

The idea of comparison functions was introduced by Berinde [30]. The function $\Upsilon : (0,\infty) \to (0,\infty)$ is said to be a comparison function if:

(i) Υ is monotone increasing, that is, $\sigma < \varsigma$ implies that $\Upsilon(\sigma) \leq \Upsilon(\varsigma)$;

(ii) $\lim_{n\to\infty} \Upsilon^n(t) = 0$ for all $t > 0$, where Υ^n stands for the nth iterate of Υ.

If Υ is a comparison function, then $\Upsilon(t) < t$ for every $t > 0$. Examples of comparison functions can be seen in Reference [30].

Lemma 1. *[26] Let $\Lambda : (0,\infty) \to (0,\infty)$ be a continuous and non-decreasing function such that $\inf_{t\in(0,\infty)} \Lambda(t) = 0$. Then, for a positive sequence $\{t_k\}$,*

$$\lim_{k\to\infty} \Lambda(t_k) = 0 \text{ if and only if } \lim_{k\to\infty} t_k = 0.$$

Definition 3. *Let $\Im$ be a non-empty set and $\Re$ be a binary relation on $\Im$. Then $\Re$ is transitive if $(\sigma_1,\sigma_2) \in \Re$ and $(\sigma_2,\sigma_3) \in \Re$ implies that $(\sigma_2,\sigma_3) \in \Re$, for all $\sigma_1,\sigma_2,\sigma_3 \in \Im$.*

Definition 4. *[31,32] Let ϕ be a self-mapping on $\Im$ a non-empty set. A binary relation $\Re$ on $\Im$ is said to be ϕ-closed if for all $\sigma,\varsigma \in \Im$, we have*

$$(\sigma,\varsigma) \in \Re \Rightarrow (\phi(\sigma),\phi(\varsigma)) \in \Re.$$

Definition 5. *[32] Let $\sigma,\varsigma \in \Im$ and $\Re$ be a binary relation $\Im$. A path (of length $n \in \mathbb{N}$) in $\Re$ from σ to ς is a sequence $\{\sigma_0,\sigma_1,\sigma_2,...,\sigma_n\} \subseteq \Im$ such that*
(i) $\sigma_0 = \sigma$ and $\sigma_n = \varsigma$;
(ii) $(\sigma_j,\sigma_{j+1}) \in \Re$ for all $j \in \{0,1,2,...,n-1\}$.

Denote by $\Gamma(\sigma,\varsigma,\Re)$ the set of all paths from σ to ς in $\Re$. The path of length n involves $n+1$ elements of $\Im$, although they are not necessarily distinct.

Definition 6. *[14] The metric space $(\Im,d)$ equipped with a binary relation $\Re$ is $\Re$-regular if for each sequence $\{\zeta_n\}$ in $\Im$,*

$$(\zeta_n,\zeta_{n+1}) \in \Re \text{ and } \zeta_n \to \zeta \in \Im \text{ implies } (\zeta_n,\zeta) \in \Re, \text{ for all } n \in \mathbb{N}.$$

Definition 7. *[33] Let $\phi,\psi : \Im \to \Im$ be two self-mappings on $\Im$ a non-empty set. Then a binary relation $\Re$ on $\Im$ is said (ϕ,ψ)-closed if for any $\sigma,\varsigma \in \Im$, we have*

$$(\sigma,\varsigma) \in \Re \Rightarrow (\phi(\sigma),\psi(\varsigma)),(\psi(\sigma),\phi(\varsigma)) \in \Re.$$

Recently, Al-Sulami et al. [34] investigated fixed points of Θ-contractions under the effect of binary relation $\Re$ in complete metric spaces. Similarly, Zada and Sarwar [33] considered F-contractions under an arbitrary binary relation $\Re$ and proved some related fixed point results.

3. Fixed Point Theorems

We shall develop an iteration method to ensure the existence of common fixed points of two self-mappings under the effect of $(\Lambda,\Upsilon,\Re)$ contraction, where, $\Re$ is a binary relation. This process will then be explained through an example. Finally, we shall apply the obtained results to ensure the existence of a solution of nonlinear matrix equations. We start with the following definition.

Definition 8. *Let ϕ,ψ be two self-mappings defined on the metric space $(\Im,d)$. Let $\Re$ be a binary relation on $\Im$. Consider $\chi = \{(\sigma,\varsigma) \in \Re : d(\phi(\sigma),\psi(\varsigma)) > 0\}$. The pair (ϕ,ψ) is said to be $(\Lambda,\Upsilon,\Re)$ contraction if there exists a continuous comparison function Υ and $\Lambda \in D$ such that, for all $(\sigma,\varsigma) \in \chi$*

$$\Lambda(d(\phi(\sigma),\psi(\varsigma))) \le \Upsilon\left(\Lambda\left(\max\left\{d(\sigma,\varsigma),\frac{d(\sigma,\psi(\varsigma))\,d(\varsigma,\phi(\sigma))}{1+d(\sigma,\varsigma)}\right\}\right)\right). \tag{1}$$

Denote by $M((\phi,\psi);\Re)$ the set of all order pairs $(\sigma,\varsigma) \in M \times M$ such that $(\phi(\sigma),\psi(\varsigma)) \in \Re$.
Our main results:

Theorem 4. *Let $(\Im,d)$ be a complete metric space, $\Re$ be a binary relation on $\Im$ and $\phi,\psi : \Im \to \Im$ be two mappings. If ϕ and ψ satisfy the following conditions:*

(a) $M((\phi,\psi);\Re)$ *is non-empty,*
(b) $\Re$ *is* (ϕ,ψ)*-closed,*
(c) (ϕ,ψ) *is* $(\Lambda,\Upsilon,\Re)$ *contraction,*
(d) ϕ *and* ψ *are continuous.*

Then ϕ and ψ have a common fixed point in $\Im$.

