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Hermite-Hadamard-Fejér Type Inequalities for Preinvex Functions Using Fractional Integrals

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Abstract: In this paper, we have established the Hermite–Hadamard–Fejér inequality for fractional integrals involving preinvex functions. The results presented here provide new extensions of those given in earlier works as the weighted estimates of the left and right hand side of the Hermite–Hadamard inequalities for fractional integrals involving preinvex functions doesn't exist previously.

Keywords: Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality; Riemann–Liouville fractional integral; preinvex function

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1. Introduction

The Hermite–Hadamard inequality for convex functions has been widely addressed due to it's importance in developing a relationship between the theory of convex functions and integral inequalities. Many generalizations of convex functions have been developed in the recent past and estimates for the Hermite–Hadamard inequality have been obtained for these generalized definitions.

Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $x_1, x_2 \in I$ with $x_1 < x_2$, then

$$g(\frac{x_1+x_2}{2}) \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2} g(x)dx \leq \frac{g(x_1)+g(x_2)}{2}. \tag{1}$$

The inequality is known as Hermite–Hadamard inequality for convex functions.

In [1], Fejér gave a weighted generalization of the inequality (1) as follows:

$$g(\frac{x_1+x_2}{2})\int_{x_1}^{x_2} w(x)dx \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2} w(x)g(x)dx \leq \frac{g(x_1)+g(x_2)}{2}\int_{x_1}^{x_2} w(x)dx, \tag{2}$$

where $w : [x_1; x_2] \to \mathbb{R}$ is non-negative, integrable and symmetric about $x = \frac{x_1+x_2}{2}$.

Recently, fractional calculus has proved to be a powerful tool in different fields of sciences. Because of the wide application of fractional calculus and Hermite–Hadamard inequalities, researchers have extended their work on Hermite–Hadamard inequalities in the fractional domain. Hermite–Hadamard inequalities involving fractional integrals for different classes of functions have been established.

In [2], Sarikaya et al. presented Hermite–Hadamard's inequalities for fractional integral as follows.

Theorem 1. *Let $g : [x_1, x_2] \subseteq \mathbb{R} \to \mathbb{R}$ with $0 \leq x_1 < x_2$ and $g \in L[x_1, x_2]$. If g is positive and a convex function on $[x_1, x_2]$, then the following inequalities for fractional integrals hold*

$$g\left(\frac{x_1+x_2}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(x_2-x_1)^\alpha}\left[J^\alpha_{x_2^-}g(x_1)+J^\alpha_{x_1^+}g(x_2)\right] \le \frac{g(x_1)+g(x_2)}{2},$$

with $\alpha > 0$. Here, the symbols $J^\alpha_{x_1^+}$ and $J^\alpha_{x_2^-}$ denote the left-sided and right-sided Riemann–Liouville fractional intagrals of the order $\alpha \in \mathbb{R}^+$ that are defined in [3]

$$J^\alpha_{x_1^+}g(x) = \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x}(x-t)^{\alpha-1}g(t)dt,\ 0 \le x_1 < x \le x_2$$

and

$$J^\alpha_{x_2^-}g(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{x_2}(t-x)^{\alpha-1}g(t)dt,\ 0 \le x_1 \le x < x_2.$$

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

The concept of invex sets was given by T. Antczak [4].

Definition 1. *A set $H \subseteq \mathbb{R}^n$ is invex with respect to the map $\eta : H \times H \to \mathbb{R}^n$ if for every $x_1, x_2 \in I$ and $t \in [0,1]$, $x_2 + t\eta(x_1, x_2) \in H$. The invex set H is also called an η-connected set. Every convex set is an invex set but its converse is not true.*

In 1998, Weir and Mond [5], defined preinvex functions as a generalization of convex functions as given below:

Definition 2. *Let $H \subseteq \mathbb{R}^n$ be an invex set and a function $g : H \to \mathbb{R}$ is said to be preinvex w.r.t η if*

$$g(x_2 + t\eta(x_1, x_2)) \le tg(x_1) + (1-t)g(x_2).$$

$\forall\ x_1, x_2 \in H$ and $t \in [0,1]$.

If $\eta(x_1, x_2) = x_1 - x_2$, then in classical sense, the preinvex functions become convex functions. A function g is called preconcave if its negative is preinvex.

In 2012, Imdat Iscan ([6]) presented following inqualities for preinvex function in fractional domain.

