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Large Contractions on Quasi-Metric Spaces with an Application to Nonlinear Fractional Differential Equations

Erdal Karapınar [1,*], Andreea Fulga [2], Maliha Rashid [3], Lariab Shahid [3] and Hassen Aydi [4]

1 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
2 Department of Mathematics and Computer Sciences, Universitatea Transilvania Brasov, 500036 Brasov, Romania; afulga@unitbv.ro
3 Department of Mathematics & Statistics, International Islamic University, Islamabad H-10 44000, Pakista; maliha.rashid@iiu.edu.pk (M.R.); laraib246@hotmail.com (L.S.)
4 Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia; hmaydi@iau.edu.sa or hassen.aydi@isima.rnu.tn
* Correspondence: erdalkarapinar@yahoo.com or karapinar@mail.cmuh.org.tw

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Abstract: In this manuscript, we introduce a new notion: a Berinde type (α, ψ)-contraction mapping. Thereafter, we investigate not only the existence, but also the uniqueness of a fixed point of such mappings in the setting of right-complete quasi-metric spaces. The result, presented here, not only generalizes a number of existing results, but also unifies several ones on the topic in the literature. An application of nonlinear fractional differential equations is given.

Keywords: nonlinear fractional differential equations; Berinde type contraction; quasi-metric space; admissible mappings; fixed point

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1. Introduction and Preliminaries

Fixed point results have been studied in various directions since the introduction of Banach contraction theorem. Mathematicians have studied fixed point results in different spaces using various contractive conditions. Several new contractive conditions have been developed in an attempt to obtain more refined fixed point results. One of the significant results, from which our results are inspired, was reported in 2004 by Berinde [1]. More precisely, Berinde [1] introduced the concept of (θ, L)-weak contractions and studied some related fixed point results. We also mention the notion of α-admissibility, defined by Samet et al. [2] and improved by Popescu [3]. Some related fixed point results are known as α-ψ contraction type results. For more details, see [3–25].

In the present paper, inspired from the result of Berinde [1] and Popescu [3], we propose a new contraction, and then we discuss fixed point existence problems for such mappings. The immediate consequences and possible further conclusions are also discussed. An example is also provided in support of the results. Moreover, we solve a nonlinear fractional differential equation using the obtained results.

We first consider some basic requirements for the sake of completeness. From now on, let $\mathcal{M}$ be a non-empty set.

Definition 1. *A functional $\rho : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is said to be a quasi-metric if it satisfies the triangle inequality axiom, and reflexivity, that is, $\rho(\theta, \vartheta) = 0 \Leftrightarrow \theta = \vartheta$. Here, a pair $(\mathcal{M}, \rho)$ denotes a non-empty set M equipped with a quasi-metric ρ. In short, a quasi-metric space is written as (qms).*

Note that each quasi-metric ρ on a non-empty set M yields a metric by letting $d(\theta, \vartheta) = \max\{\rho(\theta, \vartheta), \rho(\vartheta, \theta)\}$. The basic topological notions are observed by a slight modification. We say that a sequence $\{\theta_n\}$ in a (q.m.s.) $(\mathcal{M}, \rho)$ converges to $\theta \in \mathcal{M}$ if

$$\lim_{n\to\infty} \rho(\theta_n, \theta) = \lim_{n\to\infty} \rho(\theta, \theta_n) = 0. \tag{1}$$

Notice that the limit of a sequence, in a (q.m.s.), is unique. If $\theta_n \to \theta$, we have for all $\vartheta \in \mathcal{M}$

$$\lim_{n\to\infty} \rho(\theta_n, \vartheta) = \rho(\theta, \vartheta) \text{ and } \lim_{n\to\infty} \rho(\vartheta, \theta_n) = \rho(\vartheta, \theta).$$

A sequence $\{\theta_n\}$ in a (q.m.s.) $(\mathcal{M}, \rho)$ is called left-Cauchy if for each $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$, so that $\rho(\theta_n, \theta_m) < \epsilon$ for all $n \geq m > N$. The notion of right-Cauchy is defined analogously. It is called Cauchy if it is both left-Cauchy and right-Cauchy. If each left-Cauchy sequence in $\mathcal{M}$ is convergent, then, we say that a (q.m.s.) $(\mathcal{M}, \rho)$ is left-complete. The concept of right-completeness is defined analogously. Notice that "right completeness" in this context is equivalent to "Smyth completeness". We say that $(\mathcal{M}, \rho)$ is complete if it is both left and right complete. A self-mapping T on a (q.m.s.) $(\mathcal{M}, \rho)$ is called right-continuous if $(T\theta_n, T\theta) \to 0$ for all sequence $\{\theta_n\}$ in $\mathcal{M}$ and all $\theta \in \mathcal{M}$ such that $\rho(\theta_n, \theta) \to 0$. The left-continuity of T is defined analogously. As is expected, T is called continuous, if it is both right-continuous and left-continuous, simultaneously.

