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$(\mathcal{C}, \Psi^*, G)$ Class of Contractions and Fixed Points in a Metric Space Endowed with a Graph

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Abstract: In this paper, we introduce the $(\mathcal{C}, \Psi^*, G)$ class of contraction mappings using C-class functions and some improved control functions for a pair of set valued mappings as well as a pair of single-valued mappings, and prove common fixed point theorems for such mappings in a metric space endowed with a graph. Our results unify and generalize many important fixed point results existing in literature. As an application of our main result, we have derived fixed point theorems for a pair of α-admissible set valued mappings in a metric space.

Keywords: fixed point; common fixed point; directed graph; edge preserving; transitivity property

MSC: 47H10; 54H25

1. Introduction and Preliminaries

In [1], Ran and Reurings proved the existence of fixed points for single-valued mappings in partially ordered metric spaces, and their results were extended by Neito and Lopez [2]. However, it became clear that the concept of a graph gives a better vision of fixed points instead of partial ordering, and the first attempt in this direction was done by Jachymsky [3]. He defined the Banach G- contraction for single-valued mapping, which was later extended by Beg et al. [4] for the multivalued mappings. After these, there was a lot of work done in the direction of fixed points in metric spaces endowed with graphs, see [5–14].

In 1973, Geraghty [15] defined Θ as the class of functions $\theta : [0, \infty) \to [0, 1)$ such that

$$\theta(t_n) \to 1 \Rightarrow t_n \to 0, \tag{1}$$

and also showed generalizations of the Banach-Neumann contractive mapping principle.

We now recall the following class of functions:

Ψ denotes the class of all continuous and non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$, such that:

- $\psi(t) = 0$ if, and only if $t = 0$.

Φ denotes the class of all lower semi-continuous functions $\phi : [0, \infty) \to [0, \infty)$, such that:

- $\phi(t) = 0$ if, and only if $t = 0$.

For more results on contraction principles involving the above said control functions, we refer the reader to [16–18].

In [19], the family of $\mathcal{C}$-class functions were introduced as follows: $F : [0, \infty)^2 \to \mathbb{R}$ belongs to the $\mathcal{C}$-class functions if:

- F is continuous,
- $F(s, t) \leq s$,
- $F(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.

In [20], Samet et al. introduced the concept of α-admissible mappings, and proved fixed point theorems for α-ψ contractive-type mappings, which paved a way to prove new results and generalise existing results in the fixed point theory. For some recent results on fixed point theorems of α-admissible mappings, the reader may refer to [21–24].

In this work, we utilised the C-class functions to give modified versions of contraction principles involving Ψ class functions and Φ class functions in the sense that we have relaxed the condition $\alpha(t) = 0$ if, and only if $t = 0$ in the Ψ and Φ class functions to $\alpha(t) = 0 \Rightarrow t = 0$. As an application, we have also deduced some common fixed point theorems for a pair of α-admissible mappings.

Throughout this work, (X_G, d_G) will denote the metric space endowed with a directed graph G with $V(G) = X_G$ and $\Delta \subseteq E(G)$, where $V(G)$ denotes the set of vertices, $E(G)$ denotes the set of edges of the graph G and $\Delta = \{(x_G, x_G) : x_G \in X\}$.

Definition 1. *[3] (X_G, d_G) is said to have property A if $x_n \to x_G$ and $(x_n, x_{n+1}) \in E(G)$ implies $(x_n, x_G) \in E(G)$, for all sequences $\{x_n\}_{n \in \mathbb{N}}$ in X_G.*

Definition 2. *[16] The pair (f, g) of self mappings of X_G is g-edge preserving in G, if*

$$(gx, gy) \in E(G) \Rightarrow (fx, fy) \in E(G). \tag{2}$$

Definition 3. *[11] $E(G)$ satisfies transitivity property if, and only if for all $x, y, z \in X_G$, $(x, z) \in E(G)$ and $(z, y) \in E(G)$ implies $(x, y) \in E(G)$.*

Let mappings $S, T : X_G \to CL(X_G)$ be given. We will make use of the following notations:

- $COFIX\{S, T\} = \{u \in X_G : u \in Su \bigcap Tu\}$ is the set of all common fixed points of S and T
- $FIX\{T\} = \{u \in X_G : u \in Tu\}$ is the set of all fixed points of T.

2. Main Results

Let Θ^* be the set of all continuous functions $\theta : [0, \infty) \to [0, 1)$.

Ψ^* be the set of all continuous and non decreasing functions $\psi : [0, \infty) \to [0, \infty)$, such that:

- $\psi(t) = 0 \Rightarrow t = 0$.

Let Φ^* be the set of all lower semi-continuous functions $\phi : [0, \infty) \to [0, \infty)$, such that:

- $\phi(t) = 0 \Rightarrow t = 0$.