Theorem 2. *Let $H \subseteq \mathbb{R}$, be an open invex subset with respect to $\eta : H \times H \to \mathbb{R}$ and $x_1, x_2 \in H$ with $x_1 < x_1 + \eta(x_2, x_1)$. Suppose $g : [x_1, x_1 + \eta(x_2, x_1)] \to (0, \infty)$ is a preinvex function, $g \in L[x_1, x_1 + \eta(x_2, x_1)]$ then for every $x_1, x_2 \in H$ with $\eta(x_2, x_1) \neq 0$ the following equality holds:*

$$g\left(\frac{2x_1+\eta(x_2,x_1)}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(\eta(x_2,x_1))^\alpha}\left[J^\alpha_{a+}g(x_1+\eta(x_2,x_1)) + J^\alpha_{(x_1+\eta(x_2,x_1))-}g(x_1)\right]$$
$$\le \frac{g(x_1)+g(x_1+\eta(x_2,x_1))}{2} \le \frac{g(x_1)+g(x_2)}{2}$$

where $\alpha > 0$.

For more estimates of Hermite–Hadamard–Fejér type inequalities for generalized convex functions see [7,8].

In this paper, we present two new Hermite–Hadamard–Fejér identities for preinvex functions in fractional domains. Using the new identities, we obtain some new weighted estimates connected with the left and right hand side of the Hermite–Hadamard type inequalities for the fractional integrals involving preinvex functions.

2. Main Results

Throughout this section, we will let $\|h\|_{\infty} = \sup_{t\in[x_1,x_2]} |h(x)|$, where $h : [x_1; x_2] \to \mathbb{R}$ is a continuous function, $g^{'}$ is the derivative of g w.r.t variable t and $L[x_1, x_2]$ is the collection of all real-valued Riemann integrable functions defined on $[x_1, x_2]$.

Lemma 1. *Let H be an open invex set where $H \subseteq \mathbb{R}$ and η is mapping such that $\eta : H \times H \to \mathbb{R}$. Suppose there is a differentiable mapping $g : H \to \mathbb{R}$ such that $g^{'} \in L[x_1, x_1 + \eta(x_2, x_1)]$ and $x_1, x_2 \in H$ with $x_1 < x_1 + \eta(x_2, x_1)$. If $h : [x_1, x_1 + \eta(x_2, x_1)] \to [0, \infty)$ is an integrable mapping, then $\forall\ x_1, x_2 \in H$ with $\eta(x_2, x_1) \neq 0$ the following equality holds:*

$$\begin{aligned}
&\frac{g(x_1 + \frac{1}{2}\eta(x_2, x_1))\Gamma(\alpha)}{(\eta(x_2, x_1))^{\alpha+1}}\left[J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-}h(x_1) + J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+}h(x_1 + \eta(x_2, x_1))\right] \\
&-\frac{\Gamma(\alpha)}{(\eta(x_2, x_1))^{\alpha+1}}\left[J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-}(gh)(x_1) + J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+}(gh)(x_1 + \eta(x_2, x_1))\right] \\
= &\int_0^1 w(t)g^{'}(x_1 + t\eta(x_2, x_1))dt, \qquad (3)
\end{aligned}$$

where

$$w(t) = \begin{cases} \int_0^t u^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du, & t \in [0, \frac{1}{2}) \\ \int_1^t (1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. Consider:

$$\begin{aligned}
&\int_0^1 w(t)g^{'}(x_1 + t\eta(x_2, x_1))dt \\
= &\int_0^{\frac{1}{2}}\left(\int_0^t u^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right)g^{'}(x_1 + t\eta(x_2, x_1))dt \\
&+\int_{\frac{1}{2}}^1\left(\int_1^t (1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right)g^{'}(x_1 + t\eta(x_2, x_1))dt \\
= &I_1 + I_2.
\end{aligned}$$

From the first integral,

$$\begin{aligned}
I_1 = &\int_0^{\frac{1}{2}}\left(\int_0^t u^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right)g^{'}(x_1 + t\eta(x_2, x_1))dt \\
= &\frac{1}{\eta(x_2, x_1)}\left(\left(\int_0^t u^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right)g(x_1 + t\eta(x_2, x_1)\right)\Bigg|_0^{\frac{1}{2}} \\
&-\frac{1}{\eta(x_2, x_1)}\int_0^{\frac{1}{2}} t^{\alpha-1}h(x_1 + t\eta(x_2, x_1))g(x_1 + t\eta(x_2, x_1))dt \\
= &\frac{g(x_1 + \frac{1}{2}\eta(x_2, x_1))}{\eta(x_2, x_1)}\int_0^{\frac{1}{2}} t^{\alpha-1}h(x_1 + t\eta(x_2, x_1))dt \\
&-\frac{1}{\eta(x_2, x_1)}\int_0^{\frac{1}{2}} t^{\alpha-1}g(x_1 + t\eta(x_2, x_1))h(x_1 + t\eta(x_2, x_1))dt. \qquad (4)
\end{aligned}$$