A self-mapping φ, on the non-negative real numbers, is called a comparison function ([26]) if it is non-decreasing and satisfies $\lim\limits_{n\to\infty} \varphi^n(s) = 0$, for each $s \in [0, \infty)$. The letter Φ stands for the class of comparison functions. It was proved by Rus [26] that for each comparison function φ, the k^{th}-iteration also forms a comparison function, that is, φ^k is also a comparison function, for each $k \in \mathbb{N}$. It was also proved in [26] that each comparison function φ is continuous at 0 and the inequality $\varphi(s) < s$ holds for each $s > 0$.

A self-mapping ψ, on a non-negative real numbers, is called a $c-$comparison if it is non-decreasing and satisfies $\sum\limits_{n=0}^{\infty} \psi^n(s) < \infty$, for each $s \in (0, \infty)$. The letter Ψ stands for the family of $c-$comparison functions. Since $\Psi \subset \Phi$, for each $\psi \in \Psi$, we have $\psi(s) < s$ for each $s > 0$. More details about comparison functions and further examples can be found in [26,27].

Definition 2. *Let $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ be a function. We say that a mapping $T : \mathcal{M} \to \mathcal{M}$ is $\alpha-$ admissible if for all $\theta, \vartheta \in \mathcal{M}$, we have*

$$\alpha(\theta, \vartheta) \geq 1 \Rightarrow \alpha(T\theta, T\vartheta) \geq 1.$$

Inspired by the concept of admissible mappings [2], Popescu [3] proposed the notion of α-orbital admissibility. Let $\mathcal{M}$ be a non-empty set and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ be a mapping. A function $T : \mathcal{M} \to \mathcal{M}$ is called α-orbital admissible if

$$\alpha(\theta, T\theta) \geq 1 \Rightarrow \alpha(T\theta, T^2\theta) \geq 1.$$

Moreover, we say that a (q.m.s.) $(\mathcal{M}, \rho)$ is regular with respect to a function $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ if $\alpha(\theta_{n-1}, \theta_n) \geq 1$ and $\alpha(\theta_n, \theta_{n-1}) \geq 1$ yield $\alpha(\theta_n, \theta) \geq 1$, for all n, where $\{\theta_n\}$ is a sequence in $\mathcal{M}$ with $\lim_{n\to\infty} \rho(\theta_n, \theta) = 0$.

2. Main Results

This section starts with the definition of our new notion concerning a Berinde type (α, ψ) contraction which will be our primary interest.

For a (q.m.s.) $(\mathcal{M}, \rho)$, a self-mapping T is called a Berinde type (α, ψ) contraction if there exist $\psi \in \Psi$ and $L \geq 0$ such that

$$\alpha(\theta, \vartheta)\rho(T\theta, T\vartheta) \leq \psi\left(M(\theta, \vartheta)\right) + L \cdot B(\theta, \vartheta), \tag{2}$$

for all $\theta, \vartheta \in \mathcal{M}$, where $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is a function and

$$\begin{aligned} M(\theta, \vartheta) &= \max \left\{ \begin{array}{c} \rho(\theta, \vartheta), \rho(\theta, T\theta), \rho(\vartheta, T\vartheta), \frac{\rho(\vartheta, T\theta)+\rho(\theta, T\vartheta))}{2}, \\ \\ \frac{\rho(\vartheta, T\vartheta)[1+\rho(\theta, T\theta)]}{1+\rho(\theta, \vartheta)}, \frac{\rho(\vartheta, T\theta)[1+\rho(\theta, T\vartheta)]}{1+\rho(\theta, \vartheta)} \end{array} \right\}, \\ B(\theta, \vartheta) &= \min\left\{\rho(\theta, T\theta), \rho(\vartheta, T\vartheta), \rho(\theta, T\vartheta), \rho(\vartheta, T\theta)\right\}. \end{aligned} \tag{3}$$

Theorem 1. *Suppose that a self-mapping T on a right-complete (q.m.s.) $(\mathcal{M}, \rho)$ is $\alpha-$orbital admissible and it forms a Berinde type (α, ψ) contraction. Suppose that either T is continuous or $(\mathcal{M}, \rho)$ is regular with respect to the function α. If there exists $\theta_0 \in \mathcal{M}$ such that $\alpha(\theta_0, T\theta_0) \geq 1$, then T possesses a fixed point.*