Proof. Let $(\sigma,\varsigma)\in M((\phi,\psi);\Re)$ then $(\phi(\sigma),\psi(\varsigma))\in\Re$. Define the sequence $\{\zeta_n\}$ in $\Im$ by

$$\begin{cases} \zeta_{2n+1}=\phi(\zeta_{2n}), \\ \zeta_{2n+2}=\psi(\zeta_{2n+1}) \end{cases}, \text{ where } n\in\mathbb{N}\cup\{0\}.$$

If $\zeta_{2n^*+1}=\zeta_{2n^*}$ for some $n^*\in\mathbb{N}\cup\{0\}$. Then ζ_{2n^*} is a common fixed point of ϕ and ψ. If $\zeta_{2n+1}\neq\zeta_{2n}$ for all $n\in\mathbb{N}\cup\{0\}$. Then $d(\phi(\zeta_{2n}),\psi(\zeta_{2n+1}))>0$ for all $n\in\mathbb{N}\cup\{0\}$. From (b), we get

$$\begin{array}{ll} (\zeta_1,\zeta_2)=(\phi(\zeta_0),\psi(\zeta_1))\in\Re, & (\zeta_2,\zeta_3)=(\psi(\zeta_1),\phi(\zeta_2))\in\Re, \\ (\zeta_3,\zeta_4)=(\phi(\zeta_2),\psi(\zeta_3))\in\Re, & (\zeta_4,\zeta_5)=(\psi(\zeta_3),\phi(\zeta_4))\in\Re, \\ \quad\ldots \qquad\qquad\qquad\qquad\qquad, & (\zeta_{2n},\zeta_{2n+1})=(\psi(\zeta_{2n-1}),\phi(\zeta_{2n}))\in\Re. \end{array}$$

Thus, $(\zeta_{2n},\zeta_{2n+1})\in\chi$, for all $n\in\mathbb{N}\cup\{0\}$. By setting $\sigma=\zeta_{2n}$, $\varsigma=\zeta_{2n-1}$ in (1), and by using (c), we have

$$\begin{aligned} \Lambda(d(\zeta_{2n},\zeta_{2n+1})) &= \Lambda(d(\zeta_{2n+1},\zeta_{2n})) \\ &= \Lambda(d(\phi(\zeta_{2n}),\psi(\zeta_{2n-1}))) \\ &\leq \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} d(\zeta_{2n},\zeta_{2n-1}), \\ \frac{d(\zeta_{2n},\psi(\zeta_{2n-1}))d(\zeta_{2n-1},\phi(\zeta_{2n}))}{1+d(\zeta_{2n},\zeta_{2n-1})} \end{array}\right\}\right)\right) \\ &= \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} d(\zeta_{2n},\zeta_{2n-1}), \\ \frac{d(\zeta_{2n},\zeta_{2n})d(\zeta_{2n-1},\zeta_{2n+1})}{1+d(\zeta_{2n},\zeta_{2n-1})} \end{array}\right\}\right)\right) \\ &= \Upsilon(\Lambda(d(\zeta_{2n},\zeta_{2n-1}))), \end{aligned}$$

for all $n\in\mathbb{N}\cup\{0\}$. Similarly, setting $\sigma=\zeta_{2n}$, $\varsigma=\zeta_{2n+1}$ in (1), and again from (c), we get

$$\begin{aligned} \Lambda(d(\zeta_{2n+1},\zeta_{2n+2})) &= =\Lambda(d(\phi(\zeta_{2n}),\psi(\zeta_{2n+1}))) \\ &\leq \Upsilon\left(\Lambda\left(\max\left\{d(\zeta_{2n},\zeta_{2n+1}),\frac{d(\zeta_{2n},\psi(\zeta_{2n+1}))\,d(\zeta_{2n+1},\phi(\zeta_{2n}))}{1+d(\zeta_{2n},\zeta_{2n+1})}\right\}\right)\right) \\ &= \Upsilon\left(\Lambda\left(\max\left\{d(\zeta_{2n},\zeta_{2n+1}),\frac{d(\zeta_{2n},\zeta_{2n+2})\,d(\zeta_{2n+1},\zeta_{2n+1})}{1+d(\zeta_{2n},\zeta_{2n+1})}\right\}\right)\right) \\ &= \Upsilon(\Lambda(d(\zeta_{2n},\zeta_{2n+1}))). \end{aligned}$$

In general, we have

$$\Lambda(d(\zeta_n,\zeta_{n+1}))\leq\Upsilon(\Lambda(d(\zeta_{n-1},\zeta_n))), \text{ for all } n\in\mathbb{N}.$$

This implies

$$\begin{aligned} \Lambda(d(\zeta_n,\zeta_{n+1})) &\leq \Upsilon(\Lambda(d(\zeta_n,\zeta_{n+1})))\leq\Upsilon^2(\Lambda(d(\zeta_{n-1},\zeta_n))) \\ &\leq \ldots\leq\Upsilon^n(\Lambda(d(\zeta_0,\sigma))). \end{aligned}$$

Letting $n\to\infty$ in the above inequality, we get

$$0\leq\lim_{n\longrightarrow\infty}\Lambda(d(\zeta_n,\zeta_{n+1}))\leq\lim_{n\longrightarrow\infty}\Upsilon^n(\Lambda(d(\zeta_0,\sigma)))=0.$$

This implies that

$$\lim_{n\longrightarrow\infty}\Lambda\left(d\left(\zeta_n,\zeta_{n+1}\right)\right)=0.$$

By (C_2) and Lemma 1, we get

$$\lim_{n\to\infty}d\left(\zeta_n,\zeta_{n+1}\right)=0. \tag{2}$$

We claim that $\{\zeta_n\}$ is a Cauchy sequence. We argue by contradiction. Assume that there are $\varepsilon>0$ and sequences $\left\{\hat{h}_n\right\}_{n=1}^{\infty}$ and $\{\hat{\jmath}_n\}_{n=1}^{\infty}$ such that for each $n\in\mathbb{N}\cup\{0\}$, $\hat{h}_n>\hat{\jmath}_n>n$, we have $d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)\geq\varepsilon$, $d\left(\zeta_{\hat{h}(n)-1},\zeta_{\hat{\jmath}(n)}\right)<\varepsilon$. Thus,

$$\begin{aligned}\varepsilon &\leq d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)\leq d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)-1}\right)+d\left(\zeta_{\hat{h}(n)-1},\zeta_{\hat{\jmath}(n)}\right) \\ &< \varepsilon+d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)-1}\right).\end{aligned} \tag{3}$$

Taking $n\to\infty$ in (3), we get

$$\lim_{n\to\infty}d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)=\varepsilon. \tag{4}$$

Again using triangle inequality, we have

$$d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)\leq d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)+1}\right)+d\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)+1}\right)+d\left(\zeta_{\hat{\jmath}(n)+1},\zeta_{\hat{\jmath}(n)}\right) \tag{5}$$

and

$$d\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)+1}\right)\leq d\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{h}(n)}\right)+d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)+d\left(\zeta_{\hat{\jmath}(n)},\zeta_{\hat{\jmath}(n)+1}\right). \tag{6}$$

Taking $n\to\infty$ in (5) and (6), we get

$$\lim_{n\to\infty}d\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)+1}\right)=\varepsilon. \tag{7}$$