Definition 4. *Let $S, T : X_G \to CB(X_G)$ be two given mappings. We say that the pair (S, T) belongs to the class of $(\mathcal{C}, \Psi^*, G)$ contractions if, and only if for all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$, the following conditions are satisfied:*

(4.1) For $u_G \in Sx_G$, there exists $v_G \in Ty_G$ such that $(u_G, v_G) \in E(G)$

(4.2) For $u_G \in Tx_G$, there exists $v_G \in Sy_G$ such that $(u_G, v_G) \in E(G)$

(4.3) there exists $F \in \mathcal{C}$, $\psi \in \Psi^*$, such that

$$\psi(H(Sx_G, Ty_G)) \leq F(\psi(M(x_G, y_G)), M(x_G, y_G)) \text{ and}$$
$$\psi(H(Tx_G, Sy_G)) \leq F(\psi(M(x_G, y_G)), M(x_G, y_G))$$

where

$$M(x_G, y_G) = \max\left\{d_G(x_G, y_G), d_G(Sx_G, x_G), d_G(Ty_G, y_G), \frac{d_G(y_G, Sx_G) + d_G(x_G, Ty_G)}{2}\right\}$$

Theorem 1. *Let (X_G, d_G) be complete and $S, T : X_G \to CB(X_G)$ satisfy the following:*

(1.1) There exists $x_{G0}, x_{G1} \in X_G$ such that $x_{G1} \in Tx_{G0} \bigcup Sx_{G0}$ and $(x_{G0}, x_{G1}) \in E(G)$,
(1.2) $E(G)$ satisfy transitivity property,
(1.3) $(S, T) \in (\mathcal{C}, \Psi^*, G)$ for some $F \in \mathcal{C}^*$ and $\psi \in \Psi^*$.

Then $COFIX\{S, T\} \neq \phi$.

Proof. By condition (1.1), suppose $x_{G0} \in X_G$, and $x_{G1} \in S(x_{G0})$. By condition (4.1), we can find $x_{G2} \in T(x_{G1})$ with $(x_{G1}, x_{G2}) \in E(G)$ and

$$\psi(d_G(x_{G1}, x_{G2})) \leq \psi(H(S(x_{G0}), T(x_{G1}))) \qquad \leq F(\psi(M(x_{G0}, x_{G1})), \phi(M(x_{G0}, x_{G1})))$$

Now again by condition (4.2), for $x_{G2} \in T(x_{G1})$, there exists $x_{G3} \in S(x_{G2})$ with $(x_{G2}, x_{G3}) \in E(G)$ and

$$\begin{aligned}\psi(d_G(x_{G2}, x_{G3})) &\leq \psi(H(Tx_{G1}, Sx_{G2})) \\ &\leq F(\psi(M(x_{G1}, x_{G2})), M(x_{G1}, x_{G2})).\end{aligned}$$

Continuing inductively, we construct the sequence $\{x_{Gn}\}$ recursively as for $n \geq 0$, as

$$x_{G2n+1} \in S(x_{G2n}), x_{G2n} \in T(x_{G2n-1}) \tag{3}$$

as well as $(x_{Gn}, x_{Gn+1}) \in E(G)$. Our first task is to establish that $COFIX\{S, T\} \neq \phi$. Note that if $M(x_{Gm}, x_{Gn}) = 0$ for any n,$m \in N$ then

$$\begin{aligned}M(x_{Gm}, x_{Gn}) = \max\Big\{&d_G(x_{Gm}, x_{Gn}), d_G(Sx_{Gm}, x_{Gm}), d_G(Tx_{Gn}, x_{Gn}), \\ &\frac{d_G(x_{Gn}, Sx_{Gm}) + d_G(x_{Gm}, Tx_{Gn})}{2}\Big\} = 0\end{aligned}$$

which shows that $x_{Gm} = x_{Gn} \in COFIX\{S, T\}$, and our first task will be complete. So let $M(x_{Gm}, x_{Gn}) \neq 0$ for any $n, m \in N$. Then, by definition of ψ, $\psi(M(x_{Gn-1}, x_{Gn})) \neq 0$.

If n is odd, we have

$$\begin{aligned}\psi(d_G(x_{Gn}, x_{Gn+1})) &\leq \psi(H(Sx_{Gn-1}, Tx_{Gn})) \\ &\leq F(\psi(M(x_{Gn-1}, x_{Gn})), M(x_{Gn-1}, x_{Gn}))\end{aligned} \tag{4}$$

Since $\psi(M(x_{Gn-1}, x_{Gn})) \neq 0$, we have

$$F(\psi(M(x_{Gn-1}, x_{Gn})), M(x_{Gn-1}, x_{Gn})) < \psi(M(x_{Gn-1}, x_{Gn}))$$

Then, by (4), we get

$$\psi(d_G(x_{Gn}, x_{Gn+1})) < \psi(M(x_{Gn-1}, x_{Gn})) \tag{5}$$

where

$$\begin{aligned}
M(x_{Gn-1}, x_{Gn}) = \max\Big\{&d_G(x_{Gn-1}, x_{Gn}), d_G(Sx_{Gn-1}, x_{Gn-1}), d_G(Tx_{Gn}, x_{Gn}),\\
&\frac{d_G(x_{Gn}, Sx_{Gn-1}) + d_G(x_{Gn-1}, Tx_{Gn})}{2}\Big\}\\
\leq \max\Big\{&d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn}, x_{Gn-1}), d_G(x_{Gn+1}, x_{Gn}),\\
&\frac{d_G(x_{Gn}, x_{Gn}) + d_G(x_{Gn-1}, x_{Gn+1})}{2}\Big\}\\
\leq \max\Big\{&d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn+1}, x_{Gn}), \frac{d_G(x_{Gn-1}, x_{Gn+1})}{2}\Big\}\\
\leq \max\Big\{&d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn+1}, x_{Gn}),\\
&\frac{d_G(x_{Gn-1}, x_{Gn}) + d_G(x_{Gn}, x_{Gn+1})}{2}\Big\}\\
\leq \max\{&d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn}, x_{Gn+1})\}
\end{aligned}$$