Proof. Let $\theta_0 \in \mathcal{M}$ arbitrary. Starting with this initial point, we build a sequence $\{\theta_n\} \subset \mathcal{M}$ by $\theta_n = T\theta_{n-1} = T^{n-1}\theta_0$ for all $n \in \mathbb{N}$. If for some $n_0 \in \mathbb{N}$, we have $\theta_{n_0} = \theta_{n_0+1}$, then θ_{n_0} is a fixed point of T, that is, $T\theta_{n_0} = \theta_{n_0}$. For this reason, from now, we suppose that $\theta_{n+1} \neq \theta_n$ for all $n \in \mathbb{N}$. Hence,

$$\rho\left(\theta_{n+1}, \theta_n\right) > 0 \text{ and } \rho\left(\theta_n, \theta_{n+1}\right) > 0.$$

On the other hand, we assumed that there is $\theta_0 \in \mathcal{M}$ such that $\alpha(\theta_0, T\theta_0) \geq 1$. Since T is $\alpha-$orbital admissible, we find

$$\alpha(\theta_0, T\theta_0) \geq 1 \Rightarrow \alpha(\theta_1, T\theta_2) = \alpha(T\theta_0, T^2\theta_0) \geq 1,$$

and recursively,

$$\alpha(\theta_{n-1}, \theta_n) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{4}$$

By replacing $\theta = \theta_n$ and $\vartheta = \theta_{n-1}$ in (2) and taking into account (4), we find, for all $n \geq 1$, that

$$\begin{aligned} \rho(\theta_n, \theta_{n+1}) &= \rho(T\theta_{n-1}, T\theta_n) \leq \alpha(\theta_{n-1}, \theta_n)\rho(T\theta_{n-1}, T\theta_n) \\ &\leq \psi\left(M(\theta_{n-1}, \theta_n)\right) + L \cdot B(\theta_{n-1}, \theta_n) \end{aligned} \tag{5}$$

where

$$\begin{aligned} B(\theta_{n-1}, \theta_n) &= \min\left\{\rho(\theta_{n-1}, T\theta_{n-1}), \rho(\theta_n, T\theta_n), \rho(\theta_n, T\theta_{n-1}), \rho(\theta_{n-1}, T\theta_n)\right\} \\ &= \min\left\{\rho(\theta_n, \theta_{n+1}), \rho(\theta_{n-1}, \theta_n), \rho(\theta_n, \theta_n), \rho(\theta_{n-1}, \theta_{n+1})\right\} \\ &= 0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} M(\theta_{n-1}, \theta_n) &= \max \left\{ \begin{array}{c} \rho(\theta_{n-1}, \theta_n), \rho(\theta_{n-1}, T\theta_{n-1}), \rho(\theta_n, T\theta_n), \frac{\rho(\theta_{n-1}, T\theta_n)+\rho(\theta_n, T\theta_{n-1})}{2}, \\ \frac{\rho(\theta_n, T\theta_n)[1+\rho(\theta_{n-1}, T\theta_{n-1})]}{1+\rho(\theta_{n-1}, \theta_n)}, \frac{\rho(\theta_n, T\theta_{n-1})[1+\rho(\theta_{n-1}, T\theta_n)]}{1+\rho(\theta_{n-1}, \theta_n)} \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} \rho(\theta_{n-1}, \theta_n), \rho(\theta_n, \theta_{n+1}), \rho(\theta_{n-1}, \theta_n), \frac{\rho(\theta_{n-1}, \theta_{n+1})+\rho(\theta_n, \theta_n)}{2}, \\ \frac{\rho(\theta_n, \theta_{n+1})[1+\rho(\theta_{n-1}, \theta_n)]}{1+\rho(\theta_{n-1}, \theta_n)}, \frac{\rho(\theta_n, \theta_n)[1+\rho(\theta_{n-1}, \theta_{n+1})]}{1+\rho(\theta_{n-1}, \theta_n)} \end{array} \right\} \\ &\leq \max\left\{\rho(\theta_{n-1}, \theta_n), \rho(\theta_n, \theta_{n+1})\right\}, \end{aligned} \tag{7}$$

since $\rho(\theta_{n-1},\theta_{n+1}) \le \rho(\theta_{n-1},\theta_n) + \rho(\theta_n,\theta_{n+1})$.