Let $\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)})\in\Re$. Since $\Re$ is (ϕ,ψ)-closed, we have $\left(\phi\left(\zeta_{\hat{h}(n)}\right),\psi\left(\zeta_{\hat{\jmath}(n)}\right)\right)=\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)+1}\right)\in\Re$ and from (7), we have $d\left(\phi\left(\zeta_{\hat{h}(n)+1}\right),\psi\left(\zeta_{\hat{\jmath}(n)+1}\right)\right)>0$. Thus $\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)+1}\right)\in\chi$. Similarly, by (2) and (4), we can choose a positive integer $n_0\geq 1$ such that for all $n\geq n_0$ we have from (1)

$$\begin{aligned}0 &< \Lambda\left(d\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)+1}\right)\right)\leq\Lambda\left(d\left(\phi\left(\zeta_{\hat{h}(n)}\right),\psi\left(\zeta_{\hat{\jmath}(n)}\right)\right)\right) \\ &\leq \Upsilon\left(\Lambda\left(\max\left\{d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right),\frac{d\left(\zeta_{\hat{h}(n)},\psi\left(\zeta_{\hat{\jmath}(n)}\right)\right)d\left(\zeta_{\hat{\jmath}(n)},\phi\left(\zeta_{\hat{h}(n)}\right)\right)}{1+d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)}\right\}\right)\right) \\ &= \Upsilon\left(\Lambda\left(\max\left\{d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right),\frac{d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)+1}\right)d\left(\zeta_{\hat{\jmath}(n)},\zeta_{\hat{h}(n)+1}\right)}{1+d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)}\right\}\right)\right) \\ &\leq \Upsilon\left(\Lambda\left(\max\left\{d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right),\frac{\left[d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)+1}\right)+d\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)+1}\right)\right]d\left(\zeta_{\hat{\jmath}(n)},\zeta_{\hat{h}(n)+1}\right)}{1+d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)}\right\}\right)\right) \\ &= \Upsilon\left(\Lambda\left(\max\left\{d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right),\frac{d\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)+1}\right)d\left(\zeta_{\hat{\jmath}(n)},\zeta_{\hat{h}(n)+1}\right)}{1+d\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)}\right\}\right)\right),\end{aligned}$$

for all $n \geq n_0$. Taking the limit as $n \to \infty$ and using (2), (4) and (7), we get

$$\begin{aligned}\Lambda(\varepsilon) &\leq \Lambda\left(\lim_{n\to\infty} d\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right)\right) \leq \lim_{n\to\infty} \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} d\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right), \\ \frac{d\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) d\left(\zeta_{\hat{j}(n)}, \zeta_{\hat{h}(n)+1}\right)}{1+d\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right)} \end{array}\right\}\right)\right) \\ &= \Upsilon\left(\Lambda(\varepsilon, \frac{\varepsilon.\varepsilon}{1+\varepsilon})\right) \leq \Upsilon\left(\Lambda(\varepsilon, \frac{\varepsilon^2}{\varepsilon})\right) = \Upsilon\left(\Lambda(\varepsilon)\right) < \Lambda(\varepsilon).\end{aligned}$$

This is a contradiction. Therefore, $\{\zeta_n\}$ is a Cauchy sequence in $\Im$. Due to completeness of $\Im$, we can find $\zeta^* \in \Im$ such that

$$\lim_{n\to\infty} d\left(\zeta_n, \zeta^*\right) = 0. \tag{8}$$

Next, we prove that $\psi\left(\zeta^*\right) = \phi\left(\zeta^*\right) = \zeta^*$. Since ψ and ϕ are continuous and

$$\lim_{n\to\infty} d\left(\zeta_{2n}, \zeta^*\right) = 0 = \lim_{n\to\infty} d\left(\zeta_{2n-1}, \zeta^*\right),$$

we find that

$$\lim_{n\to\infty} d\left(\zeta_{2n+1}, \phi\left(\zeta^*\right)\right) = \lim_{n\to\infty} d\left(\phi\left(\zeta_{2n}\right), \phi\left(\zeta^*\right)\right) = 0, \tag{9}$$

and

$$\lim_{n\to\infty} d\left(\zeta_{2n}, \psi\left(\zeta^*\right)\right) = \lim_{n\to\infty} d\left(\psi\left(\zeta_{2n-1}\right), \psi\left(\zeta^*\right)\right) = 0.$$

In view of the limit uniqueness, we obtain that $\phi\left(\zeta^*\right) = \zeta^*$ and $\psi\left(\zeta^*\right) = \zeta^*$, which yields that $\phi\left(\zeta^*\right) = \psi\left(\zeta^*\right) = \zeta^*$. Thus, ϕ and ψ have a common fixed point $\zeta^* \in \Im$. □

The following theorem shows that if the set $\Gamma(\sigma, \varsigma, \Re)$ is non-empty, then the common fixed point of mappings ϕ and ψ is unique.

Theorem 5. *Let $(\Im, d)$ be a complete metric space and $\Re$ be a transitive relation on $\Im$. Let $\phi, \psi : \Im \to \Im$. Suppose that*
(a) $M(\ (\phi, \psi)\ ; \Re)$ and $\Gamma(\sigma, \varsigma, \Re)$ are non-empty,
(b) there exist a continuous comparison function Υ and a function $\Lambda \in \mathcal{D}$ so that for all $(\sigma, \varsigma) \in \chi$,

$$\Lambda\left(d\left(\phi\left(\sigma\right), \psi\left(\varsigma\right)\right)\right) \leq \Upsilon\left(\Lambda\left(\max\left\{\frac{1}{2}d\left(\sigma, \varsigma\right), \frac{d\left(\sigma, \psi\left(\varsigma\right)\right) d\left(\varsigma, \phi\left(\sigma\right)\right)}{2\left[1 + d\left(\sigma, \varsigma\right)\right]}\right\}\right)\right), \tag{10}$$

(c) ϕ and ψ are continuous,
(d) $\Re$ is (ϕ, ψ)-closed.

Then there is a unique common fixed point of ϕ and ψ in $\Im$.

Proof. As in the proof of Theorem 4, there is a common fixed point of ϕ and ψ. If υ and υ^* are two common fixed points of ϕ and ψ such that $\upsilon \neq \upsilon^*$, then since $\Gamma((\upsilon, \upsilon^*, \Re)$ is the class of paths in $\Re$ from υ to υ^*, there is a path of finite length L, that is, there is a sequence $\{F_0, F_1, F_3, ..., F_L\}$ in $\Re$ from ν to υ^* with

$$F_0 = \upsilon, F_L = \upsilon^*,\ (F_j, F_{j+1}) \in \Re \text{ for } j = 0, 1, 2, ..., (L-1).$$

Using the transitivity of $\Re$,

$$(\upsilon, F_1) \in \Re, (F_1, F_2) \in \Re, ..., (F_{k-1}, \upsilon^*) \in \Re \text{ implies } (\upsilon, \upsilon^*) \in \Re.$$

Now from (10) with $\sigma = \upsilon$, $\varsigma = \upsilon^*$, we have,

$$\Lambda\left(d\left(\phi\left(\upsilon\right),\psi\left(\upsilon^*\right)\right)\right) \leq \Upsilon\left(\Lambda\left(\max\left\{\frac{1}{2}d\left(\upsilon,\upsilon^*\right),\frac{d\left(\upsilon,\psi\left(\upsilon^*\right)\right)d\left(\upsilon^*,\phi\left(\upsilon\right)\right)}{2\left[1+d\left(\upsilon,\upsilon^*\right)\right]}\right\}\right)\right).$$