Substituting $x = x_1 + t\eta(x_2, x_1)$ in (4),

$$\begin{aligned} I_1 &= \frac{g(x_1 + \frac{1}{2}\eta(x_2, x_1))}{(\eta(x_2, x_1))^{\alpha+1}} \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} (x - x_1)^{\alpha-1} h(x)dx \\ &\quad - \frac{1}{(\eta(x_2, x_1))^{\alpha+1}} \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} (x - x_1)^{\alpha-1} g(x)h(x)dx \\ &= \frac{g(x_1 + \frac{1}{2}\eta(x_2, x_1))\Gamma(\alpha)}{(\eta(x_2, x_1))^{\alpha+1}} J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-} h(x_1) \\ &\quad - \frac{\Gamma(\alpha)}{(\eta(x_2, x_1))^{\alpha+1}} J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-} (gh)(x_1). \end{aligned} \tag{5}$$

From the second integral,

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^{1} \left(\int_{1}^{t} (1-u)^{\alpha-1} h(x_1 + u\eta(x_2, x_1))du \right) g'(x_1 + t\eta(x_2, x_1))dt \\ &= \frac{1}{\eta(x_2, x_1)} \left(\left(\int_{1}^{t} (1-u)^{\alpha-1} h(x_1 + u\eta(x_2, x_1))du \right) g(x_1 + t\eta(x_2, x_1)) \right) \Big|_{\frac{1}{2}}^{1} \\ &\quad - \frac{1}{\eta(x_2, x_1)} \int_{\frac{1}{2}}^{1} (1-t)^{\alpha-1} g(x_1 + t\eta(x_2, x_1))h(x_1 + t\eta(x_2, x_1))dt \\ &= -\frac{g\left(x_1 + \frac{1}{2}\eta(x_2, x_1)\right)}{\eta(x_2, x_1)} \int_{1}^{\frac{1}{2}} (1-t)^{\alpha-1} h(x_1 + t\eta(x_2, x_1))dt \\ &\quad - \frac{1}{\eta(x_2, x_1)} \int_{\frac{1}{2}}^{1} ((1-t)^{\alpha-1} g(x_1 + t\eta(x_2, x_1))h(x_1 + t\eta(x_2, x_1))dt. \end{aligned} \tag{6}$$

Substituting $x = x_1 + t\eta(x_2, x_1)$ in (6),

$$\begin{aligned} &= -\frac{g\left(x_1 + \frac{1}{2}\eta(x_2, x_1)\right)}{(\eta(x_2, x_1))^{\alpha+1}} \int_{x_1+\eta(x_2,x_1)}^{x_1+\frac{1}{2}\eta(x_2,x_1)} (x_1 + \eta(x_2, x_1) - x)^{\alpha-1} h(x)dx \\ &\quad - \frac{1}{(\eta(x_2, x_1))^{\alpha+1}} \int_{x_1+\frac{1}{2}\eta(x_2,x_1)}^{x_1+\eta(x_2,x_1)} (x_1 + \eta(x_2, x_1) - x)^{\alpha-1} g(x)h(x)dt \\ &= \frac{g\left(x_1 + \frac{1}{2}\eta(x_2, x_1)\right)\Gamma(\alpha)}{(\eta(x_2, x_1))\,\alpha+1} J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+} h(x_1 + \eta(x_2, x_1)) \\ &\quad - \frac{\Gamma(\alpha)}{(\eta(x_2, x_1))^2} J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+} (gh)(x_1 + \eta(x_2, x_1)). \end{aligned} \tag{7}$$

Upon adding (5) and (7), we get the required result. □

Lemma 2. *If $h : [x_1, x_1 + \eta(x_2, x_1)] \to \mathbb{R}$ is an integrable function which is also symmetric about $x_1 + \frac{1}{2}\eta(x_2, x_1)$ with $x_1 < x_1 + \eta(x_2, x_1)$, then*

$$\begin{aligned} J^{\alpha}_{x_1+} h(x_1 + \eta(x_2, x_1)) &= J^{\alpha}_{(x_1+\eta(x_2,x_1))-} h(x_1) \\ &= \frac{1}{2}\left[J^{\alpha}_{x_1+} h(x_1 + \eta(x_2, x_1)) + J^{\alpha}_{(x_1+\eta(x_2,x_1))-} h(x_1) \right], \end{aligned} \tag{8}$$

where $\alpha > 0$.

Proof. Since h is symmetric about $x_1 + \frac{1}{2}\eta(x_2, x_1)$, we have $h(2x_1 + \eta(x_2, x_1) - x) = h(x)$, for all $x \in [x_1, x_1 + \eta(x_2, x_1)]$. Taking $2x_1 + \eta(x_2, x_1) - t = x$

$$\begin{aligned} J^{\alpha}_{x_1+}h(x_1 + \eta(x_2, x_1)) &= \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_1+\eta(x_2,x_1)} [(x_1 + \eta(x_2, x_1) - t]^{\alpha-1} h(t)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_1+\eta(x_2,x_1)} (x - x_1)^{\alpha-1} h(2x_1 + \eta(x_2, x_1) - x)dx \\ &= \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_1+\eta(x_2,x_1)} (x - x_1)^{\alpha-1} h(x)dx \\ &= J^{\alpha}_{(x_1+\eta(x_2,x_1))-}h(x_1). \end{aligned}$$