Notice that if $\rho(\theta_{n-1},\theta_n) \le \rho(\theta_n,\theta_{n+1})$ for some n, then the equation yields from (5) that $\rho(\theta_n,\theta_{n+1}) \le \psi(\rho(\theta_n,\theta_{n+1})) < \rho(\theta_n,\theta_{n+1})$, a contradiction. Accordingly, we have $\rho(\theta_n,\theta_{n+1}) < \rho(\theta_{n-1},\theta_{n+1})$ for each $n \ge 1$. Further, (5) implies that

$$\rho(\theta_n,\theta_{n+1}) \le \psi\left(\rho(\theta_{n-1},\theta_n)\right), \forall n \ge 1. \tag{8}$$

Recursively, we derive

$$\rho(\theta_n,\theta_{n+1}) \le \psi(\rho(\theta_n,\theta_{n-1})) \le \psi^2(\rho(\theta_{n-1},\theta_{n-2}))... \le \psi^n(\rho(\theta_0,\theta_1)), \forall n \ge 1. \tag{9}$$

Let us prove now that the sequence $\{\theta_n\}$ is right-Cauchy. By using the triangular inequality and (9), for all $p \ge 1$, we get

$$\begin{aligned}\rho(\theta_n,\theta_{n+p}) &\le \rho(\theta_n,\theta_{n+1}) + \rho(\theta_{n+1},\theta_{n+2})... + \rho(\theta_{n+p-1},\theta_{n+p})\\ &\le \psi^n(\rho(\theta_0,\theta_1)) + \psi^{n+1}(\rho(\theta_0,\theta_1)) + ... + \psi^{n+p-1}(\rho(\theta_0,\theta_1))\\ &\le \sum_{k=n}^{n+p-1} \psi^k(\rho(\theta_0,\theta_1))\\ &\le \sum_{k=n}^{\infty} \psi^k(\rho(\theta_0,\theta_1)) \to 0 \text{ as } n \to \infty.\end{aligned} \tag{10}$$

Hence, $\rho(\theta_n,\theta_{n+p}) \to 0$ as $n \to \infty$, that is, $\{\theta_n\}$ is a right-Cauchy sequence in $(\mathcal{M},\rho)$, which is a right-complete (q.m.s.), so there exists $\omega \in \mathcal{M}$ such that

$$\lim_{n\to\infty} \rho(\theta_n,\omega) = \lim_{n\to\infty} \rho(\omega,\theta_n) = 0. \tag{11}$$

We have

$$\rho(\omega,T\omega) \le \rho(\omega,\theta_{n+1}) + \rho(\theta_{n+1},T\omega) = \rho(\omega,\theta_{n+1}) + \rho(T\theta_n,T\omega).$$

In the case that T is continuous, we have at the limit,

$$\rho(\omega,T\omega) = 0. \tag{12}$$

When $(\mathcal{M},\rho)$ is regular with respect to α, there is a subsequence $\{\theta_{n(k)}\}$ of $\{\theta_n\}$, such that $\alpha(\theta_{n(k)},\omega) \ge 1$ for each k. Accordingly, by replacing $\theta = \theta_{n(k)}$ and $\vartheta = \omega$ in (2) we find

$$\begin{aligned}\rho(\theta_{n(k)+1},T\omega) &\le \alpha\left(\theta_{n(k)},\omega\right)\rho(T\theta_{n(k)},T\omega)\\ &\le \psi\left[\max\left\{\begin{array}{c}\rho(\theta_{n(k)},\omega),\rho(\theta_{n(k)},T\theta_{n(k)}),\rho(\omega,T\omega),\frac{\rho(\omega,T\theta_{n(k)})+\rho(\theta_{n(k)},T\omega))}{2},\\ \frac{\rho(\omega,T\omega)\left[1+\rho(\theta_{n(k)},T\theta_{n(k)})\right]}{1+\rho(\theta_{n(k)},\omega)},\frac{\rho(\omega,T\theta_{n(k)})\left[1+\rho(\theta_{n(k)},T\omega)\right]}{1+\rho(\omega,\theta_{n(k)-1})}\end{array}\right\}\right]\\ &\quad +L\min\left\{\rho(\omega,T\omega),\rho(\theta_{n(k)},T\theta_{n(k)}),\rho(\omega,T\theta_{n(k)}),\rho(\theta_{n(k)},T\omega)\right\}\\ &\le \psi\left[\max\left\{\begin{array}{c}\rho(\theta_{n(k)},\omega),\rho(\omega,T\omega),\rho(\theta_{n(k)},\theta_{n(k)+1}),\frac{\rho(\omega,\theta_{n(k)+1})+\rho(\theta_{n(k)},T\omega))}{2},\\ \frac{\rho(\omega,T\omega)\left[1+\rho(\theta_{n(k)-1},\theta_{n(k)})\right]}{1+\rho(\theta_{n(k)},\omega)},\frac{\rho(\omega,\theta_{n(k+1)})\left[1+\rho(\theta_{n(k),T\omega})\right]}{1+\rho(\theta_{n(k)},\omega)}\end{array}\right\}\right]\\ &\quad +L\min\left\{\rho(\omega,T\omega),\rho(\theta_{n(k)},\theta_{n(k)+1}),\rho(\theta_{n(k)},T\omega),\rho(\omega,\theta_{n(k)})\right\}.\end{aligned}$$