This implies,

$$\begin{aligned}
\Lambda\left(d\left(\upsilon,\upsilon^*\right)\right) &\leq \Upsilon\left(\Lambda\left(\max\left\{\frac{1}{2}d\left(\upsilon,\upsilon^*\right),\frac{d\left(\upsilon,\upsilon^*\right)d\left(\upsilon^*,\upsilon\right)}{2\left[1+d\left(\upsilon,\upsilon^*\right)\right]}\right\}\right)\right)\\
&< \Upsilon\left(\Lambda\left(\max\left\{\frac{1}{2}d\left(\upsilon,\upsilon^*\right),\frac{d\left(\upsilon,\upsilon^*\right)}{2}\right\}\right)\right)\\
&= \Upsilon\left(\Lambda\left(\frac{1}{2}d\left(\upsilon,\upsilon^*\right)\right)\right)\\
&< \Upsilon\left(\Lambda\left(d\left(\upsilon,\upsilon^*\right)\right)\right)\\
&< \Lambda\left(d\left(\upsilon,\upsilon^*\right)\right),
\end{aligned}$$

which is a contradiction. Hence, $\upsilon = \upsilon^*$ and υ is the unique common fixed point of ϕ and ψ. □

In the next theorem, we relaxed continuity of mappings ϕ,ψ and considered the metric space $(\Im, d)$ to be $\Re$-regular.

Theorem 6. *Let $(\Im, d)$ be a complete metric space and $\Re$ be a binary relation on $\Im$. Let $\phi,\psi : \Im \to \Im$. Suppose that*
(a) $M(\,(\phi,\psi)\,;\Re)$ is non-empty;
(b) there exist a continuous comparison function Υ and a function $\Lambda \in \mathcal{D}$ so that for all $(\sigma,\varsigma) \in \chi$, the inequality (1) is true,
(c) $(\Im, d)$ is $\Re$-regular,
(d) $\Re$ is (ϕ,ψ)-closed.

Then ϕ and ψ have a common fixed point in $\Im$.

Proof. In the proof of Theorem 4, we have proved that $(\zeta_n,\zeta_{n+1})\in\Re$ and $\zeta_n \to \delta$ as $n \to \infty$, for all $n \in \mathbb{N}\cup\{0\}$. As $(\Im, d)$ is $\Re$-regular, so $(\zeta_n,\delta) \in \Re$, for all $n \in \mathbb{N}$. We have two cases depending on

$$\mathcal{A} = \{n \in \mathbb{N} : \phi\left(\zeta_{2n}\right) = \psi\left(\delta\right) \text{ and } \psi\left(\zeta_{2n+1}\right) = \phi\left(\delta\right)\}.$$

Case (1): If $\mathcal{A}$ is finite, then there exists $n_0 \in\mathbb{N}\cup\{0\}$ such that $\phi\left(\zeta_{2n}\right) \neq \psi\left(\delta\right)$ and $\psi\left(\zeta_{2n+1}\right) \neq \phi\left(\delta\right)$, for all $n \geq n_0$. As $\zeta_{2n} \neq \delta$ and $\zeta_{2n+1} \neq \delta$ imply that $d\left(\zeta_{2n},\delta\right) > 0$, $d\left(\zeta_{2n+1},\delta\right) > 0$ and $d\left(\phi\left(\zeta_{2n}\right),\psi\left(\delta\right)\right) > 0$, $d\left(\psi\left(\zeta_{2n+1}\right),\phi\left(\delta\right)\right) > 0$, for all $n \geq n_0$. By (1), with $\sigma = \delta$ and $\varsigma = \zeta_{2n+1}$, we have

$$\Lambda\left(d\left(\phi\left(\delta\right),\psi\left(\zeta_{2n+1}\right)\right)\right) \leq \Upsilon\left(\Lambda\left(\max\left\{d\left(\delta,\zeta_{2n+1}\right),\frac{d\left(\delta,\psi\left(\zeta_{2n+1}\right)\right)d\left(\zeta_{2n+1},\phi\left(\delta\right)\right)}{1+d\left(\delta,\zeta_{2n+1}\right)}\right\}\right)\right).$$

This implies that

$$\begin{aligned}
\Lambda\left(d\left(\phi\left(\delta\right),\zeta_{2n+2}\right)\right) &\leq \Upsilon\left(\Lambda\left(\max\left\{d\left(\delta,\zeta_{2n+1}\right),\frac{d\left(\delta,\zeta_{2n+2}\right)d\left(\zeta_{2n+1},\phi\left(\delta\right)\right)}{1+d\left(\delta,\zeta_{2n+1}\right)}\right\}\right)\right)\\
&< \Lambda\left(\max\left\{d\left(\delta,\zeta_{2n+1}\right),\frac{d\left(\delta,\zeta_{2n+2}\right)d\left(\zeta_{2n+1},\phi\left(\delta\right)\right)}{1+d\left(\delta,\zeta_{2n+1}\right)}\right\}\right).
\end{aligned}$$

But $\{\zeta_n\} = \left\{\max\left\{d\left(\delta,\zeta_{2n+1}\right),\frac{d(\delta,\zeta_{2n+2})d(\zeta_{2n+1},\phi(\delta))}{1+d(\delta,\zeta_{2n+1})}\right\}\right\}$ is a positive sequence with $\lim\limits_{n\to\infty}\zeta_n = 0$, hence by (C_2) and Lemma 1, we get $\lim\limits_{n\to\infty}\Lambda\left(\zeta_n\right) = 0$ and thus, $\lim\limits_{n\to\infty}\Lambda\left(d\left(\phi\left(\delta\right),\zeta_{2n+2}\right)\right) = 0$. Again by (C_2)

and Lemma 1, we get $\lim_{n\to\infty} d\left(\phi\left(\delta\right), \zeta_{2n+2}\right) = 0$, and $\lim_{n\to\infty} d\left(\delta, \zeta_{2n+2}\right) = 0$. Hence by the uniqueness of the limit,

$$\phi\left(\delta\right) = \delta, \tag{11}$$

hence δ is a fixed point of ϕ. Similarly, by (1) with $\sigma = \zeta_{2n}$ and $\varsigma = \delta$, we prove that $\lim_{n\to\infty} \Lambda\left(d\left(\zeta_{2n+1}, \psi\left(\delta\right)\right)\right) = 0$. By (C_2) and Lemma 1, we get, $\lim_{n\to\infty} d\left(\zeta_{2n+1}, \psi\left(\delta\right)\right) = 0$. Also, $\lim_{n\to\infty} \Lambda\left(d\left(\zeta_{2n+1}, \delta\right)\right) = 0$, so by the uniqueness of the limit,

