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# New Inertial Forward-Backward Mid-Point Methods for Sum of Infinitely Many Accretive Mappings, Variational Inequalities, and Applications

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**Abstract:** Some new inertial forward-backward projection iterative algorithms are designed in a real Hilbert space. Under mild assumptions, some strong convergence theorems for common zero points of the sum of two kinds of infinitely many accretive mappings are proved. New projection sets are constructed which provide multiple choices of the iterative sequences. Some already existing iterative algorithms are demonstrated to be special cases of ours. Some inequalities of metric projection and real number sequences are widely used in the proof of the main results. The iterative algorithms have also been modified and extended from pure discussion on the sum of accretive mappings or pure study on variational inequalities to that for both, which complements the previous work. Moreover, the applications of the abstract results on nonlinear capillarity systems are exemplified.

**Keywords:** m-accretive mapping; strongly positive mapping; $\mu$-inversely strongly accretive mapping; $\tau$-Lipschitz continuous mapping; variational inequalities; capillarity systems

**MSC:** 47H05; 47H09

## 1. Introduction and Preliminaries

Suppose $H$ is a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $K$ be the non-empty closed and convex subset of $H$. We use $\to$ and $\rightharpoonup$ to denote the strong and weak convergence in $H$, respectively.

We know that Hilbert space $H$ satisfies Opial's condition in the sense that $liminf_{n\to\infty}\|x_n - z\| < liminf_{n\to\infty}\|x_n - y\|$ for $\{x_n\} \subset H$ with $x_n \rightharpoonup z$ and $y \neq z$ (see [1]).

The inclusion problem for finding $u \in H$ such that

$$0 \in Su + Tu \tag{1}$$

is studied intensively, where $S : H \to H$ is a mapping and $T : H \to 2^H$ is a multi-valued mapping. This is mainly because many problems appear in convex programming, variational inequalities, split feasibility problems, minimization problem, inverse problem and image processing can be modeled by (1).

A mapping $T : D(T) \subset H \to 2^H$ is said to be an accretive mapping (see [2]) if for each $x, y \in D(T)$, there exist $u \in Tx$ and $v \in Ty$ such that $\langle x - y, u - v\rangle \geq 0$. An accretive mapping $T : D(T) \subset H \to 2^H$ is said to be m-accretive if $R(I + kT) = H$, for $k > 0$.

A mapping $S : D(S) \subset H \to H$ is said to be $\mu$-inversely strongly accretive mapping (see [3]) if for each $x, y \in D(S)$ and $\mu > 0$, $\langle x - y, Sx - Sy\rangle \geq \mu\|Sx - Sy\|^2$.

For a mapping $W : D(W) \subset H \to H$, a point $x \in D(W)$ is called a zero point of $W$ if $Wx = 0$. The set of zero points of $W$ is denoted by $W^{-1}0$. If $x \in D(W) \subset H$ satisfies that $Wx = x$, then $x$ is called a fixed point of $W$. The set of fixed points of $W$ is denoted by $Fix(W)$.

The study of the special case of inclusion problem (1), where $T$ is accretive and $S$ is $\mu$-inversely strongly accretive, has been a hot topic during the past few years. In particular, the constructions of the iterative algorithms for approximating the zero point of the sum of $T$ and $S$ are focused, see [3–12] and the references therein. The inertial forward-backward splitting method is one of the important iterative algorithms studied by some authors, see [7–9,13,14].

In 2015, Lorenz and Pock [9] proposed the following inertial forward-backward algorithm for approximating zero points of $T + S$, where $T : H \to 2^H$ is m-accretive and $S : H \to H$ is $\mu$-inversely strongly accretive:

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily,} \\ v_n = u_n + \theta_n(u_n - u_{n-1}), \\ u_{n+1} = (I + r_nT)^{-1}(v_n - r_nSv_n), \quad n \in \mathbb{N}. \end{cases} \tag{2}$$

In addition, the result that $u_n \rightharpoonup p \in (T + S)^{-1}0$, as $n \to \infty$, is proved under some conditions.

To get strong convergence, Dong et al. proposed the following inertial forward-backward projection algorithm in Hilbert spaces in [14]:

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily,} \\ v_n = u_n + \alpha_n(u_n - u_{n-1}), \\ w_n = (I + r_nT)^{-1}(v_n - r_nSv_n), \\ C_n = \{p \in H : \|w_n - p\|^2 \leq \|u_n - p\|^2 - 2\alpha_n\langle u_n - p, u_{n-1} - u_n\rangle \\ \qquad\qquad + \alpha_n^2\|u_{n-1} - u_n\|^2\}, \\ Q_n = \{p \in H : \langle u_n - p, u_n - u_0\rangle \leq 0\}, \\ u_{n+1} = P_{C_n \cap Q_n}(u_0), \ n \in \mathbb{N}, \end{cases} \tag{3}$$

where $T$ and $S$ are the same as those in (2) and $P_{C_n \cap Q_n}$ is the metric projection whose meaning can be seen in Definition 1. The projection sets $C_n$ and $Q_n$ play an important role in the iterative construction to ensure the strong convergence. The result that $u_n \to P_{(T+S)^{-1}0}(u_0)$, as $n \to \infty$, is proved under some conditions.

In 2018, Khan et al. proposed the following one in which the projection set $Q_n$ is deleted (see [7]):

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily,} \\ v_n = u_n + \theta_n(u_n - u_{n-1}), \\ w_n = \alpha_nu_n + (1 - \alpha_n)(I + r_nT)^{-1}(v_n - r_nSv_n), \\ C_1 := H, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq \|u_n - p\|^2 \\ \qquad\qquad - 2\theta_n(1 - \alpha_n)\langle u_n - p, u_{n-1} - u_n\rangle + 2\theta_n^2\|u_{n-1} - u_n\|^2\}, \\ u_{n+1} = P_{C_{n+1}}(u_0), \ n \in \mathbb{N}, \end{cases} \tag{4}$$

where $T$ and $S$ are the same as those in (3). The strong convergence that $u_n \to P_{(T+S)^{-1}0}(u_0)$, as $n \to \infty$, is also obtained under some conditions.

On the other hand, the inclusion problem (1) is extended to the system of inclusion problems:

$$0 \in S_iu + T_iu, \tag{5}$$

where $T_i$ is m-accretive and $S_i$ is $\mu_i$-inversely strongly accretive for $i \in \{1, 2, \cdots, m\}$ or $i \in \mathbb{N}$. In addition, some iterative algorithms for approximating common zero points of $T_i + S_i$ are constructed in [3,15–17]. In particular, Wei et al. proposed the following implicit mid-point forward-backward projection algorithm in [17]:

$$\begin{cases} u_1 \in H \text{ chosen arbitrarily}, \\ w_n = \alpha_n \eta f(u_n) + (I - \alpha_n F)u_n, \\ v_n = \beta_n w_n + \delta_n \sum_{i=1}^{\infty} c_{n,i}(I + r_{n,i}T_i)^{-1}[\frac{w_n+v_n}{2} - r_{n,i}S_i(\frac{w_n+v_n}{2})] + \lambda_n e_n, \\ C_1 = H, \\ C_{n+1} = \{p \in C_n : \|v_n - p\|^2 \leq \frac{1+\beta_n-\lambda_n}{2-\delta_n}\|w_n - p\|^2 + \frac{2\lambda_n}{2-\delta_n}\|e_n - p\|^2\}, \\ u_{n+1} = P_{C_{n+1}}(u_1), \quad n \in \mathbb{N}, \end{cases} \tag{6}$$

where $f : H \to H$ is a contraction, $F : H \to H$ is strongly positive linear bounded mapping, $e_n$ is the computational error, $T_i$ is m-accretive and $S_i$ is $\mu_i$-inversely strongly accretive for $i \in \mathbb{N}$. The result that $u_n \to P_{\bigcap_{i=1}^{\infty}(T_i+S_i)^{-1}0}(u_1)$, as $n \to \infty$, is proved under some conditions.

Recall that $f : H \to H$ is called a contraction (see [17]) if there exists a constant $l \in (0, 1)$ such that $\|f(x) - f(y)\| \leq l\|x - y\|$ for $x, y \in H$.

A mapping $F : H \to H$ is called strongly positive (see [17]) if there exists $\xi > 0$ such that $\langle x, Fx\rangle \geq \xi\|x\|^2$ for $x \in H$. In this case,

$$\|aI - bF\| = sup_{\|x\|\leq 1}|\langle (aI - bF)x, x\rangle|,$$

where $I$ is the identity mapping, $a \in [0, 1]$ and $b \in [-1, 1]$.

A mapping $U : H \to H$ is said to be non-expansive (see [17]) if for each $x, y \in H$, $\|Ux - Uy\| \leq \|x - y\|$.