Assume that $\rho(\omega,T\omega) > 0$. Letting $k \to \infty$, we derive that

$$\rho(\omega,T\omega) \le \psi\left[\max\left\{\rho(\omega,T\omega),\frac{\rho(\omega,T\omega)}{2}\right\}\right] = \psi\left[\rho(\omega,T\omega)\right] < \rho(\omega,T\omega),$$

which is a contradiction. Hence, $\rho(T\omega, \omega) = 0$, i.e., ω is a fixed point of T. □

To have the uniqueness of the fixed point whose existence is assured in Theorem 1, we must suitably strengthen its premises.

Theorem 2. *Additional to the premises of Theorem 1, if the following condition:*

(U) if ω and v are two fixed points of T, then $\alpha(\omega, v) \geq 1$,

is fulfilled, then the fixed point of T, postulated in Theorem 1 is unique.

Proof. Suppose there are two distinct fixed points of T, say θ and ϑ. By condition (U), we have $\alpha(\theta, \vartheta) \geq 1$. By (2), we have

$$\begin{aligned} \rho(\theta, \vartheta) =& \rho(T\theta, T\vartheta) \\ \leq& \alpha(\theta, \vartheta)\rho(T\theta, T\vartheta) \\ \leq& \psi\left(M(\theta, \vartheta)\right) + L \cdot B(\theta, \vartheta), \end{aligned}$$

where

$$\begin{aligned} M(\theta, \vartheta) &= \max\left\{ \begin{array}{c} \rho(\theta, \vartheta), \rho(\theta, T\theta), \rho(\vartheta, T\vartheta), \frac{\rho(\vartheta, T\theta) + \rho(\theta, T\vartheta))}{2}, \\ \\ \frac{\rho(\vartheta, T\vartheta)[1 + \rho(\theta, T\theta)]}{1 + \rho(\theta, \vartheta)}, \frac{\rho(\vartheta, T\theta)[1 + \rho(\theta, T\vartheta)]}{1 + \rho(\theta, \vartheta)} \end{array} \right\} = \rho(\theta, \vartheta), \\ B(\theta, \vartheta) &= \min\left\{\rho(\theta, T\theta), \rho(\vartheta, T\vartheta), \rho(\theta, T\vartheta), \rho(\vartheta, T\theta)\right\} = 0. \end{aligned}$$

Consequently,

$$0 < \rho(\theta, \vartheta) \leq \psi(\rho(\theta, \vartheta)) < \rho(\theta, \vartheta),$$

which is a contradiction. Thus, there is a unique a fixed point of T. □

Example 1. *Let* $\mathcal{M} = [0, 1]$ *and* $\rho(\theta, \vartheta) = \begin{cases} \theta - \vartheta & \text{if } \theta \geq \vartheta \\ 2(\vartheta - \theta) & \text{if } \theta < \vartheta \end{cases}$ *for all* $\theta, \vartheta \in \mathcal{M}$*, then* $(\mathcal{M}, \rho)$ *is a right-complete quasi-metric space. Let*

$$T\theta = \begin{cases} \frac{\theta}{2} & \text{if } 0 \leq \theta < \frac{1}{2} \\ \theta - \frac{1}{2} & \text{if } \frac{1}{2} \leq \theta \leq 1 \end{cases}$$

and

$$\alpha(\theta, \vartheta) = \begin{cases} 1 & \text{if } \left(\theta, \vartheta\right) \in \left\{\left[0, \frac{1}{2}\right) \times \left[0, \frac{1}{2}\right)\right\} \cup \left\{\{0\} \times (\frac{1}{2}, 1]\right\}, \\ 0 & \text{otherwise}. \end{cases}$$

Accordingly, for $\psi(t) = \frac{t}{2}$*, the axioms of Theorem 1 are fulfilled. Note that,* 0 *is the required unique fixed point for* T*.*

Corollary 1. *Suppose that a self-mapping T on a right-complete (q.m.s.) $(\mathcal{M}, \rho)$ is $\alpha-$orbital admissible and there exists $\psi \in \Psi$ such that*

$$\alpha(\theta, \vartheta)\rho(T\theta, T\vartheta) \leq \psi\left[N(\theta, \vartheta)\right] + L \cdot B(\theta, \vartheta) \tag{13}$$

for all $\theta, \vartheta \in \mathcal{M}$, where

$$N(\theta, \vartheta) = \max\left\{\rho(\theta, \vartheta), \rho(\theta, T\theta), \rho(\vartheta, T\vartheta), \frac{\rho(\vartheta, T\theta) + \rho(\theta, T\vartheta))}{2}\right\}. \tag{14}$$

Suppose also that either T is continuous or $(\mathcal{M}, \rho)$ is regular with respect to the mapping α. If there exists $\theta_0 \in \mathcal{M}$ such that $\alpha(\theta_0, T\theta_0) \geq 1$, then T possesses a fixed point. If, additionally, condition (U) is satisfied, then the assured fixed point is unique.