If $d_G(x_{Gn+1}, x_{Gn}) > d_G(x_{Gn-1}, x_{Gn})$, then $M(x_{Gn-1}, x_{Gn}) \leq d_G(x_{Gn+1}, x_{Gn})$. Then (5) gives

$$\psi(d_G(x_{Gn}, x_{Gn+1})) < \psi(d_G(x_{Gn}, x_{Gn+1}))$$

a contradiction. So, we have

$$d_G(x_{Gn}, x_{Gn+1}) \leq d_G(x_{Gn-1}, x_{Gn}) \tag{6}$$

For an even number n, a similar argument leads to inequality (6). Thus, $\{d_G(x_{n+1}, x_{Gn})\}$ is a monotonically non-increasing sequence which is bounded below, and thereby,

$$\lim_{n\to\infty} d_G(x_{Gn}, x_{Gn+1}) = \lim_{n\to\infty} M(x_{Gn-1}, x_{Gn}) = r \geq 0.$$

Assume that $r > 0$, so that $\psi(r) > 0$. Taking $\liminf$ on both sides of the inequality (5), we obtain

$$\psi(r) < \psi(r)$$

a contradiction. Hence $r = 0$. Consequently, we have

$$\lim_{n\to\infty} d_G(x_{Gn}, x_{Gn+1}) = 0. \tag{7}$$

Next, we prove that $\{x_{Gn}\}$ is a Cauchy sequence. By (7), it is enough if we show that the subsequence $\{x_{G2n}\}$ is a Cauchy sequence. Suppose, if possible, $\{x_{G2n}\}$ is not a Cauchy sequence. Then, there exists $\epsilon > 0$ and subsequences $\{x_{G2m(k)}\}$ and $\{x_{G2n(k)}\}$, such that $n(k)$ is the smallest index for which $n(k) > m(k) > k, d_G(x_{G2m(k)}, x_{G2n(k)}) \geq \epsilon$. That is,

$$d_G(x_{G2m(k)}, x_{G2n(k)-2}) < \epsilon \tag{8}$$

Now, we have

$$\begin{aligned}
\epsilon &\leq d_G(x_{G2m(k)}, x_{G2n(k)})\\
&\leq d_G(x_{G2m(k)}, x_{G2n(k)-2}) + d_G(x_{G2n(k)-2}, x_{G2n(k)-1}) + d_G(x_{G2n(k)-1}, x_{G2n(k)})\\
&< \epsilon + d_G(x_{G2n(k)-2}, x_{G2n(k)-1}) + d_G(x_{G2n(k)-1}, x_{G2n(k)})
\end{aligned}$$

AS $k \to \infty$, we get

$$\lim_{k\to\infty} d_G(x_{G2m(k)}, x_{G2n(k)}) = \epsilon \tag{9}$$

Also, we have

$$\begin{aligned} &|d_G(x_{G2m(k)}, x_{G2n(k)+1}) - d_G(x_{G2m(k)}, x_{G2n(k)})| \leq d_G(x_{G2n(k)}, x_{G2n(k)+1}) \\ \text{and} \quad &|d_G(x_{G2m(k)-1}, x_{G2n(k)}) - d_G(x_{G2m(k)}, x_{G2n(k)})| \leq d_G(x_{G2m(k)}, x_{2m(k)-1}). \end{aligned}$$

Letting $k \to \infty$ and using (7) and (9), we get

$$\lim_{k\to\infty} d_G(x_{G2m(k)-1}, x_{G2n(k)}) = \lim_{k\to\infty} d_G(x_{G2m(k)}, x_{G2n(k)+1}) = \epsilon. \tag{10}$$

From

$$|d_G(x_{G2m(k)-1}, x_{G2n(k)+1}) - d_G(x_{G2m(k)-1}, x_{G2n(k)})| \leq d_G(x_{G2n(k)}, x_{G2n(k)+1})$$

and making use of (7) and (10), we get

$$\lim_{k\to\infty} d_G(x_{G2m(k)-1}, x_{G2n(k)+1}) = \epsilon \tag{11}$$

Also, from the definition of M and from (7) and (9)–(11), we have

$$\lim_{k\to\infty} M(x_{G2m(k)-1}, x_{G2n(k)}) = \epsilon \tag{12}$$

Also by the transitivity property of G, we have $(x_{G2m(k)-1}, x_{G2n(k)}) \in E(G)$. Thus, we have

$$\begin{aligned} \psi(d_G(x_{G2m(k)}, x_{G2n(k)+1})) &= \psi(H(Tx_{G2m(k)-1}, Sx_{G2n(k)})) \\ &\leq F(\psi(M(x_{G2m(k)-1}, x_{G2n(k)})), (M(x_{G2m(k)-1}, x_{G2n(k)})) \end{aligned}$$

Letting $k \to \infty$ and making use of (10) and (11), the above inequality yields

$$\psi(\epsilon) < \psi(\epsilon)$$

a contradiction. Thus, $\{x_{Gn}\}$ is a Cauchy sequence. By completeness of X_G , we can find $u_G \in X_G$, such that $x_{Gn} \to u_G$ as $n \to \infty$.