In 2018, Wei et al. proposed some new hybrid iterative algorithms to approximate the common element of the set of zero points of infinitely many m-accretive mappings $T_i : H \to H$ and the set of fixed points of infinitely many non-expansive mappings $B_i : H \to H$. A special case (see Corollary 3.6 in [18]) is presented as follows:

$$\begin{cases} u_1 \in H \text{ chosen arbitrarily}, \\ y_n = \alpha_n u_n + (1 - \alpha_n)\sum_{i=1}^{\infty} c_{n,i}(I + r_{n,i}T_i)^{-1}(u_n + e_n), \\ z_n = \beta_n u_n + (1 - \beta_n)\sum_{i=1}^{\infty} b_i B_i y_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|y_n - p\|^2 \leq \alpha_n\|u_n - p\|^2 + (1 - \alpha_n)\|u_n + e_n - p\|^2, \\ \qquad \|z_n - p\|^2 \leq \beta_n\|u_n - p\|^2 + (1 - \beta_n)\|y_n - p\|^2\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|u_1 - P_{C_{n+1}}(u_1)\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{7}$$

The result that $u_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in (\bigcap_{i=1}^{\infty} T_i^{-1}0) \bigcap (\bigcap_{i=1}^{\infty} Fix(B_i))$, as $n \to \infty$, is proved under some conditions. We may notice that infinite choices of $\{u_n\}$ can be made, which is totally different from traditional projection iterative algorithms, e.g., (3).

In 2016, Wei et al. proposed an implicit forward-backward mid-point iterative algorithm for approximating common zero points of $T_i + S_i$, where $T_i$ is m-accretive and $S_i$ is $\mu_i$-inversely strongly accretive, for $i \in \mathbb{N}$. A special case in [19] in the frame of Hilbert space is presented as follows:

$$\begin{cases} u_1 \in K \subset H \text{ chosen arbitrarily}, \\ y_n = P_K[(1-\alpha_n)(u_n + e'_n)], \\ z_n = \delta_n y_n + \beta_n \sum_{i=1}^{\infty} a_i (I + r_{n,i}T_i)^{-1}[(\frac{y_n+z_n}{2} - r_{n,i}S_i(\frac{y_n+z_n}{2})] + \xi_n e''_n, \\ u_{n+1} = \gamma_n \eta f(u_n) + (I - \gamma_n F)z_n + e'''_n, \quad n \in \mathbb{N}, \end{cases} \tag{8}$$

where $f$ and $F$ are the same as those in (6), $\{e'_n\}$, $\{e''_n\}$ and $\{e'''_n\}$ are the error sequences. Under some conditions, $\{u_n\}$ is proved to be convergent strongly to $u_0 \in \bigcap_{i=1}^{\infty}(T_i + S_i)^{-1}0$, which also satisfies the following variational inequality:

$$\langle Fu_0 - \eta f(u_0), u_0 - z \rangle \leq 0, \quad \forall z \in \bigcap_{i=1}^{\infty}(T_i + S_i)^{-1}0. \tag{9}$$

We may notice that the connection between the common element of $(T_i + S_i)^{-1}0$ for $i \in \mathbb{N}$ and the solution of one kind variational inequality is set up in [19].

In this paper, our main purpose is formulated as follows: (1) obtain strong convergence theorems instead of weak ones; (2) construct new projection sets, which ensure that infinitely many iterative sequences can be generated compared to traditional projection iterative algorithms (3), (4) and (6); (3) inject the idea of inertial forward-backward algorithm into the iterative construction, compared to iterative algorithms (6)–(8); (4) set up the connection between the common zero point of the sum of two kinds of infinitely many accretive mappings and the solution of one kind variational inequality, which complements the corresponding work since rare studies of the projection iterative algorithms (e.g., (3)–(7)) have mentioned that; (5) provide the application of the abstract result to capillarity systems.

To begin our study, we need some preliminaries.

**Definition 1.** *(see [2]) For the Hilbert space $H$ and its non-empty closed and convex subset $K$, there exists a unique point $x_0 \in K$ such that $\|x - x_0\| = inf\{\|x - y\| : y \in K\}$, for each $x \in H$. In this case, the metric projection mapping $P_K : H \to K$ is defined by $P_K x = x_0$, for $\forall x \in H$.*

**Definition 2.** *(see [20]) Let $\{K_n\}$ be a sequence of non-empty closed and convex subsets of $H$. Then*

*(1) the strong lower limit of $\{K_n\}$, $s - liminf K_n$, is defined as the set of all $x \in H$ such that there exists $x_n \in K_n$ for almost all $n$ and it tends to $x$ as $n \to \infty$ in the norm;*

*(2) the weak upper limit of $\{K_n\}$, $w - limsup K_n$, is defined as the set of all $x \in H$ such that there exists a subsequence $\{K_{n_m}\}$ of $\{K_n\}$ and $x_{n_m} \in K_{n_m}$ for every $n_m$ and it tends to $x$ as $n_m \to \infty$ in the weak topology;*

*(3) the limit of $\{K_n\}$, $limK_n$, is the common value when $s - liminf K_n = w - limsup K_n$.*

**Lemma 1.** *(see [20]) Let $\{K_n\}$ be a decreasing sequence of closed and convex subsets of $H$, i.e., $K_n \subset K_m$ if $n \geq m$. Then $\{K_n\}$ converges in $H$ and $limK_n = \bigcap_{n=1}^{\infty} K_n$.*

**Lemma 2.** *(see [21]) Suppose $H$ is a real Hilbert space. If $limK_n$ exists and is not empty, then $P_{K_n}x \to P_{limK_n}x$ for every $x \in H$, as $n \to \infty$.*

**Lemma 3.** *(see [2,19]) If $B : H \to H$ is accretive, then $(I + rB)^{-1} : H \to H$ is non-expansive.*

**Lemma 4.** *(see [22]) If $H$ is a real Hilbert space with $K$ its non-empty closed and convex subset, $T_i : K \to K$ is non-expansive for $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} a_i = 1$ for $\{a_i\} \subset (0,1)$, then $\sum_{i=1}^{\infty} a_i T_i$ is non-expansive with $Fix(\sum_{i=1}^{\infty} a_i T_i) = \bigcap_{i=1}^{\infty} Fix(T_i)$ under the assumption that $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$.*

**Lemma 5.** *(see [19]) If $H$ is a real Hilbert space with $K$ its non-empty closed and convex subset, $S : K \to H$ is a single-valued mapping and $T : H \to 2^H$ is an m-accretive mapping, then*

$$Fix((I + rT)^{-1}(I - rS)) = (T + S)^{-1}0,$$

*for $\forall r > 0$.*

**Lemma 6.** *(see [23]) Let $H$ be a real Hilbert space and $r \in (0, +\infty)$. Then there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to [0, +\infty)$ with $g(0) = 0$ such that $\|kx + (1-k)y\|^2 \leq k\|x\|^2 + (1-k)\|y\|^2 - k(1-k)g(\|x - y\|)$, for $k \in [0,1]$, $x, y \in H$ with $\|x\| \leq r$ and $\|y\| \leq r$.*

**Lemma 7.** *(see [24]) Let $K$ be the non-empty closed and convex subset of Hilbert space $H$ and $P_K : H \to K$ be the metric projection. Then*

*(1) for $\forall x \in H$ and $\forall y \in K$, $\|P_K x - y\|^2 + \|P_K x - x\|^2 \leq \|y - x\|^2$.*
*(2) $y = P_K x$ if and only if there holds the following inequality $\langle x - y, y - z \rangle \geq 0$, for $\forall z \in K$.*

**Lemma 8.** *(see [25]) If $f : H \to H$ is a contraction, then there is a unique element $x \in H$ such that $f(x) = x$.*

## 2. Some Inertial Forward-Backward Algorithms

In this section, unless otherwise stated, we always assume that:

(1) $H$ is a real Hilbert space;
(2) $A_i : H \to H$ is $\mu_i$-inversely strongly accretive and $B_i : H \to H$ is m-accretive, for each $i \in \mathbb{N}$. In addition, $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \neq \emptyset$;
(3) $\{e_n\} \subset H$ is the computational error;
(4) $\{\sigma_n\}$, $\{s_{n,i}\}$ and $\{\mu_i\}$ are three real number sequences in $(0, +\infty)$ for $i, n \in \mathbb{N}$;
(5) $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real number sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n \equiv 1$, for $n \in \mathbb{N}$;
(6) $\{\omega_{n,i}\}$ is a real number sequence in $(0,1)$ with $\sum_{i=1}^{\infty} \omega_{n,i} = 1$, for $n \in \mathbb{N}$;
(7) $\{k_n\}$ is a real number sequence in $[0, k]$ for some $k \in [0,1)$.

### *2.1. New Inertial Forward-Backward Projection Algorithms*

**Theorem 1.** *Let $\{u_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily}, \quad e_1 \in H, \\ v_n = u_n + k_n(u_n - u_{n-1}), \\ w_n = \alpha_n u_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)v_n + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq (1-\gamma_n)\|u_n - p\|^2 + \gamma_n\|e_n - p\|^2 \\ \qquad\qquad + k_n^2\|u_n - u_{n-1}\|^2 - 2\beta_n k_n \langle u_n - p, u_{n-1} - u_n \rangle\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{10}$$

*Under the assumptions that: (i) $s_{n,i} \leq \mu_i$ for $i, n \in \mathbb{N}$; (ii) $\sigma_n \to 0, \gamma_n \to 0$, as $n \to \infty$; (iii) $0 < inf_n \beta_n < 1$; (iv) there exists $M > 0$ such that $\|e_n\| \leq M$, for $n \in \mathbb{N}$, we have: $u_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1) = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, as $n \to \infty$.*

**Proof.** We split the proof into nine steps.

Step 1. $\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i) : H \to H$ is non-expansive, for $n \in \mathbb{N}$.