Proof. Since $N(\theta, \vartheta) \leq M(\theta, \vartheta)$, the proof follows from Theorem 2. □

Corollary 2. *Suppose that a self-mapping T on a right-complete (q.m.s.) $(\mathcal{M}, \rho)$ is $\alpha-$orbital admissible and there exists a function $\psi \in \Psi$ such that*

$$\alpha(\theta, \vartheta)\rho(T\theta, T\vartheta) \leq \psi\left[\rho(\theta, \vartheta)\right] + L \cdot B(\theta, \vartheta), \tag{15}$$

for all $\theta, \vartheta \in \mathcal{M}$. Suppose also that either T is continuous or $(\mathcal{M}, \rho)$ is regular with respect to mapping α. If there exists $\theta_0 \in \mathcal{M}$ such that $\alpha(\theta_0, T\theta_0) \geq 1$, then T has a fixed point. If, additionally, condition (U) is satisfied, then the assured fixed point is unique.

Other consequences of our main result can be obtained by taking $\alpha(\theta, \vartheta) = 1$.

Corollary 3. *Let $\psi \in \Psi$ and $L \geq 0$. Suppose that a self-mapping T on a right-complete (q.m.s.) $(\mathcal{M}, \rho)$ provides*

$$\rho(T\theta, T\vartheta) \leq \psi\left[M(\theta, \vartheta)\right] + L \cdot B(\theta, \vartheta), \tag{16}$$

for all $\theta, \vartheta \in \mathcal{M}$, or

$$\rho(T\theta, T\vartheta) \leq \psi\left[N(\theta, \vartheta)\right] + L \cdot B(\theta, \vartheta), \tag{17}$$

for all $\theta, \vartheta \in \mathcal{M}$. Then T has a unique fixed point.

Corollary 4. *Let $\psi \in \Psi$ and $L \geq 0$. Suppose that a self-mapping T on a right-complete (q.m.s.) $(\mathcal{M}, \rho)$ fulfills*

$$\rho(T\theta, T\vartheta) \leq \psi\left[\rho(\theta, \vartheta)\right] + L \cdot B(\theta, \vartheta), \tag{18}$$

for all $\theta, \vartheta \in \mathcal{M}$. Then T possesses a unique fixed point.

Corollary 5. *Let $\psi \in \Psi$ and $L \geq 0$. Assume that a self-mapping T on a right-complete (q.m.s.) $(\mathcal{M}, \rho)$ satisfies the inequality*

$$\alpha(\theta, \vartheta)\rho(T\theta, T\vartheta) \leq \psi\left[M(\theta, \vartheta)\right] + L \cdot B(\theta, \vartheta), \tag{19}$$

for all $\theta, \vartheta \in \mathcal{M}$. Then T has a fixed point.

Corollary 6. *Let $\{A_i\}_{i=1}^{2}$ be non-empty closed subsets of a right-complete (q.m.s.) $(\mathcal{M}, \rho)$ and $T : Y \to Y$ be a continuous mapping, where $Y = A_1 \cup A_2$. Suppose that*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$;

(II) There exist $\psi \in \Psi$ and $L \geq 0$ such that

$$\rho(T\theta, T\vartheta) \leq \psi\left[M(\theta, \vartheta)\right] + L \cdot B(\theta, \vartheta), \tag{20}$$

for all $(\theta, \vartheta) \in A_1 \times A_2$. Then T has a fixed point that belongs to $A_1 \cap A_2$.

Proof. Since A_1 and A_2 are closed subsets of a right-complete (q.m.s.), then (Y, ρ) is also a right-complete (q.m.s.). Define the mapping $\alpha : Y \times Y \to [0, \infty)$ by

$$\alpha(\theta, \vartheta) = \begin{cases} 1 \text{ if } (\theta, \vartheta) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 \text{ otherwise.} \end{cases}$$

From (II) and the definition of α, we can write

$$\alpha(\theta,\vartheta)\rho(T\theta,T\vartheta) \leq \psi\left[M(\theta,\vartheta)\right] + L \cdot B(\theta,\vartheta),$$

for all $(\theta,\vartheta) \in A_1 \times A_2$. Thus, T is a Berinde type (α,ψ) contraction. From the definition of α, all the premises of Theorem 1 are easily satisfied (see also e.g., [18]). Indeed, T possesses a fixed point in $A_1 \cap A_2$. □

Notice that by letting $L = 0$ in Theorem 1, Corollarys 1–6, we can get several existing results. Furthermore, by choosing α properly, like in Corollarys 5 and 6, we get some more consequences. In the same way, one can also choose $\psi \in \Psi$ in a suitable way to get more results.