We will now prove that $u_G \in COFIX\{S, T\}$. Note that $(x_{G2n+1}, u_G) \in E(G)$, and so

$$\begin{aligned} \psi(d_G(x_{G2n+1}, Tu_G)) &\leq \psi(H(Sx_{G2n}, Tu_G)) \\ &\leq F(\psi(M(x_{G2n}, u_G)), M(x_{G2n}, u_G)) \end{aligned} \tag{13}$$

where

$$M(x_{G2n}, u_G) = \max\Big\{d_G(x_{G2n}, u_G), d_G(Sx_{G2n}, x_{G2n}), d_G(Tu_G, u_G), \frac{d_G(x_{G2n}, Tu_G) + d_G(u_G, Sx_{G2n})}{2}\Big\}$$

Note that as $n \to \infty$, $d_G(Sx_{G2n}, x_{G2n}) \to 0$, $d_G(u_G, Sx_{G2n}) \to 0$, and so $M(u_G, x_{G2n}) \to d_G(Tu_G, u_G)$ Now, if $d_G(Tu_G, u_G) \neq 0$, then from (13) as $n \to \infty$, we have

$$\psi(d_G(Tu_G, u_G) < \psi(d_G(Tu_G, u_G)),$$

again, a contradiction. Thus, $d_G(Tu_G, u_G) = 0$, which implies that $u_G \in \overline{Tu_G}$, and since Tu_G is closed, we have $u_G \in Tu_G$.

Now again, we have $M(u_G, u_G) = d_G(u_G, Su_G)$, and if $d_G(u_G, Su_G) \neq 0$, since $(u_G, u_G) \in \Delta \subset E(G)$, we get

$$\begin{aligned} \psi(d_G(Su_G, u_G)) &\leq \psi(H(Su_G, Tu_G)) \\ &\leq F(\psi(M(u_G, u_G))M(u_G, u_G)) \\ &< \psi(d_G(u_G, Su_G)) \end{aligned}$$

a contradiction, and thereby, $d_G(u_G, Su_G) = 0$ or $u_G \in Su_G$. Hence, $COFIX\{S, T\} \neq \phi$. □

We will deduce the following important results from Theorem 1:

Corollary 1. *Let (X_G, d_G) be complete and $S, T : X_G \to CB(X_G)$ satisfy conditions (4.1), condition (4.2), condition (1.1), condition (1.2), and the following:*

(1.1) For all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$

$$\psi(H(Sx_G, Ty_G)) \leq \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)) \text{ and}$$
$$\psi(H(Tx_G, Sy_G)) \leq \psi(M(x_G, y_G)) - \phi(M(x_G, y_G))$$

where $\psi \in \Psi^*$, $\phi \in \Phi^*$ and $M(x_G, y_G)$ is as in Definition 4. Then, $COFIX\{S, T\} \neq \phi$.

Proof. Take $F(r, t) = r - \phi(t)$ in Theorem 1. □

Corollary 2. *Let (X_G, d_G) be complete and $S, T : X_G \to CB(X_G)$ satisfy the conditions (4.1), condition (4.2), condition (1.1), condition (1.2), and the following:*

(2.1) For all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$

$$\psi(H(Sx_G, Ty_G)) \leq \theta(M(x_G, y_G))\psi(M(x_G, y_G)) \text{ and}$$
$$\psi(H(Tx_G, Sy_G)) \leq \theta(M(x_G, y_G))\psi(M(x_G, y_G))$$

where $\psi \in \Psi^*$, $\theta \in \Theta^*$ and $M(x_G, y_G)$ is as in Definition 4. $COFIX\{S, T\} \neq \phi$.

Proof. Take $F(r, t) = \theta(t).r$ in Theorem 1. □

Corollary 3. *Let (X_G, d_G) be complete and $S, T : X_G \to CB(X_G)$ satisfy the conditions (4.1), (4.2), (1.1) and (1.2), and the following:*

(3.1) For all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$, there exist $0 < \lambda < 1$, such that

$$H(Sx_G, Ty_G) \leq \lambda(M(x_G, y_G)) \text{ and}$$
$$H(Tx_G, Sy_G) \leq \lambda(M(x_G, y_G))$$

where $M(x_G, y_G)$ is as in Definition 4. Then $COFIX\{S, T\} \neq \phi$.

Proof. For some $k > 0$, set $k^* = k(1 - \lambda)$. Then,

$$H(Sx_G, Ty_G)) \leq \lambda(M(x_G, y_G)) \text{ and}$$
$$H(Tx_G, Sy_G) \leq \lambda(M(x_G, y_G))$$

implies

$$H(Sx_G, Ty_G) \leq \frac{k - k^*}{k}(M(x_G, y_G)) \text{ and}$$
$$H(Tx_G, Sy_G) \leq \frac{k - k^*}{k}(M(x_G, y_G))$$

or

$$kH(Sx_G, Ty_G) + 1 \le kM(x_G, y_G) + 1 - k^*M(x_G, y_G) \text{ and}$$
$$kH(Tx_G, Sy_G) \le kM(x_G, y_G) - k^*M(x_G, y_G)$$

Now, let $\psi(t) = kt + 1$ and $\phi(t) = k^*(t)$. Then, the above inequality leads to

$$\psi(H(Sx_G, Ty_G)) \le \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)) \text{ and}$$
$$\psi(H(Tx_G, Sy_G)) \le \psi(M(x_G, y_G)) - \phi(M(x_G, y_G))$$

Thus, all conditions of Corollary 1 are satisfied, and hence, $COFIX\{S, T\} \neq \phi$. □

Corollary 4. *Let (X_G, d_G) be complete and $T : X_G \to CB(X_G)$ satisfy the following:*

(4.1) There exists $x_{G0}, x_{G1} \in X_G$, such that $x_{G1} \in Tx_{G0}$ and $(x_{G0}, x_{G1}) \in E(G)$;
(4.2) For any $u \in Tx_G$, there exists $w \in Ty_G$, such that $(u, w) \in E(G)$;
(4.3) $E(G)$ satisfies the transitivity property;
(4.4) $\psi(H(Tx_G, Ty_G)) \le \psi(d_G(x_G, y_G)) - \phi(d_G(x_G, y_G))$

where $\psi \in \Psi^*$, $\phi \in \Phi^*$. Then, $FIX\{T\} \neq \phi$.