$$\psi\left(\delta\right) = \delta. \tag{12}$$

By (11) and (12), we get that

$$\phi\left(\delta\right) = \psi\left(\delta\right) = \delta.$$

Case (2): If $\mathcal{A}$ is infinite, there is $\left\{\zeta_{2n(k)}\right\}$ of $\{\zeta_n\}$ with $\zeta_{2n(k)+1} = \phi(\zeta_{2n(k)}) = \psi\left(\delta\right)$ such that $\zeta_{2n(k)+2} = \psi(\zeta_{2n(k)}) = \phi\left(\delta\right)$ for all $k \in \mathbb{N} \cup \{0\}$. But, $\lim_{n\to\infty} d\left(\zeta_{2n(k)+1}, \delta\right) = 0$ and $\lim_{n\to\infty} d\left(\zeta_{2n(k)+2}, \delta\right) = 0$. The uniqueness of the limit implies that

$$\phi\left(\delta\right) = \psi\left(\delta\right) = \delta.$$

Thus, in both cases, δ is a common fixed point of ϕ and ψ. □

The next theorem is an analogue of Theorem 6.

Theorem 7. *Let $(\Im, d)$ be a complete metric space and $\Re$ be a transitive relation on $\Im$. Let $\phi, \psi : \Im \to \Im$. Suppose that*
(a) $M(\,(\phi, \psi)\,; \Re)$ and $\Gamma((\sigma, \varsigma, \Re)$ are non-empty;
(b) there exist a continuous comparison function Υ and a function $\Lambda \in \mathcal{D}$ so that for all $(\sigma, \varsigma) \in \chi$,

$$\Lambda\left(d\left(\phi\left(\sigma\right), \psi\left(\varsigma\right)\right)\right) \leq \Upsilon\left(\Lambda\left(\max\left\{\frac{1}{2}d\left(\sigma, \varsigma\right), \frac{d\left(\sigma, \psi\left(\varsigma\right)\right) d\left(\varsigma, \phi\left(\sigma\right)\right)}{2\left[1 + d\left(\sigma, \varsigma\right)\right]}\right\}\right)\right), \tag{13}$$

(c) $(\Im, d)$ is $\Re$-regular,
(d) $\Re$ is (ϕ, ψ)-closed.

Then there is a unique common fixed point of ϕ and ψ.

Proof. It follows immediately from the proofs of Theorem 6 and Theorem 5. □

The following results are induced results for a single mapping.

Corollary 1. *Let $\phi : \Im \to \Im$ be defined on a complete metric space $(\Im, d)$ and $\Re$ be a binary relation on $\Im$. Suppose that*
(a) $M(\phi; \Re)$ is non-empty,
(b) $\Re$ is ϕ-closed,
(c) either ϕ is continuous, or $(\Im, d)$ is $\Re$-regular;
(d) there exist a continuous comparison function Υ and a function $\Lambda \in \mathcal{D}$ such that for all $(\sigma, \varsigma) \in \chi$,

$$\Lambda\left(d\left(\phi\left(\sigma\right), \phi\left(\varsigma\right)\right)\right) \leq \Upsilon\left(\Lambda\left(\max\left\{d\left(\sigma, \varsigma\right), \frac{d\left(\sigma, \phi\left(\varsigma\right)\right) d\left(\varsigma, \phi\left(\sigma\right)\right)}{1 + d\left(\sigma, \varsigma\right)}\right\}\right)\right).$$

Then ϕ has a fixed point.

Proof. Set $\phi = \psi$ in Theorem 4 and Theorem 6. □

Corollary 2. *Let $\phi : \Im \to \Im$ be defined on a complete metric space $(\Im, d)$ and $\Re$ be a transitive relation on $\Im$. Suppose that*
(a)$M(\phi; \Re)$ and $\Gamma(\sigma, \varsigma, \Re)$ are non-empty;
(b) $\Re$ is ϕ-closed;
(c) ϕ is continuous or $(\Im, d)$ is $\Re$-regular;
(d) if there exist a continuous comparison function Υ and a function $\Lambda \in \mathcal{D}$ such that for all $(\sigma, \varsigma) \in \chi$, we have

$$\Lambda \left(d \left(\phi \left(\sigma \right), \phi \left(\varsigma \right) \right) \right) \leq \Upsilon \left(\Lambda \left(\max \left\{ \frac{1}{2} d \left(\sigma, \varsigma \right), \frac{d \left(\sigma, \phi \left(\varsigma \right) \right) d \left(\varsigma, \phi \left(\sigma \right) \right)}{2 \left[1 + d \left(\sigma, \varsigma \right) \right]} \right\} \right) \right).$$

Then there is a unique fixed point of ϕ.

Proof. Set $\phi = \psi$ in Theorem 5 and Theorem 7. □

Now, we give an easy example explaining our main result.

Example 1. *Let $\Im = [0, 2]$. Consider $d : \Im \times \Im \to [0, +\infty)$ defined by $d(\sigma, \varsigma) = |\sigma - \varsigma|$, for all $\sigma, \varsigma \in \Im$. Then $(\Im, d)$ is a complete metric space. Consider $\Lambda : (0, \infty) \to (0, \infty)$ as $\Lambda(t) = te^t$, for each $t > 0$. Note that $\Lambda \in \mathcal{D}$. Let $\Upsilon(t) = \frac{t}{2}$, which is a continuous comparison function. Define the binary relation*

$$\Re = \left\{ (0,0), (0, \frac{1}{5}), (\frac{1}{5}, 0), (0,1), (1,0), (1,1), (0,2), (2,0) \right\},$$

on $\Im$. Consider the mappings $\phi, \psi : \Im \to \Im$ by

$$\phi \left(\zeta \right) = \begin{cases} x, & 0 \leq \zeta \leq \frac{1}{5}, \\ \frac{1}{5}, & \frac{1}{5} < \zeta \leq 2, \end{cases} \quad \text{and } \psi \left(\zeta \right) = 0, \text{ for all } \zeta \in \Im.$$

Clearly, $\Re$ is (ϕ, ψ)-closed and ϕ, ψ are continuous. Let

$$\chi = \left\{ (\sigma, \varsigma) \in \Re : \left| \phi \left(\sigma \right) - \psi \left(\varsigma \right) \right| > 0 \right\},$$

then

$$\chi = \left\{ (1.0), (1,1), (2,0) \right\}.$$

Now, for all $(\sigma, \varsigma) \in \chi$,

$$\Lambda \left(d \left(\phi \left(\sigma \right), \psi \left(\varsigma \right) \right) \right) \leq \Upsilon \left(\Lambda \left(\max \left\{ \begin{array}{c} d \left(\sigma, \varsigma \right), \\ \frac{d(\sigma, \psi(\varsigma)) d(\varsigma, \phi(\sigma))}{1 + d(\sigma, \varsigma)} \end{array} \right\} \right) \right).$$

Thus hypotheses of Theorem 4 are satisfied, and so ϕ and ψ have a common fixed point in $\Im$.