□

Lemma 3. *Let H be an open invex set where $H \subseteq \mathbb{R}$ and $\eta : H \times H \to \mathbb{R}$ is a mapping. Suppose there is a differentiable mapping $g : H \to \mathbb{R}$ on H such that $g' \in L[x_1, x_1 + \eta(x_2, x_1)]$ and $x_1, x_2 \in H$ with $x_1 < x_1 + \eta(x_2, x_1)$. If $h : [x_1, x_1 + \eta(x_2, x_1)] \to [0, \infty)$ is an integrable mapping, then $\forall\, x_1, x_2 \in H$ with $\eta(x_2, x_1) \neq 0$ the following equality holds:*

$$\begin{aligned} &\left[\frac{g(x_1) + g(x_1 + \eta(x_2, x_1))}{2}\right]\left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}h(x_1) + J^{\alpha}_{x_1+}h(x_1 + \eta(x_2, x_1))\right] \\ &- \left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}(gh)(x_1) + J^{\alpha}_{x_1+}(gh)(x_1 + \eta(x_2, x_1))\right] \\ = &\frac{(\eta(x_2, x_1))^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 w(t)g'(x_1 + t\eta(x_2, x_1))dt, \end{aligned} \tag{9}$$

where

$$w(t) = \int_1^t (1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du + \int_0^t (1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du, t \in [0, 1].$$

Proof. Let us consider

$$\begin{aligned} &\int_0^1 w(t)g'(x_1 + t\eta(x_2, x_1))dt \\ = &\int_0^1 \left[\int_1^t (1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du + \right. \\ &\left. \int_0^t (1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right] g'(x_1 + t\eta(x_2, x_1))dt \\ = &\int_0^1 \left(\int_1^t (1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right) g'(x_1 + t\eta(x_2, x_1))dt \\ &+ \int_0^1 \left(\int_0^t (1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right) g'(x_1 + t\eta(x_2, x_1))dt \\ = &I_1 + I_2. \end{aligned} \tag{10}$$

From the first integral,

$$
\begin{aligned}
I_1 &= \int_0^1 \left(\int_1^t (1-u)^{\alpha-1} h(x_1 + u\eta(x_2,x_1)) du \right) g'(x_1 + t\eta(x_2,x_1)) dt \\
&= \frac{1}{\eta(x_2,x_1)} \left(\left(\int_1^t (1-u)^{\alpha-1} h(x_1 + u\eta(x_2,x_1)) du \right) g(x_1 + t\eta(x_2,x_1)) \right) \Big|_0^1 \\
&\quad - \frac{1}{\eta(x_2,x_1)} \int_0^1 (1-t)^{\alpha-1} h(x_1 + t\eta(x_2,x_1)) g(x_1 + t\eta(x_2,x_1)) dt \\
&= \frac{g(x_1)}{\eta(x_2,x_1)} \int_0^1 (1-t)^{\alpha-1} h(x_1 + t\eta(x_2,x_1)) dt \\
&\quad - \frac{1}{\eta(x_2,x_1)} \int_0^1 (1-t)^{\alpha-1} g(x_1 + t\eta(x_2,x_1)) h(x_1 + t\eta(x_2,x_1)) dt. \qquad (11)
\end{aligned}
$$

Substituting $x = x_1 + t\eta(x_2,x_1)$ in (11),

$$
\begin{aligned}
I_1 &= \frac{g(x_1)}{(\eta(x_2,x_1))^{\alpha+1}} \int_{x_1}^{x_1+\eta(x_2,x_1)} ((x_1 + \eta(x_2,x_1) - x)^{\alpha-1} h(x) dx \\
&\quad - \frac{1}{(\eta(x_2,x_1))^{\alpha+1}} \int_{x_1}^{x_1+\eta(x_2,x_1)} (x_1 + \eta(x_2,x_1) - x)^{\alpha-1} g(x) h(x) dx \\
&= \frac{g(x_1)\Gamma(\alpha)}{(\eta(x_2,x_1))^{\alpha+1}} J_{x_1+}^{\alpha} h(x_1 + \eta(x_2,x_1)) - \frac{\Gamma(\alpha)}{(\eta(x_2,x_1))^{\alpha+1}} J_{x_1+}^{\alpha} (gh)(x_1 + \eta(x_2,x_1)). \qquad (12)
\end{aligned}
$$