The proof of Step 1 is essentially from that of Step 1 in Theorem 2.1 of [17]. For the sake of completeness, we present it as follows.

Since $s_{n,i} \le 2\mu_i$, then for each $x, y \in H$,

$$\begin{aligned}
&\|(I - s_{n,i}A_i)x - (I - s_{n,i}A_i)y\|^2 = \|(x - y) - s_{n,i}(A_i x - A_i y)\|^2 \\
&= \|x - y\|^2 - 2s_{n,i}\langle x - y, A_i x - A_i y\rangle + s_{n,i}^2\|A_i x - A_i y\|^2 \\
&\le \|x - y\|^2 + s_{n,i}(s_{n,i} - 2\mu_i)\|A_i x - A_i y\|^2 \le \|x - y\|^2.
\end{aligned}$$

Thus, $(I - s_{n,i}A_i) : H \to H$ is non-expansive, for $i, n \in \mathbb{N}$. It then follows from Lemmas 3 and 4 that $\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i) : H \to H$ is non-expansive, for $n \in \mathbb{N}$. Combining with Lemma 5, $\bigcap_{i=1}^{\infty} Fix((I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)) = \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$.

Step 2. $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \subset C_n$, for $n \in \mathbb{N}$.

If $n = 1$, it is obvious that $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \subset C_1$.

Now, $\forall q \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, suppose the result is true for $n = m + 1$, then if $n = m + 2$, it follows from (10) that

$$\begin{aligned}
&\|v_{m+1} - q\|^2 \\
&= \|u_{m+1} - q\|^2 + 2k_{m+1}\langle u_{m+1} - q, u_{m+1} - u_m\rangle + k_{m+1}^2\|u_{m+1} - u_m\|^2.
\end{aligned} \tag{11}$$

Using Step 1, we have:

$$\begin{aligned}
&\|w_{m+1} - q\|^2 \\
&\le \alpha_{m+1}\|u_{m+1} - q\|^2 + \beta_{m+1}\|v_{m+1} - q\|^2 + \gamma_{m+1}\|e_{m+1} - q\|^2.
\end{aligned} \tag{12}$$

Combining (11) and (12),

$$\begin{aligned}
&\|w_{m+1} - q\|^2 \le (\alpha_{m+1} + \beta_{m+1})\|u_{m+1} - q\|^2 \\
&+\gamma_{m+1}\|e_{m+1} - q\|^2 + k_{m+1}^2\|u_{m+1} - u_m\|^2 - 2k_{m+1}\beta_{m+1}\langle u_{m+1} - q, u_m - u_{m+1}\rangle,
\end{aligned}$$

which ensures that $q \in C_{m+2}$. Then by induction, $q \in C_n$, for $n \in \mathbb{N}$.

Step 3. $C_n$ is a closed and convex subset of $H$, for each $n \in \mathbb{N}$.

It is not difficult to see that

$$\begin{aligned}
&\|w_n - p\|^2 \le (1 - \gamma_n)\|u_n - p\|^2 + \gamma_n\|e_n - p\|^2 \\
&+k_n^2\|u_n - u_{n-1}\|^2 - 2\beta_n k_n\langle u_n - p, u_{n-1} - u_n\rangle \\
&\Longleftrightarrow \|w_n\|^2 - (1 - \gamma_n)\|u_n\|^2 - \gamma_n\|e_n\|^2 - k_n^2\|u_n - u_{n-1}\|^2 + 2\beta_n k_n\langle u_n, u_{n-1} - u_n\rangle \\
&\le 2\langle p, w_n\rangle - 2(1 - \gamma_n)\langle p, u_n\rangle - 2\gamma_n\langle p, e_n\rangle + 2\beta_n k_n\langle p, u_{n-1} - u_n\rangle.
\end{aligned}$$

Then $C_n$ is a closed and convex subset of $H$, for each $n \in \mathbb{N}$.

Step 4. $Q_n$ is non-empty for each $n \in \mathbb{N}$, which ensures that $\{u_n\}$ is well-defined.

From Step 3 and the definition of metric projection, for $\sigma_{n+1}$, there exists $\delta_{n+1} \in C_{n+1}$ such that $\|u_1 - \delta_{n+1}\|^2 \le (inf_{z \in C_{n+1}}\|u_1 - z\|)^2 + \sigma_{n+1} = \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}$. Thus, $Q_{n+1} \ne \emptyset$, for $n \in \mathbb{N}$. And then $\{u_n\}$ is well-defined.

Step 5. $P_{C_{n+1}}(u_1) \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

The proof of Step 5 is similar to Step 2 of Theorem 3.1 in [18]. It follows from Lemma 1 that $limC_n$ exists and $limC_n = \bigcap_{n=1}^{\infty} C_n \ne \emptyset$. Then Lemma 2 implies that $P_{C_{n+1}}(u_1) \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Step 6. $u_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Since $u_{n+1} \in Q_{n+1} \subset C_{n+1}$ and $C_n$ is a convex subset of $H$, then for $\forall t \in (0, 1)$, $tP_{C_{n+1}}(u_1) + (1 - t)u_{n+1} \in C_{n+1}$, which implies that

$$\|P_{C_{n+1}}(u_1) - u_1\| \le \|tP_{C_{n+1}}(x_1) + (1 - t)u_{n+1} - u_1\|. \tag{13}$$

Using Lemma 6, we have

$$\begin{aligned}&\|tP_{C_{n+1}}(u_1)+(1-t)u_{n+1}-u_1\|^2=\|t(P_{C_{n+1}}(u_1)-u_1)+(1-t)(u_{n+1}-u_1)\|^2\\&\le t\|P_{C_{n+1}}(u_1)-u_1\|^2+(1-t)\|u_{n+1}-u_1\|^2-t(1-t)g(\|P_{C_{n+1}}(u_1)-u_{n+1}\|).\end{aligned}\tag{14}$$

Then (13) and (14) ensure that $tg(\|P_{C_{n+1}}(u_1)-u_{n+1}\|)\le\|u_{n+1}-u_1\|^2-\|P_{C_{n+1}}(u_1)-u_1\|^2\le\sigma_{n+1}$. Letting $t\to1$ first and then $n\to\infty$, we know that $P_{C_{n+1}}(u_1)-u_{n+1}\to0$ as $n\to\infty$. From Step 5, $u_n\to P_{\bigcap_{m=1}^\infty C_m}(u_1)$, as $n\to\infty$.

Step 7. $v_n\to P_{\bigcap_{m=1}^\infty C_m}(u_1)$ and $w_n\to P_{\bigcap_{m=1}^\infty C_m}(u_1)$, as $n\to\infty$.

Since $u_{n+1}-u_n\to0$ and $u_{n+1}\in Q_{n+1}\subset C_{n+1}$, then from (10), $\|u_{n+1}-v_{n+1}\|=k_{n+1}\|u_{n+1}-u_n\|\to0$, as $n\to\infty$. Thus, $v_n\to P_{\bigcap_{m=1}^\infty C_m}(u_1)$, as $n\to\infty$. Since $u_{n+1}\in Q_{n+1}\subset C_{n+1}$, then

$$\begin{aligned}&\|w_n-u_{n+1}\|^2\le(1-\gamma_n)\|u_{n+1}-u_n\|^2+\gamma_n\|e_n-u_{n+1}\|^2\\&+k_n^2\|u_n-u_{n-1}\|^2-2\beta_nk_n\langle u_n-u_{n+1},u_{n-1}-u_n\rangle\\&\le(1-\gamma_n)\|u_{n+1}-u_n\|^2+\gamma_n\|e_n-u_{n+1}\|^2+k_n^2\|u_n-u_{n-1}\|^2+2\|u_{n+1}-u_n\|\|u_{n-1}-u_n\|\to0,\end{aligned}$$

as $n\to\infty$. Thus, $w_n\to P_{\bigcap_{m=1}^\infty C_m}(u_1)$, as $n\to\infty$.

Step 8. $P_{\bigcap_{m=1}^\infty C_m}(u_1)\in\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0$.

In fact, if, otherwise, $P_{\bigcap_{m=1}^\infty C_m}(u_1)\overline{\in}\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0$. Then $P_{\bigcap_{m=1}^\infty C_m}(u_1)\neq\sum_{i=1}^\infty\omega_{n,i}(I+s_{n,i}B_i)^{-1}(I-s_{n,i}A_i)P_{\bigcap_{m=1}^\infty C_m}(u_1)$.

Since $w_n=\alpha_nu_n+\beta_n\sum_{i=1}^\infty\omega_{n,i}(I+s_{n,i}B_i)^{-1}(I-s_{n,i}A_i)v_n+\gamma_ne_n$, then $\beta_n[\sum_{i=1}^\infty\omega_{n,i}(I+s_{n,i}B_i)^{-1}(I-s_{n,i}A_i)v_n-w_n]=\alpha_n(w_n-u_n)+\gamma_n(w_n-e_n)\to0$, as $n\to\infty$.

Since $inf_n\beta_n>0$, then there exists a subsequence of $\{n\}$, which is still denoted by $\{n\}$ such that $\sum_{i=1}^\infty\omega_{n,i}(I+s_{n,i}B_i)^{-1}(I-s_{n,i}A_i)v_n-w_n\to0$, as $n\to\infty$.