3. Ulam-Stability

In this section, we discuss a standard application of fixed point theory: Ulam stability.

Definition 3. *Let $(\mathcal{M},\rho)$ be a (q.m.s.) and let T be a self-mapping on $\mathcal{M}$. We say that the fixed point equation*

$$\theta = T\theta,\ \theta \in \mathcal{M} \tag{21}$$

is generalized Ulam-stable if for each $\varepsilon > 0$ and for any $v \in \mathcal{M}$ satisfying the inequality

$$\rho(v,Tv) \leq \varepsilon, \tag{22}$$

there exist an increasing function $\beta : [0,\infty) \to [0,\infty)$ continuous at 0, with $\beta(0) = 0$ and $\omega \in \mathcal{M}$ a solution of the Equation (21) such that

$$\rho(v,\omega) \leq \beta(\varepsilon) \quad \textit{and} \quad \rho(\omega,v) \leq \beta(\varepsilon). \tag{23}$$

If we consider $\beta(s) = as$ for all $s \geq 0$, where $a > 0$, then the fixed point Equation (21) is said to be Ulam-stable.

Theorem 3. *Let $(\mathcal{M},\rho)$ be a right-complete (q.m.s.). Let the function $\beta : [0,\infty) \to [0,\infty)$ be defined by $\beta(s) := s - \psi(s)$, with $\psi \in \Psi$. Suppose that the premises of Corollary 2 are satisfied. Then the fixed point Equation (21) is generalized Ulam-stable.*

Proof. According to the Corollary 2, there exists a unique $\omega \in \mathcal{M}$ such that $T\omega = \omega$, which means that ω is a solution of the fixed point Equation (21). Let $v \in \mathcal{M}$. By triangle inequality and (15), for $\theta = v$ and $\vartheta = \omega$, we have

$$\begin{aligned} \rho(v,\omega) &\leq \rho(v,Tv) + \rho(Tv,T\omega) + \rho(T\omega,\omega) \leq \rho(v,Tv) + \alpha\,(v,\omega)\,\rho(Tv,T\omega) + \rho(T\omega,\omega) \\ &\leq \rho(v,Tv) + \psi\left[\rho(v,\omega)\right] + L \cdot B(v,\omega) \\ &= \rho(v,Tv) + \psi\left[\rho(v,\omega)\right] \leq \varepsilon + \psi\left[\rho(v,\omega)\right]. \end{aligned} \tag{24}$$

We get

$$\rho(v,\omega) - \psi(q(v,\omega)) = \beta(\rho(v,\omega)) \leq \varepsilon.$$

Similarly,

$$\beta(\rho(\omega,v)) \leq \varepsilon.$$

Equivalently,

$$\rho(v,\omega) \leq \beta^{-1}(\varepsilon) \quad \text{and} \quad \rho(\omega,v) \leq \beta^{-1}(\varepsilon).$$

The function $\beta : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing, so β^{-1} is also continuous, increasing and satisfies $\beta^{-1}(0) = 0$. Hence, the fixed point Equation (21) is generalized Ulam-stable. $\square$

4. An Application

In this section, we consider the following nonlinear fractional differential equation:

$$D^{\delta}(\xi(t)) = f(t, \xi(t)), \tag{25}$$

where $0 \le t \le 1$, $\delta > 1$, under the two-point boundary value condition

$$\xi(0) = \xi(1) = 0,$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function (for more details, see [28–32]). Clearly, a solution to Equation (25) is a fixed point of the integral equation

$$F\xi(t) \quad = \quad \int_0^1 G(t,s) f(ds, \xi(s)),$$

where $G(t,s)$ is the Green function associated to the problem (25) defined as

$$G(t,s) = \begin{cases} [t(1-s)]^{\delta-1} - (t-s)^{\delta-1} & \text{if } \ 0 \le s \le t \le 1 \\ \frac{[t(1-s)]^{\delta-1}}{\Gamma(\delta)} & \text{if } \ 0 \le t \le s \le 1, \end{cases}$$

where Γ is the gamma function.