Now for the second integral,

$$
\begin{aligned}
I_2 &= \int_0^1 \left(\int_0^t (1-u)^{\alpha-1} h(x_1 + u\eta(x_2,x_1)) du \right) g'(x_1 + t\eta(x_2,x_1)) dt \\
&= \frac{1}{\eta(x_2,x_1)} \left(\left(\int_0^t (1-u)^{\alpha-1} h(x_1 + u\eta(x_2,x_1)) du \right) g(x_1 + t\eta(x_2,x_1)) \right) \Big|_0^1 \\
&\quad - \frac{1}{\eta(x_2,x_1)} \int_0^1 (1-t)^{\alpha-1} g(x_1 + t\eta(x_2,x_1)) h(x_1 + t\eta(x_2,x_1)) dt \\
&= \frac{g(x_1 + \eta(x_2,x_1))}{\eta(x_2,x_1)} \int_0^1 (1-t)^{\alpha-1} h(x_1 + t\eta(x_2,x_1)) dt - \\
&\quad \frac{1}{\eta(x_2,x_1)} \int_0^1 (1-t)^{\alpha-1} g(x_1 + t\eta(x_2,x_1)) h(x_1 + t\eta(x_2,x_1)) dt. \qquad (13)
\end{aligned}
$$

Substituting $x = x_1 + t\eta(x_2,x_1)$ in (13),

$$
\begin{aligned}
I_2 &= \frac{g(x_1 + \eta(x_2,x_1))}{(\eta(x_2,x_1))^{\alpha+1}} \int_{x_1}^{x_1+\eta(x_2,x_1)} ((x_1 + \eta(x_2,x_1) - x)^{\alpha-1} h(x) dx \\
&\quad - \frac{1}{(\eta(x_2,x_1))^{\alpha+1}} \int_{x_1}^{x_1+\eta(x_2,x_1)} ((x_1 + \eta(x_2,x_1) - x)^{\alpha-1} g(x) h(x) dx \\
&= \frac{g(x_1 + \eta(x_2,x_1))\Gamma(\alpha)}{(\eta(x_2,x_1))^{\alpha+1}} J_{x_1+}^{\alpha} h(x_1 + \eta(x_2,x_1)) \\
&\quad - \frac{\Gamma(\alpha)}{(\eta(x_2,x_1))^{\alpha+1}} J_{x_1+}^{\alpha} (gh)(x_1 + \eta(x_2,x_1)). \qquad (14)
\end{aligned}
$$

By adding the results of (12) and (14) using (8), we get the required result. □

Theorem 3. *Let H be an open invex set where $H \subseteq \mathbb{R}$ and η is mapping such that $\eta : H \times H \to \mathbb{R}$. Suppose there is a differentiable mapping $g : H \to \mathbb{R}$ on H such that $g' \in L[x_1, x_1 + \eta(x_2, x_1)]$ and $x_1, x_2 \in H$ with $x_1 < x_1 + \eta(x_2, x_1)$. If $h : [x_1, x_1 + \eta(x_2, x_1)] \to [0, \infty)$ is an integrable mapping which is also symmetric with respect to $x_1 + \frac{1}{2}\eta(x_2, x_1)$. If $\left|g'\right|$ is preinvex function on H, then $\forall\, x_1, x_2 \in H$ with $\eta(x_2, x_1) \neq 0$ the following inequality holds:*

$$\begin{aligned} &\Big| g(x_1 + \frac{1}{2}\eta(x_2, x_1)) \left[J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-} h(x_1) + J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+} h(x_1 + \eta(x_2, x_1)) \right] \\ &- \left[J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-} (gh)(x_1) + J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+} (gh)(x_1 + \eta(x_2, x_1)) \right] \Big| \\ \leq\ & \frac{1}{\Gamma(\alpha+2)} \left(|g'(x_1)| + |g'(x_2)| \right) \|h\|_{\infty} \frac{(\eta(x_2, x_1))^{\alpha+1}}{2^{\alpha+1}}. \end{aligned} \tag{15}$$

Proof. Applying modulus on both sides of (3),

$$\begin{aligned} &\Big| \frac{g(x_1 + \frac{1}{2}\eta(x_2, x_1)).\Gamma(\alpha)}{(\eta(x_2, x_1))^{\alpha+1}} \left[J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-} h(x_1) + J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+} h(x_1 + \eta(x_2, x_1)) \right] \\ &- \frac{\Gamma(\alpha)}{(\eta(x_2, x_1))^{\alpha+1}} \left[J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-} (gh)(x_1) + J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+} (gh)(x_1 + \eta(x_2, x_1)) \right] \Big| \\ =\ & \Big| \int_0^{1/2} \left(\int_0^t u^{\alpha-1} h(x_1 + u\eta(x_2, x_1)) du \right) g'(x_1 + t\eta(x_2, x_1)) dt + \\ & \int_{1/2}^1 \left(-\int_t^1 ((1-u)^{\alpha-1} h(x_1 + u\eta(x_2, x_1)) du \right) g'(x_1 + t\eta(x_2, x_1)) dt \Big|. \end{aligned} \tag{16}$$