Thus, $\sum_{i=1}^\infty\omega_{n,i}(I+s_{n,i}B_i)^{-1}(I-s_{n,i}A_i)v_n\to P_{\bigcap_{m=1}^\infty C_m}(u_1)$, as $n\to\infty$.

Since $H$ satisfies Opial's condition, then

$$\begin{aligned}&liminf_{n\to\infty}\|v_n-P_{\bigcap_{m=1}^\infty C_m}(u_1)\|\\&<liminf_{n\to\infty}\|v_n-\textstyle\sum_{i=1}^\infty\omega_{n,i}(I+s_{n,i}B_i)^{-1}(I-s_{n,i}A_i)P_{\bigcap_{m=1}^\infty C_m}(u_1)\|\\&\le liminf_{n\to\infty}\|v_n-\textstyle\sum_{i=1}^\infty\omega_{n,i}(I+s_{n,i}B_i)^{-1}(I-s_{n,i}A_i)v_n\|\\&+liminf_{n\to\infty}\|v_n-P_{\bigcap_{m=1}^\infty C_m}(u_1)\|\\&\le liminf_{n\to\infty}\|v_n-P_{\bigcap_{m=1}^\infty C_m}(u_1)\|,\end{aligned}$$

which makes a contradiction! Thus, $P_{\bigcap_{m=1}^\infty C_m}(u_1)\in\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0$.

Step 9. $P_{\bigcap_{m=1}^\infty C_m}(u_1)=P_{\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0}(u_1)$.

From Step 8, $\|P_{\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0}(u_1)-u_1\|\le\|P_{\bigcap_{m=1}^\infty C_m}(u_1)-u_1\|$.

On the other hand, since $\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0\subset\bigcap_{m=1}^\infty C_m$, then $\|P_{\bigcap_{m=1}^\infty C_m}(u_1)-u_1\|\le\|P_{\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0}(u_1)-u_1\|$. Thus,

$$\|P_{\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0}(u_1)-u_1\|=\|P_{\bigcap_{m=1}^\infty C_m}(u_1)-u_1\|.$$

Using Lemma 7, we have

$$\begin{aligned}&\|P_{\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0}(u_1)-P_{\bigcap_{m=1}^\infty C_m}(u_1)\|^2+\|P_{\bigcap_{m=1}^\infty C_m}(u_1)-u_1\|^2\\&\le\|P_{\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0}(u_1)-u_1\|^2=\|P_{\bigcap_{m=1}^\infty C_m}(u_1)-u_1\|^2,\end{aligned}$$

which implies that $P_{\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0}(u_1)=P_{\bigcap_{m=1}^\infty C_m}(u_1)$. $\square$

**Remark 1.** *Compared to (3) and (4), infinitely many m-accretive mappings and infinitely many $\mu_i$-inversely strongly accretive mappings are considered in (10). Compared to (6), the idea of inertial forward-backward* *algorithm is embodied in (10). Compared to (3), (4) and (6), infinite choices of the iterative sequences $\{u_n\}$ are defined.*

**Remark 2.** *The traditional idea for choosing the unique iterative element $u_{n+1}$ as the metric projection of the initial element in iterative algorithms (3), (4) and (6) is contained in the ideas of (10) in our paper.*

In fact, if take $u_{n+1} = P_{C_{n+1}}(u_1)$, we can easily see that

$$\|u_{n+1} - u_1\|^2 \leq \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}.$$

Thus, $u_{n+1} = P_{C_{n+1}}(u_1) \in Q_{n+1}$, which means that this $u_{n+1}$ is a kind of choice of (10).

**Corollary 1.** *If $k_n \equiv 0$, then (10) in Theorem 1 becomes to the traditional forward-backward iterative algorithm:*

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily}, \ \ e_1 \in H, \\ w_n = \alpha_n u_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i} (I + s_{n,i} B_i)^{-1} (I - s_{n,i} A_i) u_n + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq (1 - \gamma_n)\|u_n - p\|^2 + \gamma_n \|e_n - p\|^2\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}. \end{cases}$$

**Corollary 2.** *If $i \equiv 1$, then (10) in Theorem 1 becomes to the following iterative algorithm:*

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily}, \ \ e_1 \in H, \\ v_n = u_n + k_n (u_n - u_{n-1}), \\ w_n = \alpha_n u_n + \beta_n (I + s_n B)^{-1} (I - s_n A) v_n + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq (1 - \gamma_n)\|u_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\ \qquad\qquad + k_n^2 \|u_n - u_{n-1}\|^2 - 2\beta_n k_n \langle u_n - p, u_{n-1} - u_n \rangle\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{15}$$

**Remark 3.** *Let $e_n \equiv 0, \alpha_n + \beta_n \equiv 1$, for $n \in \mathbb{N}$. After taking $u_{n+1} = P_{C_{n+1}}(u_1)$ in (15), we may see that $Q_{n+1}$ can be deleted which implies that (15) reduces to (4). However, the strong assumption that $\sum_{n=1}^{\infty} k_n \|u_n - u_{n-1}\| < +\infty$ in [7] is no longer needed in our paper.*

### *2.2. New Mid-Point Inertial Forward-Backward Projection Algorithms*

**Theorem 2.** *Suppose $f : H \to H$ is a contraction with $k \in (0, 1)$ and $F : H \to H$ is a strongly positive linear bounded operator with coefficient $\xi > 0$. Let $\{u_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily}, \ \ e_1 \in H, \\ z_0 = u_0, \\ z_n = \delta_n \lambda f(u_n) + (I - \delta_n F) u_n, \\ v_n = z_n + k_n (z_n - z_{n-1}), \\ w_n = \alpha_n v_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i} (I + s_{n,i} B_i)^{-1} (I - s_{n,i} A_i)(\frac{v_n + w_n}{2}) + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq \frac{2\alpha_n + \beta_n}{2 - \beta_n} \|z_n - p\|^2 + \frac{2\gamma_n}{2 - \beta_n} \|e_n - p\|^2 \\ \qquad\qquad + \frac{2\alpha_n + \beta_n}{2 - \beta_n} k_n^2 \|z_n - z_{n-1}\|^2 - 2k_n \frac{2\alpha_n + \beta_n}{2 - \beta_n} \langle z_n - p, z_{n-1} - z_n \rangle\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{16}$$

*Under the assumptions of* $(i)-(iv)$ *in Theorem 1 and* $(v)$ $\lambda > 0$ *and* $(vi)$ $\delta_n \to 0$*, as* $n \to \infty$*, we have:* $u_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1) = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1) \in \bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0$*, as* $n \to \infty$.

**Proof.** We split the proof into ten steps.

Step 1. $\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i) : H \to H$ is non-expansive, for $n \in \mathbb{N}$.

Copy Step 1 in Theorem 1.

Step 2. $\{w_n\}$ is well-defined.

Define $U : H \to H$ by

$$Ux = ty + sT(\frac{x+y}{2}) + (1-t-s)v,$$

where $T : H \to H$ is non-expansive and $t, s \in (0,1)$.

It is easy to check that $U$ is a contraction since

$$\|Ux - Uz\| \leq s\|\tfrac{x+y}{2} - \tfrac{z+y}{2}\| = \tfrac{s}{2}\|x - z\| < \|x - z\|,$$

for $\forall x, z \in H$.

In view of Lemma 8, there exists a unique element $x \in H$ such that $x = Ux$. Combining with the result of Step 1, $\{w_n\}$ is well-defined.

Step 3. $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \subset C_n$, for $n \in \mathbb{N}$.

If $n = 1$, it is obvious that $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \subset C_1$.

Now, $\forall q \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, suppose the result is true for $n = m+1$, then if $n = m+2$, in view of Lemma 4, we have:

$$\|w_{m+1} - q\|^2 \leq \alpha_{m+1}\|v_{m+1} - q\|^2 + \beta_{m+1}\|\tfrac{v_{m+1}+w_{m+1}}{2} - q\|^2 + \gamma_{m+1}\|e_{m+1} - q\|^2,$$

which ensures that

$$\|w_{m+1} - q\|^2 \leq \tfrac{2\alpha_{m+1}+\beta_{m+1}}{2-\beta_{m+1}}\|v_{m+1} - q\|^2 + \tfrac{2\gamma_{m+1}}{2-\beta_{m+1}}\|e_{m+1} - q\|^2. \tag{17}$$

It follows from (16) that

$$\begin{aligned}&\|v_{m+1} - q\|^2 \\ &= \|z_{m+1} - q\|^2 + 2k_{m+1}\langle z_{m+1} - q, z_{m+1} - z_m\rangle + k_{m+1}^2\|z_{m+1} - z_m\|^2.\end{aligned} \tag{18}$$

Combining (17) and (18),

$$\begin{aligned}&\|w_{m+1} - q\|^2 \leq \tfrac{2\alpha_{m+1}+\beta_{m+1}}{2-\beta_{m+1}}\|z_{m+1} - q\|^2 + \tfrac{2\gamma_{m+1}}{2-\beta_{m+1}}\|e_{m+1} - q\|^2 \\ &+\tfrac{2\alpha_{m+1}+\beta_{m+1}}{2-\beta_{m+1}}k_{m+1}^2\|z_{m+1} - z_m\|^2 - 2k_{m+1}\tfrac{2\alpha_{m+1}+\beta_{m+1}}{2-\beta_{m+1}}\langle z_{m+1} - q, z_m - z_{m+1}\rangle.\end{aligned}$$

Thus, $q \in C_{m+2}$. Then by induction, $q \in C_n$, for $n \in \mathbb{N}$.