Proof. Take $S = T$ in Corollary 1. □

Corollary 5. *Let (X_G, d_G) be complete and $T : X_G \to CB(X_G)$ satisfy conditions (4.1)–(4.3), and the following:*

(5.1) $\psi(H(Tx_G, Ty_G)) \le \theta(d_G(x_G, y_G))\psi(d_G(x_G, y_G))$

where $\psi \in \Psi^*$, $\theta \in \Theta^*$. Then, $FIX\{T\} \neq \phi$.

Proof. Take $F(r, t) = \theta(t).r$ in Corollary 2. □

Example 1. *Let $X_G = \{0, \frac{1}{2^n} : n \in \mathbb{N}\}$, $d_G(x_G, y_G) = |x_G - y_G|$, $G = (V, E)$, with $V(G) = X_G$ and $E(G) = \{(0, 0), (\frac{1}{2^n}, \frac{1}{2^n}), (\frac{1}{2^n}, 0)\}$ and $S, T : X_G \to CB(X_G)$ be defined by*

$$Sx_G = \begin{cases} \{0\}, & \text{if } x_G = 0 \\ \{\frac{1}{2^{n+1}}, 0\}, & \text{if } x_G = \frac{1}{2^n}. \end{cases}$$

and

$$Tx_G = \begin{cases} \{0\}, & \text{if } x_G = 0 \\ \{\frac{1}{2^{n+2}}, 0\}, & \text{if } x_G = \frac{1}{2^n}. \end{cases}$$

Define $\psi, \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = 2t + 1$ and $\phi(t) = \frac{t}{4}$ for all $t \in [0, \infty)$. Clearly, $\psi(t) \in \Psi^*$ (note that $\psi \notin \Psi$) and $\phi(t) \in \Phi^*$.

- If $x_G = \frac{1}{2^n}$ and $y_G = 0$ with $(\frac{1}{2^n}, 0) \in E(G)$, then $Sx_G = \{\frac{1}{2^{n+1}}, 0\}$, $Sy = \{0\}$, $Tx_G = \{\frac{1}{2^{n+2}}, 0\}$, $Ty_G = \{0\}$, $d_G(x_G, y_G) = \frac{1}{2^n}$, $H(Sx_G, Ty_G) = \frac{1}{2^{n+1}}$, $\psi(H(Sx_G, Ty_G)) = \frac{1}{2^n} + 1$, $M(x_G, y_G) = \frac{1}{2^n}$, $\psi(M(x_G, y_G)) = \frac{1}{2^{n-1}} + 1$, $\phi(M(x_G, y_G)) = \frac{1}{2^{n+2}}$ and

$$\psi(H(Sx_G, Ty_G)) = \frac{1}{2^n} + 1 < \frac{1}{2^{n-1}} + 1 - \frac{1}{2^{n+2}} = \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)) \text{ for all } n \in \mathbb{N}.$$

Also, for $\frac{1}{2^{n+1}} \in Sx_G$, there exists $0 \in Ty_G$, such that $(\frac{1}{2^{n+1}}, 0) \in E(G)$ and for $0 \in Sx_G$, there exists $0 \in Ty_G$ such that $(0,0) \in E(G)$.
For $\frac{1}{2^{n+2}} \in Tx_G$, there exists $0 \in Sy_G$ such that $(\frac{1}{2^{n+2}}, 0) \in E(G)$ and for $0 \in Tx_G$, there exists $0 \in Sy_G$ such that $(0,0) \in E(G)$.

- If $x_G = 0, y_G = 0$ with $(0,0) \in E(G)$, then $Sx_G = \{0\} = Sy_G = Tx_G = Ty_G, 0 = d_G(x_G, y_G) = H(Sx_G, Ty_G) = M(x_G, y_G), \psi(H(Sx_G, Ty_G)) = \psi(M(x_G, y_G)) = 1, \phi(M(x_G, y_G)) = 0$ and

$$\psi(H(Sx_G, Ty_G)) = 1 = \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)).$$

Also, for $0 \in Sx_G$, there exists $0 \in Ty_G$ such that $(0,0) \in E(G)$ and for $0 \in Sx_G$, there exists $0 \in Ty_G$, such that $(0,0) \in E(G)$.
For $0 \in Tx_G$, there exists $0 \in Sy_G$, such that $(0,0) \in E(G)$ and for $0 \in Tx_G$, there exists $0 \in Sy_G$, such that $(0,0) \in E(G)$

- Also, if $x_G = \frac{1}{2^n}, y_G = \frac{1}{2^n}$ with $(\frac{1}{2^n}, \frac{1}{2^n}) \in E(G)$, then $Sx_G = \{\frac{1}{2^{n+1}}, 0\} = Sy_G, Tx_G = \{\frac{1}{2^{n+2}}, 0\} = Ty_G, d_G(x_G, y_G) = 0, H(Sx_G, Ty_G) = \frac{1}{2^{n+2}}, \psi(H(Sx_G, Ty_G)) = \frac{1}{2^{n+1}} + 1, M(x_G, y_G) = \frac{1}{2^{n+1}}, \psi(M(x_G, y_G)) = \frac{1}{2^n} + 1, \phi(M(x_G, y_G)) = \frac{1}{2^{n+3}}$ and

$$\psi(H(Sx_G, Ty_G)) = \frac{1}{2^{n+1}} + 1 < \frac{1}{2^n} + 1 - \frac{1}{2^{n+3}} = \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)) \text{ for all } n \in \mathbb{N}.$$