From preinvexity of $|g'|$ on H and Lemma 1 , we have

$$\begin{aligned} &\Big| \frac{g(x_1 + \frac{1}{2}\eta(x_2, x_1)).\Gamma(\alpha)}{(\eta(x_2, x_1))^{\alpha+1}} \left[J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-} h(x_1) + J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+} h(x_1 + \eta(x_2, x_1)) \right] \\ &- \frac{\Gamma(\alpha)}{(\eta(x_2, x_1))^{\alpha+1}} \left[J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))-} (gh)(x_1) + J^{\alpha}_{(x_1+\frac{1}{2}\eta(x_2,x_1))+} (gh)(x_1 + \eta(x_2, x_1)) \right] \Big| \\ \leq\ & \int_0^{1/2} \left(\int_0^t u^{\alpha-1} |h(x_1 + u\eta(x_2, x_1))| du \right) \left[(1-t)|g'(x_1)| + t|g'(x_2)| \right] dt + \\ & \int_{1/2}^1 \left(\int_t^1 ((1-u)^{\alpha-1} |h(x_1 + u\eta(x_2, x_1))| du \right) \left[(1-t)|g'(x_1)| + t|g'(x_2)| \right] dt \\ =\ & I_1 + I_2. \end{aligned} \tag{17}$$

By the change of the order of integration in fisrt term of (17), we have

$$\begin{aligned} I_1 &= \int_0^{1/2} \left(\int_0^t u^{\alpha-1} h(x_1 + u\eta(x_2, x_1)) du \right) \left[(1-t)|g'(x_1)| + t|g'(x_2)| \right] dt \\ &= \int_0^{1/2} u^{\alpha-1} |h(x_1 + u\eta(x_2, x_1))| \int_u^{1/2} \left[(1-t)|g'(x_1)| + t|g'(x_2)| \right] dt du \\ &= \int_0^{1/2} u^{\alpha-1} |h(x_1 + u\eta(x_2, x_1))| \left[|g'(x_1)| \left(\frac{(1-u)^2}{2} - \frac{1}{8} \right) + \right. \\ &\left. |g'(x_2)| \left(\frac{1}{8} - \frac{u^2}{2} \right) \right] du. \end{aligned}$$

Making the change of variable $x = x_1 + u\eta(x_2, x_1)$ for $u \in [0,1]$,

$$I_1 = \frac{|g'(x_1)|}{\eta(x_2,x_1)} \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} \left(\frac{1}{2}\left(1 - \frac{x-x_1}{\eta(x_2,x_1)}\right)^2 - \frac{1}{8}\right)\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^{\alpha-1} |h(x)|\,dx + $$
$$+ \frac{|g'(x_2)|}{\eta(x_2,x_1)} \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} \left(\frac{1}{8} - \frac{1}{2}\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^2\right)\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^{\alpha-1} |h(x)|\,dx.$$

Let $\|h\|_\infty = sup_{t\in[x_1,x_2]}|h(x)|$,

$$\begin{aligned} I_1 \leq\; & \frac{|g'(x_1)|}{\eta(x_2,x_1)} \|h\|_\infty \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} \left(\frac{1}{2}\left(1 - \frac{x-x_1}{\eta(x_2,x_1)}\right)^2 - \frac{1}{8}\right)\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^{\alpha-1} dx + \\ & + \frac{|g'(x_2)|}{\eta(x_2,x_1)} \|h\|_\infty \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} \left(\frac{1}{8} - \frac{1}{2}\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^2\right)\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^{\alpha-1} dx. \end{aligned} \tag{18}$$

Similarly, by changing the order of integration in the second term and using the fact that h is symmetric to $x_1 + \frac{1}{2}\eta(x_2, x_1)$, we obtain

$$I_2 = \int_{1/2}^{1}\left(\int_t^1 (1-u)^{\alpha-1} |h(x_1 + (1-u)\eta(x_2,x_1)|)du\right)\left[(1-t)|g'(x_1)| + t|g'(x_2)|\right] dt$$
$$= \int_{1/2}^{1} (1-u)^{\alpha-1} |h(x_1 + (1-u)\eta(x_2,x_1)|) \int_{1/2}^{u} \left[(1-t)|g'(x_1)| + t|g'(x_2)|\right] dt du$$
$$= \int_{1/2}^{1} (1-u)^{\alpha-1} |h(x_1 + (1-u)\eta(x_2,x_1)|) \left[|g'(x_1)|\left(\frac{1}{8} - \frac{(1-u)^2}{2}\right)\right.$$
$$\left. + |g'(x_2)|\left(\frac{u^2}{2} - \frac{1}{8}\right)\right] du.$$