We endow the space $X =: C([0,1], \mathbb{R})$ by the quasi-metric $d : X \times X \to [0, \infty)$ defined as

$$d(\xi, \eta) = \begin{cases} \|\xi - \eta\|_\infty + \|\xi\|_\infty & \text{if } \ \xi \ne \eta \\ \|\xi\|_\infty & \text{otherwise,} \end{cases}$$

where

$$\|x\|_\infty = \sup_{t \in [0,1]} |x(t)|$$

for each $x \in X$. Note that (X, d) is a complete quasi-metric space. For $\xi, \eta \in C([0,1], \mathbb{R})$, denote by $\xi \preceq \eta$ if and only if $\xi(t) \le \eta(t)$ for each $t \in [0,1]$. Then $(X, d, \preceq)$ is a partially ordered quasi-metric space. Now, we prove the following existence theorem.

Theorem 4. *Consider the nonlinear fractional differential Equation (25). Assume that there exist $k_1, k_2 \in [0,1)$ such that all $\xi, \eta \in C([0,1], \mathbb{R})$, $\xi \preceq \eta$, we have*

(i)

$$|f(s, \eta(s)) - f(s, \xi(s))| \le k_1(\eta(s) - \xi(s))$$

and

$$|f(s, \xi(s))| \le k_2 |\xi(s)|$$

for each $s \in [0,1]$;

(ii) *f is non-decreasing with respect to its second variable with respect to the partial order $\preceq$;*

(iii) *There exists $\xi_0 \in X$ such that for each $t \in [0,1]$, we have*

$$\xi_0(t) \le \int_0^1 G(t,s) f(s, \xi_0(s))\, ds;$$

(*iv*) *If* $\{\xi_n\}$ *is a sequence in* X *such that* $\xi_n \preceq \xi_{n+1}$ *for each* n *and* $\xi_n \to \xi \in X$ *as* $n \to \infty$, *then there exists a subsequence* $\{\xi_{n(k)}\}$ *of* $\{\xi_n\}$ *such that* $\xi_{n(k)} \preceq \xi$ *for each* k.

Then, there exists a solution to Equation (25).

Proof. Let $\xi, \eta \in C([0,1], \mathbb{R})$ such that $\xi \preceq \eta$ with $\xi \neq \eta$. For $t \in [0,1]$, we have

$$\begin{aligned}
|F\xi(t) - F\eta(t)| &= |\int_0^1 G(t,s)[f(s,\xi(s)) - f(s,\eta(s))]ds| \\
&\leq \int_0^1 G(t,s)|f(s,\xi(s)) - f(s,\eta(s))|ds \\
&\leq k_1 \int_0^1 G(t,s)|\xi(s) - \eta(s)|ds \\
&\leq k_1 \|\xi - \eta\|_\infty \sup_{t\in[0,1]} \int_0^1 G(t,s)ds \\
&\leq k_1 \|\xi - \eta\|_\infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|F\xi(t)| &= |\int_0^1 G(t,s)f(s,\xi(s))ds| \\
&\leq \int_0^1 G(t,s)|f(s,\xi(s))|ds \\
&\leq k_2 \int_0^1 G(t,s)|\xi(s)|ds \\
&\leq k_2 \|\xi\|_\infty \sup_{t\in[0,1]} \int_0^1 G(t,s)ds \\
&\leq k_2 \|\xi\|_\infty.
\end{aligned}$$

For all $t \in [0,1]$ and $\xi, \eta \in C([0,1], \mathbb{R})$ such that $\xi \preceq \eta$ with $\xi \neq \eta$, we deduce

$$|F\xi(t) - F\eta(t)| + |F\xi(t)| \leq k_1\|\xi - \eta\|_\infty + k_2\|\xi\|_\infty \leq k(\|\xi - \eta\|_\infty + \|\xi\|_\infty),$$

where $k = \max\{k_1, k_2\}$. Thus,

$$d(F\xi, F\eta) \leq \psi(d(\xi, \eta)),$$

for all $\xi, \eta \in C([0,1], \mathbb{R})$ such that $\xi \preceq \eta$ with $\xi \neq \eta$. Here, $F\xi \preceq F\eta$ holds by condition (ii) with $F\xi \neq F\eta$, where $\psi(t) = kt$. The above inequality also holds for $\xi = \eta$. All conditions of Theorem 1 are verified by taking

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then, there is a solution of the nonlinear fractional differential Equation (25). The proof is completed. □

5. Conclusions

Information for contributors: In the last decades, one of the hottest research topics involves revisiting differential and integral equations in the framework of "fractional". In addition, fixed point theory plays a key role in the solution of differential and integral equations. In this paper, we combine these two trends and solve a nonlinear fractional differential equation by using the techniques of fixed point theory.

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