Also, for $\frac{1}{2^{n+1}} \in Sx_G$, there exists $0 \in Ty_G$, such that $(\frac{1}{2^{n+1}}, 0) \in E(G)$ and for $0 \in Sx_G$, there exists $0 \in Ty_G$, such that $(0,0) \in E(G)$.
For $\frac{1}{2^{n+2}} \in Tx_G$, there exists $0 \in Sy_G$, such that $(\frac{1}{2^{n+2}}, 0) \in E(G)$ and for $0 \in Tx_G$, there exists $0 \in Sy_G$, such that $(0,0) \in E(G)$.

Thus, we see that for all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$

$$\psi(H(Sx_G, Ty_G)) \leq \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)) \text{ for all } x_G, y_G \in X_G.$$

Also, for $u_G \in Sx_G$, there exists $v_G \in Ty$ such that $(u_G, v_G) \in E(G)$ and for $u_G \in Tx_G$, there exists $v_G \in Sy$, such that $(u_G, v_G) \in E(G)$. Hence, the pair $(S, T) \in (\mathcal{C}, \Psi^*, G)$ with $F(r, t) = r - \phi(t)$. Thus, all conditions of Theorem 1 are satisfied, and $COFIX\{S, T\} = \{0\}$.

Remark 1. *Corollary 3 (and hence, Corollary 1 and Theorem 1) are proper extensions and generalisations of Theorem 3.1 of [4] and Theorem 4.2 of [8].*

Remark 2. *Note that in Example 1, the graph G is a directed graph and not connected, and so Theorem 3.1 of [4] cannot be applied to neither of the mappings S or T. Also note that $(\frac{1}{2^n}, \frac{1}{2^n}) \in E(G)$, $H(S\frac{1}{2^n}, T\frac{1}{2^n}) = \frac{1}{2^{n+2}} > 0 = d_G(\frac{1}{2^n}, \frac{1}{2^n})$ and hence a simple extension of Theorem 3.1 of [4] and Theorem 4.2 of [8] to two mappings cannot be applied. However, we see that the mappings S and T satisfy the conditions of Corollary 3, and so Corollary 3 also ensures the existence of a common fixed point of S and T.*

Remark 3. *In Theorem 1, if the directed graph G is replaced with an undirected graph G' with $E(G') = E(G) \bigcup E(G^{-1})$, then Condition (4.3) in Definition 4 can be replaced with only one inequality:*

$$\psi(H(Sx_G, Ty_G)) \leq F(\psi(M(x_G, y_G)), M(x_G, y_G))$$

Similar arguments follow in Corollaries 1–3 also.

Definition 5. *Let $f, g : X_G \to X_G$. We say that the pair (f, g) belongs to the class of Jungck type $(\mathcal{C}, \Psi^*, G)$ contractions if*

(5.1) f *is* $g-$ *edge preserving in* G.
(5.2) *For all* $x_G,\ y_G \in X_G$ *with* $(gx_G, gy_G) \in E(G)$

$$\psi(d_G(fx_G, fy_G)) \leq F(\psi(M(gx_G, gy_G)), M(gx_G, gy_G)),\ for\ some\ \psi \in \Psi^*,\ F \in \mathcal{C} \tag{14}$$

$$M(gx_G, gy_G) = \max\{d_G(gx_G, gy_G), d_G(gx_G, fx_G), d_G(gy_G, fy_G), \frac{d_G(gx_G, fy_G) + d_G(gy_G, fx_G)}{2}\}.$$

Let mappings $f, g : X_G \to X_G$ be given. We will make use of the following notations:

- $X_G(f, g) : \{u \in X_G : (gu, fu) \in E(G)\}$,
- $C(f, g) : \{u \in X_G : fu = gu\}$ is the set of all coincidence points of mappings f and g,
- $C_m(f, g) : \{u \in X_G : fu = gu = u\}$ is the set of all common fixed points of mappings f and (g).
- $CS(X, d)$: Collection of all Cauchy sequences in the metric space (X, d).

Lemma 1. *Let f and g satisfy the following:*

(1.1) $x_G, y_G \in C(f, g)$ *implies* $gx_G = gy_G$
(2.1) (f, g) *is compatible*

Then, $C_m(f, g) \neq \phi$.

Proof. Let $x_G \in C(f, g)$ and $gx_G = w$. Then, since (f, g) is compatible, $gw = ggx_G = gfx_G = fgx_G = fw$, or in other words, $w \in C(f, g)$. By Lemma (1), $gw = gx_G = w$, which, in turn, shows that $w \in C_m(f, g)$. □

Theorem 2. *Let d'_G and d_G be any two metrics defined on X_G, and (X_G, d'_G) is complete. Suppose $f, g : X_G \to X_G$ satisfy the following:*

(2.1) $(f, g) \in$ *Jungck type* $(\mathcal{C}, \Psi^*, G)$ *with respect to* d_G
(2.2) g *is continuous and* $g(X_G)$ *is closed with respect to* d'_G
(2.3) $f(X_G) \subseteq g(X_G)$
(2.4) $E(G)$ *satisfies the transitivity property*
(2.5) *if* $d_G \geq d'_G$, *then* $f : (X_G, d_G) \to (X_G, d'_G)$ *is* $g-$ *Cauchy*
(2.6) f *is* $G-$ *continuous with respect to* d'_G, f *and* g *are* d'_G- *compatible.*

Then,

$$X_G(f, g) \neq \phi\ \ iff\ \ C(f, g) \neq \phi$$

Proof. Suppose that $C(f, g) \neq \phi$. Let $u \in C(f, g)$. Then, $(gu, fu) = (gu, gu) \in \Delta \subset E(G)$ and so $u \in X_G(f, g)$; that is, $X_G(f, g) \neq \phi$.