Step 4. $C_n$ is a closed and convex subset of $H$, for each $n \in \mathbb{N}$. It is not difficult to see that

$$\begin{aligned}&\|w_n - p\|^2 \leq \tfrac{2\alpha_n+\beta_n}{2-\beta_n}\|z_n - p\|^2 + \tfrac{2\gamma_n}{2-\beta_n}\|e_n - p\|^2 \\ &+k_n^2\tfrac{2\alpha_n+\beta_n}{2-\beta_n}\|z_n - z_{n-1}\|^2 - 2k_n\tfrac{2\alpha_n+\beta_n}{2-\beta_n}\langle z_n - p, z_{n-1} - z_n\rangle \\ &\Longleftrightarrow \|w_n\|^2 - \tfrac{2\alpha_n+\beta_n}{2-\beta_n}\|z_n\|^2 - \tfrac{2\gamma_n}{2-\beta_n}\|e_n\|^2 - k_n^2\tfrac{2\alpha_n+\beta_n}{2-\beta_n}\|z_n - z_{n-1}\|^2 + 2k_n\tfrac{2\alpha_n+\beta_n}{2-\beta_n}\langle z_n, z_{n-1} - z_n\rangle \\ &\leq 2\langle p, w_n\rangle - 2\tfrac{2\alpha_n+\beta_n}{2-\beta_n}\langle p, z_n\rangle - \tfrac{4\gamma_n}{2-\beta_n}\langle p, e_n\rangle + 2k_n\tfrac{2\alpha_n+\beta_n}{2-\beta_n}\langle p, z_{n-1} - z_n\rangle.\end{aligned}$$

Then $C_n$ is a closed and convex subset of $H$, for each $n \in \mathbb{N}$.

Step 5. $Q_n$ is non-empty for each $n \in \mathbb{N}$, which ensures that $\{u_n\}$ is well-defined.

Copy Step 4 in Theorem 1.

Step 6. $P_{C_{n+1}}(u_1) \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Copy Step 5 in Theorem 1.

Step 7. $u_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Copy Step 6 in Theorem 1.

Step 8. $z_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, $v_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$ and $w_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Since $z_n - u_n = \delta_n(\lambda f(u_n) - Fu_n)$ and $\delta_n \to 0$, then it is easy to see that $z_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Since $v_n = z_n + k_n(z_n - z_{n-1})$, then $v_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Since $u_{n+1} - u_n \to 0$ and $u_{n+1} \in Q_{n+1} \subset C_{n+1}$, then

$$\begin{aligned}
&\|w_n - u_{n+1}\|^2 \leq \tfrac{2\alpha_n+\beta_n}{2-\beta_n}\|z_n - u_{n+1}\|^2 + \tfrac{2\gamma_n}{2-\beta_n}\|e_n - u_{n+1}\|^2 \\
&+k_n^2 \tfrac{2\alpha_n+\beta_n}{2-\beta_n}\|z_n - z_{n-1}\|^2 - 2k_n \tfrac{2\alpha_n+\beta_n}{2-\beta_n}\langle z_n - u_{n+1}, z_{n-1} - z_n\rangle \\
&\leq \|z_n - u_{n+1}\|^2 + 2\gamma_n\|e_n - z_{n+1}\|^2 + k_n^2\|z_n - z_{n-1}\|^2 + 2\|u_{n+1} - z_n\|\|z_{n-1} - z_n\| \to 0.
\end{aligned}$$

Thus, $w_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Step 9. $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$.

In fact, if, otherwise, $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \overline{\in} \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$. Then $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \neq \sum_{i=1}^{\infty}\omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$.

Since $w_n = \alpha_n v_n + \beta_n \sum_{i=1}^{\infty}\omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)(\frac{v_n+w_n}{2}) + \gamma_n e_n$, then $\beta_n[\sum_{i=1}^{\infty}\omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)(\frac{v_n+w_n}{2}) - w_n] = \alpha_n(w_n - v_n) + \gamma_n(w_n - e_n) \to 0$, as $n \to \infty$.

Since $inf_n\beta_n > 0$, then there exists a subsequence of $\{n\}$, which is still denoted by $\{n\}$ such that $\sum_{i=1}^{\infty}\omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)(\frac{v_n+w_n}{2}) - w_n \to 0$, as $n \to \infty$.

Thus, $\sum_{i=1}^{\infty}\omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)(\frac{v_n+w_n}{2}) \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Since $H$ satisfies Opial's condition, then

$$\begin{aligned}
&liminf_{n\to\infty}\|\tfrac{v_n+w_n}{2} - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\| \\
&< liminf_{n\to\infty}\|\tfrac{v_n+w_n}{2} - \textstyle\sum_{i=1}^{\infty}\omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\| \\
&\leq liminf_{n\to\infty}\|\tfrac{v_n+w_n}{2} - \textstyle\sum_{i=1}^{\infty}\omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)\tfrac{v_n+w_n}{2}\| \\
&+liminf_{n\to\infty}\|\tfrac{v_n+w_n}{2} - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\| \\
&\leq liminf_{n\to\infty}\|\tfrac{v_n+w_n}{2} - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\|,
\end{aligned}$$

which makes a contradiction. Thus, $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$.

Step 10. $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1)$.

From Step 9, $\|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1) - u_1\| \leq \|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\|$.

On the other hand, since $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \subset \bigcap_{m=1}^{\infty} C_m$, then $\|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\| \leq \|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1) - u_1\|$. Thus,

$$\|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1) - u_1\| = \|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\|.$$

Using Lemma 7, we have

$$\begin{aligned}
&\|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1) - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\|^2 + \|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\|^2 \\
&\leq \|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1) - u_1\|^2 = \|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\|^2,
\end{aligned}$$

which implies that $P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1) = P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$. $\square$

**Remark 4.** *Similar to Remark 2, $u_{n+1} = P_{C_{n+1}}(u_1)$ is also a possible choice of $u_{n+1} \in Q_{n+1}$ in Theorem 2.*

**Corollary 3.** *If $\delta_n \equiv 0$, then (16) becomes to the following traditional mid-point inertial forward-backward projection iterative algorithm:*

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily}, \quad e_1 \in H, \\ v_n = u_n + k_n(u_n - u_{n-1}), \\ w_n = \alpha_n v_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)(\frac{v_n + w_n}{2}) + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq \frac{2\alpha_n + \beta_n}{2 - \beta_n}\|u_n - p\|^2 + \frac{2\gamma_n}{2 - \beta_n}\|e_n - p\|^2 \\ \qquad + \frac{2\alpha_n + \beta_n}{2 - \beta_n}k_n^2\|u_n - u_{n-1}\|^2 - 2k_n\frac{2\alpha_n + \beta_n}{2 - \beta_n}\langle u_n - p, u_{n-1} - u_n\rangle\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{19}$$

*If, moreover, $k_n \equiv 0$ in (19), then it becomes to the following traditional forward-backward mid-point iterative algorithm:*

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily}, \quad e_1 \in H, \\ w_n = \alpha_n u_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)(\frac{u_n + w_n}{2}) + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq \frac{2\alpha_n + \beta_n}{2 - \beta_n}\|u_n - p\|^2 + \frac{2\gamma_n}{2 - \beta_n}\|e_n - p\|^2\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{20}$$

**Corollary 4.** *If $k_n \equiv 0$ in (15), then it becomes to the following one:*

$$\begin{cases} u_0, u_1 \in H \text{ chosen arbitrarily}, \quad e_1 \in H, \\ z_n = \delta_n \lambda f(u_n) + (I - \delta_n F)u_n, \\ w_n = \alpha_n z_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)(\frac{w_n + z_n}{2}) + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq \frac{2\alpha_n + \beta_n}{2 - \beta_n}\|z_n - p\|^2 + \frac{2\gamma_n}{2 - \beta_n}\|e_n - p\|^2\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{21}$$

*If, moreover, take $u_{n+1} = P_{C_{n+1}}(u_1)$ in (21), then it becomes to (6).*

### *2.3. Relationship with Variational Inequalities*

A lot work has been done on designing iterative algorithms to approximate solution of variational inequalities due to their wide applications (e.g., [26,27]). A classical variational inequality is to find $u \in K$ such that for $\forall v \in K$,

$$\langle v - u, Tu\rangle \geq 0, \tag{22}$$

where $T : K \to H$ is a nonlinear mapping. The symbol $VI(K, T)$ denotes the solution of the above variational inequality.