4. Some Consequences

Next corollaries (Corollary 3 and Corollary 4) generalize fixed point theorems given by Jleli [24] (Theorem 3) and Al-Sulami et al. [34].

Corollary 3. *Let $\phi, \psi : \Im \to \Im$ be self-mappings defined on a complete metric space $(\Im, d)$ and $\Re$ be a relation on $\Im$. Suppose that*
(a)$M((\phi, \psi) ; \Re)$ is non-empty;
(b) $\Re$ is (ϕ, ψ)-closed;

(c) ϕ and ψ are continuous;
(d) there exist $\theta \in \Theta$ and $k \in (0,1)$ such that, for all $(\sigma, \varsigma) \in \chi$,

$$\theta\left(d\left(\phi\left(\sigma\right), \psi\left(\varsigma\right)\right)\right) \leq \left[\theta\left(\max\left\{d\left(\sigma, \varsigma\right), \frac{d\left(\sigma, \psi\left(\varsigma\right)\right) d\left(\varsigma, \phi\left(\sigma\right)\right)}{1 + d\left(\sigma, \varsigma\right)}\right\}\right)\right]^{k}.$$

Then there is a common fixed point of ϕ and ψ.

Proof. Set $\Upsilon(t) := (\ln k)\, t$ and $\Lambda(t) = \ln \theta$, in Theorem 4. □

Corollary 4. *Let $\phi : \Im \to \Im$ be a self-mapping defined on a complete metric space $(\Im, d)$ and $\Re$ be a relation on $\Im$. Suppose that*
(a) $M(\phi; \Re)$ is non-empty;
(b) $\Re$ is ϕ-closed;
(c) ϕ is continuous;
(d) if there are $\theta \in \Theta$ and $k \in (0,1)$ such that, for all $(\sigma, \varsigma) \in \chi$,

$$\theta\left(d\left(\phi\left(\sigma\right), \phi\left(\varsigma\right)\right)\right) \leq \left[\theta\left(d\left(\sigma, \varsigma\right)\right)\right]^{k}.$$

Then there is a fixed point of ϕ.

Next corollaries (Corollary 5 and Corollary 6) generalize fixed point theorems given by Wardowski [22] (Theorem 1) and Zada and Sarwar [33].

Corollary 5. *Let $\phi, \psi : \Im \to \Im$ be self-mappings defined on a complete metric space $(\Im, d)$ and $\Re$ be a relation on $\Im$. Suppose that*
(a)$M((\phi, \psi); \Re)$ is non-empty;
(b) $\Re$ is (ϕ, ψ)-closed;
(c) ϕ and ψ are continuous;
(d) there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $(\sigma, \varsigma) \in \chi$,

$$\tau + F\left(d\left(\phi\left(\sigma\right), \psi\left(\varsigma\right)\right)\right) \leq F\left(\max\left\{d\left(\sigma, \varsigma\right), \frac{d\left(\sigma, \psi\left(\varsigma\right)\right) d\left(\varsigma, \phi\left(\sigma\right)\right)}{1 + d\left(\sigma, \varsigma\right)}\right\}\right).$$

Then there is a common fixed point of ϕ and ψ.

Proof. It comes from Theorem 4 by taking $\Upsilon(t) = e^{-\tau} t$ and $\Lambda(t) = e^{F}$. □

Corollary 6. *Let $\phi, \psi : \Im \to \Im$ be self-mappings defined on a complete metric space $(\Im, d)$ and and $\Re$ be a relation on $\Im$. Suppose that*
(a)$M((\phi, \psi); \Re)$ is non-empty;
(b) $\Re$ is (ϕ, ψ)-closed;
(c) ϕ and ψ are continuous;
(d) there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $(\sigma, \varsigma) \in \chi$,

$$\tau + F\left(d\left(\phi\left(\sigma\right), \psi\left(\varsigma\right)\right)\right) \leq F\left(d\left(\sigma, \varsigma\right) + \frac{d\left(\sigma, \psi\left(\varsigma\right)\right) d\left(\varsigma, \phi\left(\sigma\right)\right)}{1 + d\left(\sigma, \varsigma\right)}\right).$$

Then there is a common fixed point of ϕ and ψ.

Let $\beta : [0, \infty) \to [0, \infty)$ be such that $\lim\limits_{r \to t^{+}} \beta(r) < 1$ for each $t \in (0, \infty)$

Corollary 7. *Let $\phi, \psi : \Im \to \Im$ be self-mappings defined on a complete metric space $(\Im, d)$ and $\Re$ be a relation on $\Im$. Suppose that*
(a)$M((\phi, \psi); \Re)$ is non-empty;
(b) $\Re$ is (ϕ, ψ)-closed;
(c) ϕ and ψ are continuous;
(d) there exists a function β such that, for all $(\sigma, \varsigma) \in \chi$,

$$d(\phi(\sigma), \psi(\varsigma)) \leq \beta\left(\max\left\{\begin{array}{c} d(\sigma, \varsigma), \\ \frac{d(\sigma,\psi(\varsigma))d(\varsigma,\phi(\sigma))}{1+d(\sigma,\varsigma)} \end{array}\right\}\right).\max\left\{\begin{array}{c} d(\sigma, \varsigma), \\ \frac{d(\sigma,\psi(\varsigma))d(\varsigma,\phi(\sigma))}{1+d(\sigma,\varsigma)} \end{array}\right\}.$$

Then there is a common fixed point of ϕ and ψ.

Proof. It follows from Theorem 4 by taking $\psi(t) := \beta(t)\, t$ and $\phi(t) = t : (0, \infty) \to (0, \infty)$. □

5. Applications to Nonlinear Matrix Equations

Let us denote, $M(p)$ the set of $p \times p$ complex matrices, $H(p)$ the set of $p \times p$ Hermitian matrices, $P(p)$ the set of $p \times p$ positive definite matrices and $H^+(p)$ the set of $p \times p$ positive semi-definite matrices. Here, $P(p) \subseteq H(p) \subseteq M(p)$, $H^+(p) \subseteq H(p)$, $\Im_1 \succ 0$ and $\Im_1 \succeq 0$ means that $\Im_1 \in P(p)$ and $\Im_1 \in H^+(p)$, respectively. For $\Im_1 - \Im_2 \succeq 0$ and $\Im_1 - \Im_2 \succ 0$, we will use $\Im_1 \succeq \Im_2$ and $\Im_1 \succ \Im_2$, respectively.