Suppose now, $X_G(f, g) \neq \phi$. Let $x_{G0}, \in X_G$, such that $(gx_{G0}, fx_{G0}) \in E(G)$. Now, since $F(X_G) \subseteq g(X_G)$, using condition (5.1) we can construct sequence $\{x_{Gn}\}$ in X_G, such that

$$gx_{Gn} = fx_{Gn-1}, \quad (gx_{Gn-1}, gx_{Gn}) \in E(G)$$

for all $n \in \mathbb{N}$. It is easy to see that if $M(x_{Gm}, x_{Gn}) = 0$ for any $m, n \in N$,, then $x_{Gm}, x_{Gn} \in C(f, g)$ and the proof is done. So we assume that for all $m, n \in \mathbb{N}$, $M(x_{Gm}, x_{Gn}) \neq 0$. Then,

$$\begin{aligned} \psi(d_G(gx_{Gn+1}, gx_{n+2})) &= \psi(d_G(fx_{Gn}, fx_{Gn+1})) \\ &\leq F(\psi(M(gx_{Gn}, gx_{Gn+1})), M(gx_{Gn}, gx_{Gn+1})) \\ &< \psi(M(gx_{Gn}, gx_{Gn+1})) \end{aligned} \tag{15}$$

We also have

$$\begin{aligned} M(gx_{Gn}, gx_{Gn+1}) &= \max\Big\{d_G(gx_{Gn}, gx_{Gn+1}), d_G(gx_{Gn}, fx_{Gn}), d_G(gx_{Gn+1}, fx_{Gn+1}), \\ &\qquad \frac{d_G(gx_{Gn}, fx_{Gn+1}) + d_G(gx_{Gn+1}, fx_{Gn})}{2}\Big\} \\ &= \max\Big\{d_G(gx_{Gn}, gx_{Gn+1}), d_G(gx_{Gn+1}, gx_{n+2}), \frac{d_G(gx_{Gn}, gx_{n+2})}{2}\Big\} \\ &\leq \max\{d_G(gx_{Gn}, gx_{Gn+1}), d_G(gx_{Gn}, gx_{n+2})\} \end{aligned}$$

If $M(gx_{Gn}, gx_{Gn+1}) = d_G(gx_{Gn+1}, gx_{n+2})$, then by (15), we obtain that

$$\psi(d_G(gx_{Gn+1}, gx_{n+2})) < \psi(d_G(gx_{Gn+1}, gx_{n+2}))$$

a contradiction. Hence,

$$M(gx_{Gn}, gx_{Gn+1}) = d_G(gx_{Gn}, gx_{Gn+1})$$

Substituting in (15), we get t $\psi(d_G(gx_{Gn+1}, gx_{n+2})) < \psi(d_G(gx_{Gn}, gx_{Gn+1}))$. So by the definition of ψ, we have

$$d_G(gx_{Gn+1}, gx_{n+2}) \leq d_G(gx_{Gn}, gx_{Gn+1}), \qquad \forall n \in \mathbb{N}$$

Hence, the sequence $\{d_G(gx_{Gn}, gx_{Gn+1})\}$ is non-negative and non-increasing, and thereby we can find $r \geq 0$, such that $\lim_{n\to\infty} d_G(gx_{Gn}, gx_{Gn+1}) = r$. We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. Letting $n \to \infty$ in (15), we obtain

$$\psi(r) \leq F(\psi(r), r) < \psi(r)$$

a contradiction. Thus,

$$\lim_{n\to\infty} d_G(gx_{Gn}, gx_{Gn+1}) = 0. \tag{16}$$

We will show that $\{gx_{Gn}\} \in CS(X_G, d_G)$. Suppose $\{gx_{Gn}\} \notin CS(X_G, d_G)$ and for $\epsilon > 0$, $k \in \mathbb{N}$, let $n(k) \in \mathbb{N}$ be the smallest integer with $n(k) > m(k) \geq k$ and

$$\begin{aligned} d_G(gx_{Gn(k)}, gx_{Gm(k)}) &\geq \epsilon \\ d_G(gx_{Gn(k)-1}, gx_{Gm(k)}) &< \epsilon. \end{aligned}$$

Then, we have

$$\begin{aligned} \epsilon &\leq d_G(gx_{Gm(k)}, gx_{Gn(k)}) \\ &\leq d_G(gx_{Gm(k)}, gx_{Gn(k)-1}) + d_G(gx_{Gn(k)-1}, gx_{Gn(k)}) \\ &< \epsilon + d_G(gx_{Gn(k)-1}, gx_{Gn(k)}) \end{aligned}$$