#### 2.3.1. The First Kind Iteration Theorems

**Definition 3.** *Let $H$ be a real Hilbert space with $K$ being its non-empty closed and convex subset. $T : K \to H$ is called a $\tau$-Lipschitz continuous mapping if $\|Tx - Ty\| \leq \tau\|x - y\|$, for $x, y \in K$.*

**Theorem 3.** *Let $H$ be a real Hilbert space with $K$ being its non-empty closed and convex subset. Suppose $A_i : K \to H$ is $\mu_i$-inversely strongly accretive and $B_i : K \to H$ is m-accretive, for each $i \in \mathbb{N}$. Suppose $T : K \to H$ is an accretive and $\tau$-Lipschitz continuous mapping. Let $\{u_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} u_0 \in K, u_1 \in K \text{ chosen arbitrarily}, \ e_1 \in H, \\ y_0 = P_K(u_0 - \lambda_0 T u_0), \\ y_n = P_K(u_n - \lambda_n T u_n), \\ v_n = y_n + k_n(y_n - y_{n-1}), \\ w_n = \alpha_n v_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_K(u_n - \lambda_n T y_n) + \gamma_n e_n, \\ C_1 = K = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \le \alpha_n \|y_n - p\|^2 + \beta_n \|u_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\ \qquad\qquad + k_n^2 \|y_n - y_{n-1}\|^2 - 2\alpha_n k_n \langle y_n - p, y_{n-1} - y_n \rangle\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \le \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{23}$$

*Under the assumptions of (i)–(iv) in Theorem 1 and the additional assumptions (v) $\lambda_n \to 0$, (vi) $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \bigcap VI(K,T) \neq \emptyset$, we have: $u_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1) = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0 \bigcap VI(K,T)}(u_1)$, as $n \to \infty$.*

**Proof.** We split the proof into nine steps.

Step 1. $\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i) : K \to K$ is non-expansive, for $n \in \mathbb{N}$.

Copy the proof of Step 1 in Theorem 1.

Step 2. $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \bigcap VI(K,T) \subset C_n$, for $n \in \mathbb{N}$.

We first show that $\forall q \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \bigcap VI(K,T)$,

$$\|P_K(u_n - \lambda_n T y_n) - q\| \le \|u_n - q\|. \tag{24}$$

In fact, in view of Lemma 7, we have:

$$\begin{aligned} &\|P_K(u_n - \lambda_n T y_n) - q\|^2 \\ &\le \|u_n - q - \lambda_n T y_n\|^2 - \|u_n - P_K(u_n - \lambda_n T y_n) - \lambda_n T y_n\|^2 \\ &= \|u_n - q\|^2 - \|u_n - P_K(u_n - \lambda_n T y_n)\|^2 \\ &+ 2\lambda_n[\langle T y_n - Tq, q - y_n\rangle + \langle Tq, q - y_n\rangle + \langle T y_n, y_n - P_K(u_n - \lambda_n T y_n)\rangle] \\ &\le \|u_n - q\|^2 - \|u_n - P_K(u_n - \lambda_n T y_n)\|^2 + 2\lambda_n \langle T y_n, y_n - P_K(u_n - \lambda_n T y_n)\rangle \\ &= \|u_n - q\|^2 - (\|u_n - y_n\|^2 + \|y_n - P_K(u_n - \lambda_n T y_n)\|^2) + 2\lambda_n \langle T u_n - T y_n, P_K(u_n - \lambda_n T y_n) - y_n\rangle \\ &+ 2\langle u_n - y_n - \lambda_n T u_n, P_K(u_n - \lambda_n T y_n) - y_n\rangle \\ &\le \|u_n - q\|^2 - (\|u_n - y_n\|^2 + \|y_n - P_K(u_n - \lambda_n T y_n)\|^2) + 2\lambda_n \tau \|u_n - y_n\| \|y_n - P_K(u_n - \lambda_n T y_n)\| \\ &\le \|u_n - q\|^2 + (\lambda_n^2 \tau^2 - 1)\|u_n - y_n\|^2 \\ &\le \|u_n - q\|^2, \end{aligned}$$

which implies that (24) is true.

Next, we can easily check the following by noticing the result of Step 1 and (24):

$$\begin{aligned} &\|w_n - q\|^2 \le \alpha_n \|v_n - q\|^2 + \beta_n \|u_n - q\|^2 + \gamma_n \|e_n - q\|^2 \\ &\le \alpha_n \|y_n - q\|^2 + \beta_n \|u_n - q\|^2 + \gamma_n \|e_n - q\|^2 + k_n^2 \|y_n - y_{n-1}\|^2 - 2\alpha_n k_n \langle y_n - q, y_{n-1} - y_n\rangle. \end{aligned}$$

Thus, by induction as that in Theorem 1, $q \in C_n$, for $n \in \mathbb{N}$.

Step 3. $C_n$ is a closed and convex subset of $H$, for each $n \in \mathbb{N}$.

It is not difficult to see that

$$\begin{aligned}
&\|w_n - p\|^2 \le \alpha_n\|y_n - p\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n\|e_n - p\|^2 \\
&+k_n^2\|y_n - y_{n-1}\|^2 - 2\alpha_n k_n\langle y_n - p, y_{n-1} - y_n\rangle \\
&\Longleftrightarrow \|w_n\|^2 - \alpha_n\|y_n\|^2 - \beta_n\|u_n\|^2 - \gamma_n\|e_n\|^2 - k_n^2\|y_n - y_{n-1}\|^2 + 2\alpha_n k_n\langle y_n, y_{n-1} - y_n\rangle \\
&\le 2\langle p, w_n\rangle - 2\alpha_n\langle p, y_n\rangle - 2\beta_n\langle p, u_n\rangle - 2\gamma_n\langle p, e_n\rangle + 2\alpha_n k_n\langle p, y_{n-1} - y_n\rangle.
\end{aligned}$$

Then $C_n$ is a closed and convex subset of $H$, for each $n \in \mathbb{N}$.

Copy the results of Steps 4–6 in Theorem 1, we have:

Step 4. $Q_n$ is non-empty for each $n \in \mathbb{N}$, which ensures that $\{u_n\}$ is well-defined.

Step 5. $P_{C_{n+1}}(u_1) \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Step 6. $u_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Step 7. $y_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, $v_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$ and $w_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

It is easy to see that $Q_n$ is a closed subset of $K$. Then $u_{n+1} \in Q_{n+1} \subset K$ and $u_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$ imply that $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in K$. Therefore,

$$\begin{aligned}
&\|y_n - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\| \le \|u_n - \lambda_n T u_n - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\| \\
&\le \|u_n - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\| + \lambda_n\|Tu_n\| \to 0,
\end{aligned}$$

as $n \to \infty$. Thus, $y_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Since $y_{n+1} - y_n \to 0$ and $v_n = y_n + k_n(y_n - y_{n-1})$, then $v_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Since $u_{n+1} \in Q_{n+1} \subset C_{n+1}$, then from (23), $\|w_n - u_{n+1}\|^2 \le \alpha_n\|y_n - u_{n+1}\|^2 + \beta_n\|u_{n+1} - u_n\|^2 + \gamma_n\|e_n - u_{n+1}\|^2 + k_n^2\|y_n - y_{n-1}\|^2 - 2\alpha_n k_n\langle y_n - u_{n+1}, y_{n-1} - y_n\rangle \to 0$, as $n \to \infty$. Thus, $w_n \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Step 8. $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \bigcap VI(K, T)$.

We shall first show that $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in VI(K, T)$.

For this , define

$$Bv = \begin{cases} Tv + N_K v, & if \;\; v \in K, \\ \emptyset, & if \;\; v \overline{\in} K, \end{cases}$$

where $N_K v = \{w \in H : \langle v - u, w\rangle \ge 0, \;\; \forall u \in K\}$ is the normal cone to $K$ at $v \in K$. It is well-known that $B : H \to H$ is m-accretive and $0 \in Bv$ if and only if $v \in VI(K, T)$ [28].

Let $z \in Bv = Tv + N_K v$, then $z - Tv \in N_K v$. From the definition of the normal cone, we have

$$\langle v - y_n, z - Tv\rangle \ge 0. \tag{25}$$

From Lemma 7, we have:

$$\langle u_n - \lambda_n T u_n - P_K(u_n - \lambda_n T u_n), P_K(u_n - \lambda_n T u_n) - v\rangle \ge 0, \;\; \forall v \in K,$$

which implies that

$$\langle v - P_K(u_n - \lambda_n T u_n), \frac{P_K(u_n - \lambda_n T u_n) - u_n}{\lambda_n} + Tu_n\rangle \ge 0, \;\; \forall v \in K. \tag{26}$$

In view of (25) and (26), we know that

$$\begin{aligned}
&\langle v - y_n, z\rangle \geq \langle v - y_n, Tv\rangle \\
&= \langle v - P_K(u_n - \lambda_n Tu_n), Tv\rangle \\
&\geq \langle v - P_K(u_n - \lambda_n Tu_n), Tv\rangle - \langle v - P_K(u_n - \lambda_n Tu_n), \tfrac{P_K(u_n-\lambda_n Tu_n)-u_n}{\lambda_n} + Tu_n\rangle \\
&= \langle v - P_K(u_n - \lambda_n Tu_n), Tv - TP_K(u_n - \lambda_n Tu_n)\rangle \\
&+\langle v - P_K(u_n - \lambda_n Tu_n), TP_K(u_n - \lambda_n Tu_n) - Tu_n\rangle \\
&-\langle v - P_K(u_n - \lambda_n Tu_n), \tfrac{P_K(u_n-\lambda_n Tu_n)-u_n}{\lambda_n}\rangle \\
&\geq \langle v - P_K(u_n - \lambda_n Tu_n), TP_K(u_n - \lambda_n Tu_n) - Tu_n\rangle \\
&-\langle v - P_K(u_n - \lambda_n Tu_n), \tfrac{P_K(u_n-\lambda_n Tu_n)-u_n}{\lambda_n}\rangle \\
&= \langle v - y_n, Ty_n - Tu_n\rangle - \langle v - y_n, \tfrac{y_n-u_n}{\lambda_n}\rangle.
\end{aligned}$$

Taking limit on both sides of the above inequality, we have: $\langle v - P_{\bigcap_{m=1}^{\infty} C_m}(u_1), z\rangle \geq 0$, which implies from the fact $B$ is m-accretive that $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in B^{-1}0$, and then $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in VI(K,T)$.