By the change of variable $x = x_1 + (1-u)\eta(x_2, x_1)$,

$$\begin{aligned} I_2 =\; & \frac{|g'(x_1)|}{\eta(x_2,x_1)} \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} \left(\frac{1}{8} - \frac{1}{2}\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^2\right)\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^{\alpha-1} |h(x)|\,dx \\ & + \frac{|g'(x_2)|}{\eta(x_2,x_1)} \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} \left(\frac{1}{2}\left(1 - \frac{x-x_1}{\eta(x_2,x_1)}\right)^2 - \frac{1}{8}\right)\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^{\alpha-1} |h(x)|\,dx. \end{aligned}$$

Knowing that $\|h\|_\infty = sup_{t\in[x_1,x_2]}|h(x)|$,

$$\begin{aligned} I_2 \leq\; & \frac{|g'(x_1)|}{\eta(x_2,x_1)} \|h\|_\infty \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} \left(\frac{1}{8} - \frac{1}{2}\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^2\right)\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^{\alpha-1} dx \\ & + \frac{|g'(x_2)|}{\eta(x_2,x_1)} \|h\|_\infty \int_{x_1}^{x_1+\frac{1}{2}\eta(x_2,x_1)} \left(\frac{1}{2}\left(1 - \frac{x-x_1}{\eta(x_2,x_1)}\right)^2 - \frac{1}{8}\right)\left(\frac{x-x_1}{\eta(x_2,x_1)}\right)^{\alpha-1} dx. \end{aligned} \tag{19}$$

Adding Equations (18) to (19) based on (17), we get our required result. □

Theorem 4. *Let H be the open invex set where $H \subseteq \mathbb{R}$ and $\eta : H \times H \to \mathbb{R}$ is a mapping. Suppose there is a differentiable mapping $g : H \to \mathbb{R}$ on H such that $g' \in L[x_1, x_1 + \eta(x_2, x_1)]$ and $x_1, x_2 \in H$ with $x_1 < x_1 + \eta(x_2, x_1)$. If $h : [x_1, x_1 + \eta(x_2, x_1)] \to [0, \infty)$ is an integrable mapping symmetric to $x_1 + \frac{1}{2}\eta(x_2, x_1)$ and also $\left|g'\right|$ is preinvex function on H, then $\forall\, x_1, x_2 \in H$ with $\eta(x_2, x_1) \neq 0$ the following inequality holds:*

$$\begin{aligned} &\left|\frac{g(x_1) + g(x_1 + \eta(x_2, x_1))}{2}\left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}h(x_1) + J^{\alpha}_{x_1+}h(x_1 + \eta(x_2, x_1))\right]\right. \\ &\left.- \left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}(gh)(x_1) + J^{\alpha}_{x_1+}(gh)(x_1 + \eta(x_2, x_1))\right]\right| \\ \leq\ & \|h\|_{\infty}\frac{(\eta(x_2, x_1))^{\alpha+1}}{\Gamma(\alpha+1)}\left(\frac{|g'(x_1)| + |g'(x_2)|}{2}\right). \end{aligned} \tag{20}$$

Proof. Applying modulus on both sides of (9),

$$\begin{aligned} &\left|\frac{g(x_1) + g(x_1 + \eta(x_2, x_1))}{2}\left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}h(x_1) + J^{\alpha}_{x_1+}h(x_1 + \eta(x_2, x_1))\right]\right. \\ &\left.- \left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}(gh)(x_1) + J^{\alpha}_{x_1+}(gh)(x_1 + \eta(x_2, x_1))\right]\right| \\ =\ & \left|\frac{(\eta(x_2, x_1))^{\alpha+1}}{\Gamma(\alpha)}\int_0^1\left(-\int_t^1(1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right.\right. \\ &\left.\left.+\int_0^t(1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right)g'(x_1 + t\eta(x_2, x_1))dt\right|. \end{aligned} \tag{21}$$

From preinvexity of $|g'|$ on H, we have

$$\begin{aligned} &\left|\frac{g(x_1) + g(x_1 + \eta(x_2, x_1))}{2}\left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}h(x_1) + J^{\alpha}_{x_1+}h(x_1 + \eta(x_2, x_1))\right]\right. \\ &\left.- \left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}(gh)(x_1) + J^{\alpha}_{x_1+}(gh)(x_1 + \eta(x_2, x_1))\right]\right| \\ \leq\ & \frac{(\eta(x_2, x_1))^{2}}{\Gamma(\alpha)}\int_0^1\left|-\int_t^1(1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du+\right. \\ &\left.\int_0^t(1-u)^{\alpha-1}h(x_1 + u\eta(x_2, x_1))du\right|\left((1-t)\left|g'(x_1)\right| + t\left|g'(x_2)\right|\right)dt. \end{aligned} \tag{22}$$