We consider the following non-linear matrix equations:

$$X = D + \sum_{i=1}^{n} A_i^* X A_i - \sum_{i=1}^{n} B_i^* X B_i, \tag{14}$$

and

$$X = D + \sum_{i=1}^{n} A_i^* \gamma(X) A_i, \tag{15}$$

where $D \in P(p)$, A_i, B_i are arbitrary $p \times p$ matrices and $\gamma : H(p) \to P(p)$ is a continuous order preserving map such that $\gamma(0) = 0$. Consider the metric induced by the norm $\|E\|_{tr} = \sum_{i=1}^{n} \theta_i(E)$, where $\theta_i(E)$, $i = 1, 2, ..., n$, are the singular values of $E \in M(p)$. The set $H(p)$ equipped with the trace norm $\|.\|_{tr}$ is a complete metric space (see [35–37]) and partially ordered with the partial ordering $\preceq$, where $E_1 \preceq E_2 \Leftrightarrow E_2 \preceq E_1$. Also, for every $E_1, E_2 \in H(p)$ there is a glb and a lub (see [36]]).

We need the two following lemmas.

Lemma 2. *[36] If $E_1, E_2 \succeq 0$ are $p \times p$ matrices, then*

$$0 \leq tr(E_1 E_2) = \|E_2\|\, tr(E_1).$$

Lemma 3. *[38] If $E \in M(p)$ with $E \prec I_n$, then $\|E\| < 1$.*

Take $\psi : H(p) \to H(p)$ by

$$\psi(\chi) = \frac{1}{2}(\psi_1(\chi) + \psi_2(\chi)),$$

where $\psi_1, \psi_2 : H(p) \to H(p)$ are given as

$$\psi_1(\chi) = D + 2\sum_{i=1}^{n} A_i^* \chi A_i \text{ and } \psi_2(\chi) = D - 2\sum_{i=1}^{n} B_i^* \chi B_i.$$

Then the solutions of the matrix Equation (14) are the fixed points of the operator ψ, which are the common fixed points of operators ψ_1 and ψ_2.

Theorem 8. *Suppose that*
$(h1)$ *there exist two positive reals* M_1 *and* M_1 *so that* $\sum\limits_{i=1}^{n} A_i A_i^* \prec \delta_1 I_n$ *and* $\sum\limits_{i=1}^{n} B_i B_i^* \prec \delta_2 I_n$;
$(h2)$ *for every* $E_1, E_2 \in H(p)$ *such that* $(E_1, E_2) \in \preceq$, *we have*

$$\|E_1\|_{tr} + \|E_2\|_{tr} \leq \frac{1}{2\delta} \frac{R(E_1, E_2)}{R(E_1, E_2) + 1},$$

where

$$R(E_1, E_2) = \max \left\{ \begin{array}{c} \|E_1 - E_2\|_{tr}, \\ \frac{\|E_1 - \psi_2(E_2)\|_{tr} \|E_2 - \psi_1(E_1)\|_{tr}}{1 + \|E_1 - E_2\|_{tr}} \end{array} \right\},$$

and $\delta = \max\{\delta_1, \delta_2\}$. *Then the non-linear matrix Equation (14) has a solution in* $H(p)$.

Proof. Since ψ_1 and ψ_2 are well defined, we have that $(E_1, E_2) \in \preceq$ implies that

$$(\psi_1(E_1), \psi_2(E_2)), (\psi_2(E_1), \psi_1(E_2)) \in \preceq,$$

so that $\preceq$ on $H(p)$ is (ψ_1, ψ_2)-closed. Now, we show that the operators ψ_1 and ψ_2 satisfy (1). Consider

$$\begin{aligned}
\|\psi_1(E_1) - \psi_2(E_2)\|_{tr} &= tr(\psi_1(E_1) - \psi_2(E_2)) = \\
&= 2tr\left(\sum_{i=1}^{n} (A_i^* E_1 A_i + B_i^* E_2 B_i)\right) \\
& \quad 2\sum_{i=1}^{n} tr(A_i^* E_1 A_i + B_i^* E_2 B_i) \\
& \quad 2\left(\sum_{i=1}^{n} tr(A_i A_i^* E_1) + \sum_{i=1}^{n} tr(B_i B_i^* E_2)\right) \\
& \quad 2\left(tr\left(\sum_{i=1}^{n} A_i A_i^* E_1\right) + tr\left(\sum_{i=1}^{n} B_i B_i^* E_2\right)\right) \\
&\leq 2\left(\left\|\sum_{i=1}^{n} A_i A_i^*\right\| \|E_1\|_{tr} + \left\|\sum_{i=1}^{n} B_i B_i^*\right\| \|E_2\|_{tr}\right) \\
&\leq 2(\delta_1 \|E_1\|_{tr} + \delta_2 \|E_2\|_{tr}) \\
&\leq 2\delta \|E_1\|_{tr} + \|E_2\|_{tr}.
\end{aligned}$$

From conditions $(h1)$ and $(h2)$, we have

$$\|\psi_1(E_1) - \psi_2(E_2)\|_{tr} < \frac{R(E_1, E_2)}{R(E_1, E_2) + 1},$$

where

$$R(E_1, E_2) = \max \left\{ \begin{array}{c} \|E_1 - E_2\|_{tr}, \\ \frac{\|E_1 - \psi_2(E_2)\|_{tr} \|E_2 - \psi_1(E_1)\|_{tr}}{1 + \|E_1 - E_2\|_{tr}} \end{array} \right\}.$$

Let $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ and $\Upsilon : (0, \infty) \longrightarrow (0, \infty)$ be the mappings defined by

$$\Lambda(t) = t, t > 0 \text{ and } \Upsilon(t) = \frac{t}{t+1}, t > 0, \text{ respectively.}$$

Then the above inequality becomes

$$\Lambda\left(\|\psi_1(E_1)-\psi_2(E_2)\|_{tr}\right)\leq \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c}\|E_1-E_2\|_{tr},\\ \frac{\|E_1-\psi_2(E_2)\|_{tr}\|E_2-\psi_1(E_1)\|_{tr}}{1+\|E_1-E_2\|_{tr}}\end{array}\right\}\right)\right).$$

Consequently,

$$\Lambda\left(d\left(\psi_1(E_1),\psi_2(E_2)\right)\right)\leq \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c}d(E_1,E_2),\\ \frac{d(E_1,\psi_2(E_2)),d(E_2,\psi_1(E_1))}{1+d(E_1,E_2)}\end{array}\right\}\right)\right).$$

Therefore, all the conditions of Theorem 4 immediately hold. Then there is a common fixed point of ψ_1 and ψ_2, say E^*. Consequently, ψ has a fixed point and hence the system of non-linear matrix Equations (14) has a solution. □

Theorem 9. *Under the hypotheses $(h1)$ and $(h2)$ of Theorem 8, the non-linear matrix Equation (14) has a unique solution if $\Re$ is transitive and $H(p)$ is $\Re$-regular.*

Proof. Using Theorem 5 and proceeding as the same arguments of Theorem 8, there is a unique solution of the non-linear matrix Equation (14). □

Define the operator $\phi : H(p)\to H(p)$ as

$$\phi(E)=D+\sum_{i=1}^{n}A_i^*\gamma(E)A_i.$$

The solutions of the matrix Equation (15) coincide with the fixed points of the operator $\phi(E)$.