Next, we shall show that $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$.

In fact, if, otherwise, $P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\overline{\in} \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$. Then $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \neq \sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$.

Since $w_n = \alpha_n v_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_K(u_n - \lambda_n Ty_n) + \gamma_n e_n$, then $\beta_n[\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_K(u_n - \lambda_n Ty_n) - w_n] = \alpha_n(w_n - v_n) + \gamma_n(w_n - e_n) \to 0$, as $n \to \infty$.

Since $inf_n\beta_n > 0$, then there exists a subsequence of $\{n\}$, which is still denoted by $\{n\}$ such that $\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_K(u_n - \lambda_n Ty_n) - w_n \to 0$, as $n \to \infty$.

Thus, $\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_K(u_n - \lambda_n Ty_n) \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$. From Step 1 and $u_n - y_n \to 0$, we can also know that $\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_K(u_n - \lambda_n Tu_n) \to P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$, as $n \to \infty$.

Since $H$ satisfies Opial's condition, $\lambda_n \to 0$ and $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in K$, then

$$\begin{aligned}
&liminf_{n\to\infty}\|u_n - \lambda_n Tu_n - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\| \\
&< liminf_{n\to\infty}\|u_n - \lambda_n Tu_n - \textstyle\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\| \\
&\leq liminf_{n\to\infty}\|u_n - \lambda_n Tu_n - \textstyle\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_K(u_n - \lambda_n Tu_n)\| \\
&+liminf_{n\to\infty}\| \textstyle\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_K(u_n - \lambda_n Tu_n) \\
&- \textstyle\sum_{i=1}^{\infty} \omega_{n,i}(I + s_{n,i}B_i)^{-1}(I - s_{n,i}A_i)P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\| \\
&\leq liminf_{n\to\infty}\|u_n - \lambda_n Tu_n - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\|,
\end{aligned}$$

which makes a contradiction! Thus, $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$.

Step 9. $P_{\bigcap_{m=1}^{\infty} C_m}(u_1) = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0\bigcap VI(K,T)}(u_1)$.

From Step 8, $\|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0\bigcap VI(K,T)}(u_1) - u_1\| \leq \|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\|$.

On the other hand, since $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \bigcap VI(K,T) \subset \bigcap_{m=1}^{\infty} C_m$, then

$$\|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\| \leq \|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0\bigcap VI(K,T)}(u_1) - u_1\|.$$

Thus

$$\|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0\bigcap VI(K,T)}(u_1) - u_1\| = \|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\|.$$

Using Lemma 7, we have

$$\begin{aligned}
&\|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0\bigcap VI(K,T)}(u_1) - P_{\bigcap_{m=1}^{\infty} C_m}(u_1)\|^2 + \|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\|^2 \\
&\leq \|P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0\bigcap VI(K,T)}(u_1) - u_1\|^2 = \|P_{\bigcap_{m=1}^{\infty} C_m}(u_1) - u_1\|^2,
\end{aligned}$$

which implies that $P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0\bigcap VI(K,T)}(u_1) = P_{\bigcap_{m=1}^{\infty} C_m}(u_1)$.

This completes the proof. □

2.3.2. The Second Kind Iteration Theorems

The following result is a special result of Lemma 10 in [19]:

**Theorem 4.** *Let $H$ be a real Hilbert space and $K$ be a non-empty closed and convex subset of $H$. Suppose $f : H \to H$ is a contraction with $k \in (0,1)$, $F : H \to H$ is a strongly positive linear bounded operator with coefficient $\xi$ and $U : H \to H$ is non-expansive mapping. If $0 < \lambda < \frac{\xi}{2k}$, then there exists $x_t$ which satisfies $x_t = t\lambda f(x_t) + (I - tF)Ux_t$, for $0 < t \leq \|F\|^{-1}$. Moreover, $x_t \to p_0$ as $t \to 0$, and $p_0$ satisfies the following variational inequality: for $\forall z \in Fix(U)$,*

$$\langle (F - \lambda f)p_0, p_0 - z\rangle \leq 0. \tag{27}$$

*In Lemma 10 of [19], we can also know that the solution of the variational inequality (27) is unique.*

**Theorem 5.** *Under the assumptions of Theorem 2, $\{u_n\}$ generated by (16) converges strongly to $P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1)$. Set $\widetilde{x} = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1)$. If $\widetilde{x} = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}[\lambda f(\widetilde{x}) - F(\widetilde{x}) + \widetilde{x}]$, then $\widetilde{x}$ satisfies the following variational inequality: $\forall z \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$,*

$$\langle (F - \lambda f)\widetilde{x}, \widetilde{x} - z\rangle \leq 0. \tag{28}$$

**Proof.** It follows from Lemma 7 that $\langle (F - \lambda f)\widetilde{x}, \widetilde{x} - z\rangle \leq 0$, $\forall z \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$. Since Theorem 4 tells us that (28) has a unique solution, then we know that $\{u_n\}$ generated by (16) converges strongly to the unique solution of variational inequality (28). □

**Remark 5.** *The assumption that $\widetilde{x} = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}[\lambda f(\widetilde{x}) - F(\widetilde{x}) + \widetilde{x}]$ is reasonable. For example, we may take $f(x) = \frac{x}{2}$ and $F(x) = \frac{\lambda x}{2}$, for $x \in H$.*

**Remark 6.** *For projection iterative algorithms such as (16), rare work can be found to show that the limit of the iterative sequences is also the solution of a kind of variational inequalities.*

## 3. Applications

### *3.1. Preparation for Discussion of Capillarity Systems*

To present some examples in this section, we need some basic definitions in Banach spaces.

Let $E$ be a real Banach space with $E^*$ being its dual space and let $\langle \cdot, \cdot \rangle$ denote the generalized duality pairing between $E$ and $E^*$.

**Definition 4.** *(see [29]) Recall that $J : E \to 2^{E^*}$ is called the normalized duality mapping if $\forall x \in E$,*

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

**Definition 5.** *(see [29]) Recall that $\mathcal{A} : E \to E^*$ is said to be a monotone mapping if $\forall x_i \in D(\mathcal{A}), i = 1, 2$, one has*

$$\langle x_1 - x_2, \mathcal{A}x_1 - \mathcal{A}x_2\rangle \geq 0.$$

*A monotone mapping $\mathcal{A} : E \to E^*$ is said to be maximal monotone if $R(J + r\mathcal{A}) = E^*$, $\forall r > 0$.*

**Definition 6.** *(see [29]) Recall that a mapping $\mathcal{B} : E \to E^*$ is said to be coercive if $\{x_n\} \subset D(\mathcal{B})$ with $\lim_{n\to\infty} \|x_n\| = +\infty$, then $\lim_{n\to\infty} \frac{\langle x_n, \mathcal{B}x_n\rangle}{\|x_n\|} = +\infty$.*

**Definition 7.** *(see [29]) Recall that a mapping $\mathcal{B} : D(\mathcal{B}) = E \to E^*$ is said to be a hemi-continuous mapping if $\mathcal{B}(x + ty) \rightharpoonup \mathcal{B}x$, as $t \to 0$, for $\forall x, y \in E$.*

**Definition 8.** *(see [29]) $\psi : E \to (-\infty, +\infty]$ is said to be a proper convex functional if there exists $u_0 \in E$ such that $\psi(u_0) < +\infty$ and $\psi((1-\lambda)u + \lambda v) \leq (1-\lambda)\psi(u) + \lambda\psi(v)$, for $\forall u, v \in E$ and $\lambda \in [0,1]$. $\psi : E \to (-\infty, +\infty]$ is said to be lower-semi-continuous: $\liminf_{y \to x} \psi(y) \geq \psi(x)$, for $\forall x \in E$. The subdifferential of $\psi$, $\partial\psi : E \to E^*$, is defined by:*

$$\partial\psi(u) = \{w \in E^* : \psi(u) - \psi(v) \leq \langle u - v, w \rangle, \ \forall v \in E\}, \quad \forall u \in E.$$

### *3.2. Applications to Capillarity Elliptic Systems*

**Example 1.** *Suppose $\Omega$ is bounded conical domain in $R^n$ $(n \in \mathbb{N})$ with $\Gamma \in C^1$, $\vartheta$ is the exterior normal derivative of $\Gamma$, $\varepsilon_i$ is a non-negative constant, $\lambda_i$ is a positive number, $f_i(x) \in L^{p_i}(\Omega)$ is a given function, for $i = 1, 2, \cdots, M$. In addition, $\beta_x : R \to R$ is the subdifferential of $\varphi_x$, where $\varphi_x = \varphi(x, \cdot) : R \to R$, for each $x \in \Gamma$. Suppose $\frac{2n}{n+1} < p_i < +\infty$, $i = 1, 2, \cdots, M$. If $p_i \geq n$ then $1 \leq q_i, r_i < +\infty$ and if $p_i < n$ then $1 \leq q_i, r_i \leq \frac{np_i}{n-p_i}$ for $i = 1, 2, \cdots, M$.*

The following capillarity system is studied in [30]:

$$\begin{cases} -div[(1 + \frac{|\nabla u^{(i)}|^{p_i}}{\sqrt{1+|\nabla u^{(i)}|^{2p_i}}})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)}] + \lambda_i(|u^{(i)}|^{q_i-2}u^{(i)} + |u^{(i)}|^{r_i-2}u^{(i)}) \\ +\varepsilon_i g_i(x, \nabla u^{(i)}, u^{(i)}) = f_i(x), \quad x \in \Omega \\ - < \vartheta, (1 + \frac{|\nabla u^{(i)}|^{p_i}}{\sqrt{1+|\nabla u^{(i)}|^{2p_i}}})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)} > \in \beta_x(u^{(i)}(x)), \quad x \in \Gamma, \quad i = 1, 2, \cdots, M, \end{cases} \tag{29}$$

where $|\cdot|$ and $< \cdot, \cdot >$ denote the norm and inner product in $R^n$, respectively.