After simplification, (22) becomes

$$\begin{aligned} &\left|\frac{g(x_1) + g(x_1 + \eta(x_2, x_1))}{2}\left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}h(x_1) + J^{\alpha}_{x_1+}h(x_1 + \eta(x_2, x_1))\right]\right. \\ &\left.- \left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}(gh)(x_1) + J^{\alpha}_{x_1+}(gh)(x_1 + \eta(x_2, x_1))\right]\right| \\ \leq\ & \frac{(\eta(x_2, x_1))^{\alpha+1}}{\Gamma(\alpha)}\int_0^1\left(\int_t^1(1-u)^{\alpha-1}\left|h(x_1 + u\eta(x_2, x_1))\right|du\right)\left((1-t)\left|g'(x_1)\right| + t\left|g'(x_2)\right|\right)dt + \\ & \frac{(\eta(x_2, x_1))^{\alpha+1}}{\Gamma(\alpha)}\int_0^1\left(\int_0^t(1-u)^{\alpha-1}\left|h(x_1 + u\eta(x_2, x_1))\right|du\right)\left((1-t)\left|g'(x_1)\right| + t\left|g'(x_2)\right|\right)dt. \end{aligned}$$

By changing the order of integration, we have

$$\begin{aligned}
&\left|\frac{g(x_1)+g\left(x_1+\eta\left(x_2,x_1\right)\right)}{2}\left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}h(x_1)+J^{\alpha}_{x_1+}h(x_1+\eta\left(x_2,x_1\right))\right]\right.\\
&\left.-\left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}\left(gh\right)\left(x_1\right)+J^{\alpha}_{x_1+}\left(gh\right)\left(x_1+\eta\left(x_2,x_1\right)\right)\right]\right|\\
\leq\;&\frac{\left(\eta\left(x_2,x_1\right)\right)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1(1-u)^{\alpha-1}\left|h(x_1+u\eta\left(x_2,x_1\right))\right|\int_0^u\left((1-t)\left|g'\left(x_1\right)\right|+t\left|g'\left(x_2\right)\right|\right)dtdu+\\
&\frac{\left(\eta\left(x_2,x_1\right)\right)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1(1-u)^{\alpha-1}\left|h(x_1+u\eta\left(x_2,x_1\right))\right|\int_u^1\left((1-t)\left|g'\left(x_1\right)\right|+t\left|g'\left(x_2\right)\right|\right)dtdu.\\
=\;&\frac{\left(\eta\left(x_2,x_1\right)\right)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1(1-u)^{\alpha-1}\left|h(x_1+u\eta\left(x_2,x_1\right))\right|\int_0^1\left((1-t)\left|g'\left(x_1\right)\right|+t\left|g'\left(x_2\right)\right|\right)dtdu\\
=\;&\frac{\left(\eta\left(x_2,x_1\right)\right)^{\alpha+1}}{\Gamma(\alpha)}\left(\frac{\left|g'\left(x_1\right)\right|+\left|g'\left(x_2\right)\right|}{2}\right)\int_0^1(1-u)^{\alpha-1}\left|h(x_1+u\eta\left(x_2,x_1\right))\right|du.
\end{aligned}$$

By changing the variable $x=x_1+u\eta\left(x_2,x_1\right)$ and reminding that $\|h\|_\infty=sup_{t\in[x_1,x_2]}\left|h(x)\right|$,

$$\begin{aligned}
&\left|\frac{g(x_1)+g\left(x_1+\eta\left(x_2,x_1\right)\right)}{2}\left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}h(x_1)+J^{\alpha}_{x_1+}h(x_1+\eta\left(x_2,x_1\right))\right]\right.\\
&\left.-\left[J^{\alpha}_{(x_1+\eta(x_2,x_1))-}\left(gh\right)\left(x_1\right)+J^{\alpha}_{x_1+}\left(gh\right)\left(x_1+\eta\left(x_2,x_1\right)\right)\right]\right|\\
\leq\;&\frac{\left(\eta\left(x_2,x_1\right)\right)^{\alpha}}{\Gamma(\alpha)}\|h\|_\infty\left(\frac{\left|g'\left(x_1\right)\right|+\left|g'\left(x_2\right)\right|}{2}\right)\int_{x_1}^{x_1+\eta(x_2,x_1)1}\left(\frac{\left(x_1+\eta\left(x_2,x_1\right)-x\right.}{\eta\left(x_2,x_1\right)}\right)^{\alpha-1}dx\\
=\;&\frac{\left(\eta\left(x_2,x_1\right)\right)^{\alpha+1}}{\Gamma(\alpha+1)}\|h\|_\infty\left(\frac{\left|g'\left(x_1\right)\right|+\left|g'\left(x_2\right)\right|}{2}\right),
\end{aligned}$$

which is as required. □

3. Conclusions

Some new estimates for the lower and upper boundaries of fractional Hermite–Hadamard–Fejér type inequalities are obtained for preinvex functions, which add up to the literature new error bounds for the lower and upper boundaries of the Hermite–Hadamard–Fejér inequality for preinvex functions in fractional domain correspondingly.

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