Theorem 10. *Suppose that*

(1) there is a real positive number δ such that $\sum_{i=1}^{n}A_iA_i^*\prec\delta I_n$;

(2) for all $E_1,E_2\in H(p)$ so that $(E_1,E_2)\in\preceq$, and $\sum_{i=1}^{n}A_i\gamma(E_1)A_i^*\neq\sum_{i=1}^{n}A_i\gamma(E_2)A_i^*$,we have

$$\|tr\left(\gamma(E_1)-\gamma(E_2)\right)\|_{tr}\leq\frac{1}{\delta}\frac{R(E_1,E_2)}{R(E_1,E_2)+1},$$

where,

$$R(E_1,E_2)=\max\left\{\begin{array}{c}\|E_1-E_2\|_{tr},\\ \frac{\|E_1-\phi(E_2)\|_{tr}\|E_2-\phi(E_1)\|_{tr}}{1+\|E_1-E_2\|_{tr}}\end{array}\right\}.$$

Then the non-linear matrix Equation (15) has a solution in $H(p)$.

Proof. Since ϕ is well defined and $(E_1, E_2) \in \preceq$ implies that $(\phi(E_1), \phi(E_2)) \in \preceq$, so $\preceq$ on $H(p)$ is ϕ-closed. Now, we show that the operator ϕ satisfies inequality (1). If $E_1, E_2 \in H(p)$, then $E_2 \preceq E_1$. But, γ is an order preserving mapping, hence $\gamma(E_2) \preceq \gamma(E_1)$. Thus,

$$\begin{aligned}
\|\phi(E_1) - \phi(E_2)\|_{tr} &= tr(\phi(E_1) - \phi(E_2)) = \\
&= tr\left(\sum_{i=1}^{n} (A_i^* [\gamma(E_1) - \gamma(E_2)] A_i)\right) \\
&= \sum_{i=1}^{n} tr(A_i^* [\gamma(E_1) - \gamma(E_2)] A_i) \\
& \quad \sum_{i=1}^{n} tr(A_i A_i^* [\gamma(E_1) - \gamma(E_2)]) \\
& \quad tr\left(\sum_{i=1}^{n} A_i A_i^*\right) [\gamma(E_1) - \gamma(E_2)] \\
&\leq \left\|\sum_{i=1}^{n} A_i A_i^*\right\| \|\gamma(E_1) - \gamma(E_2)\|_{tr}.
\end{aligned}$$

Using condition $(h2)$, we have

$$\|\phi(E_1) - \phi(E_2)\|_{tr} < \frac{\left\|\sum_{i=1}^{n} A_i A_i^*\right\|}{\delta} \frac{R(E_1, E_2)}{R(E_1, E_2) + 1},$$

where,

$$R(E_1, E_2) = \max\left\{ \begin{array}{c} \|E_1 - E_2\|_{tr}, \\ \frac{\|E_1 - \phi(E_2)\|_{tr} \|E_2 - \phi(E_1)\|_{tr}}{1 + \|E_1 - E_2\|_{tr}} \end{array} \right\}.$$

From condition $(h1)$, we get

$$\|\phi(E_1) - \phi(E_2)\|_{tr} < \frac{R(E_1, E_2)}{R(E_1, E_2) + 1},$$

Let $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ and $\Upsilon : (0, \infty) \longrightarrow (0, \infty)$ be the mappings defined by

$$\Lambda(t) = t, t > 0 \text{ and } \Upsilon(t) = \frac{t}{t+1}, t > 0, \text{ respectively.}$$

Then the above inequality becomes

$$\Lambda(d(\phi(E_1), \phi(E_2))) \leq \Upsilon\left(\Lambda\left(\max\left\{ \begin{array}{c} d(E_1, E_2), \\ \frac{d(E_1, \phi(E_2)), d(E_2, \phi(E_1))}{1 + d(E_1, E_2)} \end{array} \right\}\right)\right).$$

Therefore, all the conditions of Corollary 1 immediately hold. Then there is a fixed point of ϕ, say E. Then the non-linear matrix Equation (15) has a solution. □

Theorem 11. *Under the hypotheses $(h1)$ and $(h2)$ of Theorem 10, the non-linear matrix Equation (15) has a unique solution if $\Re$ is transitive and $H(p)$ is $\Re$-regular.*

Proof. By Corollary 2 and using the arguments of Theorem 10, there is a unique solution of the non-linear matrix Equation (15). □

Example 2. *Take nonlinear matrix equation:*

$$E = D + \sum_{i=1}^{2} A_i^* \gamma(E) A_i.$$

where D, E_1 and E_2 are given by

$$D = \begin{pmatrix} 7 & 5 & 3 & 1 \\ 5 & 7 & 5 & 3 \\ 3 & 5 & 7 & 5 \\ 1 & 3 & 5 & 7 \end{pmatrix},\ A_1 = \begin{pmatrix} 0.0241 & 0.0124 & 0.0124 & 0.0241 \\ 0.0124 & 0.0241 & 0.0241 & 0.0124 \\ 0.0124 & 0.0241 & 0.0241 & 0.0124 \\ 0.0241 & 0.0124 & 0.0124 & 0.0241 \end{pmatrix}$$

$$\text{and } A_2 = \begin{pmatrix} 0.0521 & 0.0329 & 0.0329 & 0.0521 \\ 0.0871 & 0.68 & 0.0871 & 0.68 \\ 0.0521 & 0.0329 & 0.0329 & 0.0521 \\ 0.0871 & 0.68 & 0.0871 & 0.68 \end{pmatrix}.$$

Define $\gamma : H(4) \longrightarrow H(4)$ by

$$\gamma(E) = \frac{E}{3}.$$

Consider $\phi : H(4) \longrightarrow H(4)$ as $\phi(E) = D + \sum_{i=1}^{2} A_i^* \gamma(E) A_i$. Then conditions of Theorem 10 are satisfied for $\delta = 2$.

6. Conclusions

The $(\Lambda, \Upsilon, \Re)$-contraction (under the effect of a continuous comparison function and an arbitrary binary relation) considered in this paper is in general enough to contain several corresponding contractions (θ-contractions and F-contractions). The results obtained here generalize several corresponding results and are applicable to solving nonlinear matrix equations. There is an **open problem** which states: what happens if functions Λ and Υ are not continuous.

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