The study on (29) in [30] is based on the following assumptions.

(1) $\forall x \in \Gamma$, $\varphi_x = \varphi(x, \cdot) : R \to R$ is a proper convex and lower-semi-continuous mapping with $\varphi_x(0) = 0$.

(2) $0 \in \beta_x(0)$, $\forall t \in R$, $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

(3) For each $i \in \{1, 2, \cdots, M\}$, $g_i : \Omega \times R^n \times R \to R$ satisfies Caratheodory's conditions and satisfies that
$$|g_i(x, r_1, r_2, \cdots, r_{n+1}) - g_i(x, s_1, s_2, \cdots, s_{n+1})| \leq b_i|r_{n+1} - s_{n+1}|,$$
for $\forall(r_1, r_2, \cdots, r_{n+1}), (s_1, s_2, \cdots, s_{n+1}) \in R^{n+1}$.

By using splitting method, the sufficient condition that (29) has non-trivial solution is obtained:

**Theorem 6.** *(see [30]) If $u^{(i)} \in L^{p_i}(\Omega)$ satisfies that*

$$\int_\Omega [g_i(x, \nabla u^{(i)}, u^{(i)}) - f_i]|u^{(i)}|^{p_i-2}u^{(i)}dx \geq 0,$$

*for $i = 1, 2, \cdots, M$, then $u = (u^{(1)}, u^{(2)}, \cdots, u^{(M)})$ is the non-trivial solution of capillarity system (29).*

Based on Example 1, we present the following example:

**Example 2.** *Suppose $\Omega$, $\Gamma$ and $\vartheta$ are the same as those in Example 1. Suppose $\lambda_i > 0$, $\frac{2n}{n+1} < p_i < +\infty$. If $p_i \geq n$, then suppose $1 \leq q_i, r_i < +\infty$ and if $p_i < n$, then suppose $1 \leq q_i, r_i \leq \frac{np_i}{n-p_i}$, for $i \in \mathbb{N}$.*

Now, we will discuss the following capillarity systems.

$$\begin{cases} -div[(1+\frac{|\nabla u^{(i)}|^{p_i}}{\sqrt{1+|\nabla u^{(i)}|^{2p_i}}})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)}] \\ +\lambda_i(|u^{(i)}|^{q_i-2}u^{(i)}+|u^{(i)}|^{r_i-2}u^{(i)})+u^{(i)}(x)=f_i(x), \quad x\in\Omega \\ -<\vartheta,(1+\frac{|\nabla u^{(i)}|^{p_i}}{\sqrt{1+|\nabla u^{(i)}|^{2p_i}}})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)}>=0, \quad x\in\Gamma,\ i\in\mathbb{N}. \end{cases} \tag{30}$$

Please note that (30) is the extension from the finite case of (29) to that for infinite case. However, both the capillarity equations and the boundary conditions are the special case of (29) in the sense that $\varepsilon_i \equiv 1$ and $g_i(x,\nabla u^{(i)},u^{(i)}) \equiv u^{(i)}$ for $i\in\mathbb{N}$ and $\beta_x \equiv 0$, for $x\in\Gamma$.

**Lemma 9.** *(see [30]) The mapping $U_i : W^{1,p_i}(\Omega) \to (W^{1,p_i}(\Omega))^*$ defined by*

$$\langle w, U_i u\rangle = \int_\Omega <(1+\frac{|\nabla u|^{p_i}}{\sqrt{1+|\nabla u|^{2p_i}}})|\nabla u|^{p_i-2}\nabla u,$$

$$\nabla v > dx + \lambda_i\int_\Omega |u(x)|^{q_i-2}u(x)v(x)dx + \lambda_i\int_\Omega |u(x)|^{r_i-2}u(x)v(x)dx,$$

*for $\forall u,w\in W^{1,p_i}(\Omega)$, is everywhere defined, hemi-continuous, monotone and coercive, for each $i\in\mathbb{N}$.*

**Lemma 10.** *(see [30]) Define $B_i : L^2(\Omega)\to L^2(\Omega)$ by*

$$D(B_i) = \{u\in L^2(\Omega) \mid \exists f\in L^2(\Omega) \text{ such that } f\in U_i u\}.$$

*For $u\in D(B_i)$, $B_i u = \{f\in L^2(\Omega) \mid f\in U_i u\}$. Then $B_i : L^2(\Omega)\to L^2(\Omega)$ is m-accretive, for each $i\in\mathbb{N}$.*

**Lemma 11.** *(see [30]) The mapping $A_i : D(A_i)\subset L^2(\Omega)\to L^2(\Omega)$ defined by*

$$(A_i u)(x) = u(x) - f_i(x), \quad \forall u(x)\in D(A_i),$$

*is $\mu_i$-inversely strongly accretive, for $\mu_i\in(0,1]$, for $i\in\mathbb{N}$.*

**Theorem 7.** *If $f_i(x)\equiv\lambda_i(|k|^{q_i-1}+|k|^{r_i-1})sgnk+k$, then $\{u^{(i)}\equiv k : i\in\mathbb{N}\}$ is the solution of capillarity system (30). Moreover, $\{k\} = \bigcap_{i=1}^\infty (A_i+B_i)^{-1}0$.*

**Proof.** It is easy to see that $\{u^{(i)}\equiv k : i\in\mathbb{N}\}$ is the solution of capillarity system (30) and $\{k\}\subset\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0$. Now, we shall show that $\{k\}\supset\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0$.

In fact, if $A_i u + B_i u = 0$ and $A_i v + B_i v = 0$, then $u + B_i u = v + B_i v$, which implies that

$$0\le\langle u-v, B_i u - B_i v\rangle = \langle u-v, v-u\rangle \le 0.$$

Thus, $u = v$ and then $\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0$ is a singleton. Since $k\in\bigcap_{i=1}^\infty(A_i+B_i)^{-1}0$, then the result follows. □

**Theorem 8.** *Let $f_i(x)\equiv\lambda_i(|k|^{q_i-1}+|k|^{r_i-1})sgnk+k$, for $i\in\mathbb{N}$. Suppose $A_i$ and $B_i$ are the same as those in Lemmas 10 and 11, respectively. Let $F : L^2(\Omega)\to L^2(\Omega)$ be any strongly positive linear bounded operator with coefficient $\xi > 0$ and $f : L^2(\Omega)\to L^2(\Omega)$ be a contraction with coefficient $k\in(0,1)$. Constructing the following iterative algorithm:*

$$
\begin{cases}
u_0, u_1 \in L^2(\Omega) \text{ chosen arbitrarily}, \quad e_1 \in L^2(\Omega), \\
z_0 = u_0, \\
z_n = \delta_n \lambda f(u_n) + (I - \delta_n F) u_n, \\
v_n = z_n + k_n(z_n - z_{n-1}), \\
w_n = \alpha_n v_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i} (I + s_{n,i} B_i)^{-1} (I - s_{n,i} A_i)(\frac{v_n + w_n}{2}) + \gamma_n e_n, \\
C_1 = L^2(\Omega) = Q_1, \\
C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq \frac{2\alpha_n + \beta_n}{2 - \beta_n} \|z_n - p\|^2 + \frac{2\gamma_n}{2 - \beta_n} \|e_n - p\|^2 \\
\qquad + \frac{2\alpha_n + \beta_n}{2 - \beta_n} k_n^2 \|z_n - z_{n-1}\|^2 - 2k_n \frac{2\alpha_n + \beta_n}{2 - \beta_n} \langle z_n - p, z_{n-1} - z_n \rangle \}, \\
Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|P_{C_{n+1}}(u_1) - u_1\|^2 + \sigma_{n+1}\}, \\
u_{n+1} \in Q_{n+1}, \quad n \in \mathbb{N}.
\end{cases}
$$

*Under the assumptions of Theorem 2, using the result of Theorem 5, one has* $u_n(x) \to P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1)$, *which is the unique solution of capillarity system (30) and satisfies the following variational inequality: For* $\forall z(x) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$,

$$\langle (F - \lambda f) P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1), P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(u_1) - z) \rangle \leq 0.$$

**Remark 7.** *From Theorem 8 we can easily see the relationship among the solution of capillarity system, the solution of variational inequality and the zero of sum of infinitely many m-accretive mappings and infinitely many $\mu_i$-inversely strongly accretive mappings.*

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