Using (16) in the above inequality, we get

$$\lim_{k\to\infty} d_G(gx_{Gm(k)}, gx_{Gn(k)}) = \epsilon > 0.$$

By condition (2.4) we get $(gx_{Gm(k)}, gx_{Gn(k)}) \in E(G)$. Thus, we have

$$\begin{aligned}\psi(d_G(gx_{Gm(k)+1}, gx_{Gn(k)+1})) &= \psi(d_G(fx_{Gm(k)}, fx_{Gn(k)})) \\ &\leq F(\psi(M(gx_{Gm(k)}, gx_{Gn(k)})), M(gx_{Gm(k)}, gx_{Gn(k)}))\end{aligned} \tag{17}$$

where

$$\begin{aligned}M(gx_{Gm(k)}, gx_{Gn(k)}) &= \max\Big\{d_G(gx_{Gm(k)}, gx_{Gn(k)}), d_G(gx_{Gm(k)}, fx_{Gm(k)}), d_G(gx_{Gn(k)}, fx_{Gn(k)}), \\ &\quad \frac{d_G(gx_{Gm(k)}, fx_{Gn(k)}) + d_G(gx_{Gn(k)}, fx_{Gm(k)})}{2}\Big\} \\ &= \max\Big\{d_G(gx_{Gm(k)}, gx_{Gn(k)}), d_G(gx_{Gm(k)}, gx_{m(k)+1}), d_G(gx_{Gn(k)}, gx_{n(k)+1}), \\ &\quad \frac{d_G(gx_{Gm(k)}, gx_{Gn(k)+1}) + d_G(gx_{Gn(k)}, gx_{m(k)+1})}{2}\Big\}\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} M(gx_{Gm(k)}, gx_{Gn(k)}) = \epsilon$$

By inequality (17), we get

$$\psi(\epsilon) < \psi(\epsilon)$$

a contradiction. So $\{gx_{Gn}\} \in CS(X_G, d_G)$.

We will show that $\{gx_{Gn}\} \in CS(X_G, d'_G)$. If $d_G \geq d'_G$, it is trivial. Thus, suppose $d_G \geq d'_G$. Let $\epsilon > 0$. Since $\{gx_{Gn} \in CS(X_G, d_G)\}$, by condition (2.5) we see that $\{fx_{Gn}\} \in CS(X_G, d'_G)$. Then, there exists $N_0 \in \mathbb{N}$ with

$$d'_G(gx_{Gn+1}, gx_{m+1}) = d'_G(fx_{Gn}, fx_m) < \epsilon$$

whenever $n, m \geq N_0$. So $\{gx_{Gn}\} \in CS(X_G, d'_G)$.

Since $g(X_G)$ is d'_G - closed and (X_G, d'_G) is complete, there exists $u_G = gx_G \in g(X_G)$, such that

$$\lim_{n\to\infty} gx_{Gn} = \lim_{n\to\infty} fx_{Gn} = u_G.$$

By d'_G - compatibility of f and g, we have

$$\lim_{n\to\infty} d'_G(gfx_{Gn}, fgx_{Gn}) = 0 \tag{18}$$

Then,

$$d'_G(gu_G, fu_G) \leq d'_G(gu_G, gfx_{Gn}) + d'_G(gfx_{Gn}, fgx_{Gn}) + d'_G(fgx_{Gn}, fu_G)$$

Letting $n \to \infty$ and using (18), the continuity of g, and the G- continuity of f, it follows that $d'_G(gu_G, fu_G) = 0$, which implies that $gu_G = fu_G$. So $u_G \in C(f, g)$ and the proof is complete. □

If $d_G = d'_G$, we have the following

Theorem 3. *Let (X_G, d_G) be complete and $f, g : X_G \to X_G$ satisfy the following:*

(3.1) $(f, g) \in$ *Jungck type* $(\mathcal{C}, \Psi^*, G)$
(3.2) g *is continuous and* $g(X_G)$ *is closed*
(3.3) $F(X_G) \subset g(X_G)$
(3.4) $E(G)$ *satisfies the transitivity property*
(3.5) *(a)* f *is G-continuous and* f *and* g *are* d_G*-compatible or*
(b) (X_G, d_G, G) *has property A.*

Then,

$$X_G(f, g) \neq \phi \text{ iff } C(f, g) \neq \phi.$$

Proof. Proceeding as in the proof of Theorem 2, we see that if $C(f, g) \neq \phi$ then $X_G(f, g) \neq \phi$ and if $X_G(f, g) \neq \phi$ then $\{gx_{Gn}\} \in CS(X_G, d_G)$ Now since $g(X_G)$ is closed in X_G, there exists $u_G \in X_G$, such that

$$\lim_{n\to\infty} gx_{Gn} = gu_G = \lim_{n\to\infty} fx_{Gn}. \tag{19}$$

We will show that $u_G \in C(f, g)$. Suppose $u_G \notin C(f, g)$. Then $d_G(fu_G, gu_G) > 0$. Note that if $M(x_{Gm}, u_G) = 0$ for any $m \in N$,, then $x_{Gm}, u_G \in C(f, g)$ and the proof is done. So we assume that for all $m \in \mathbb{N}$, $M(x_{Gm}, u_G) \neq 0$. If condition (3.5a) is satisfied, then proof follows from a similar argument as in Theorem 2. If condition (3.5b) is satisfied, then $(gx_{Gn}, gu) \in E(G)$ for each $n \in \mathbb{N}$. Thus, we have

$$d_G(gu_G fu_G) \leq d_G(gu_G, fx_{Gn(k)}) + d_G(fx_{Gn(k)}, fu_G)$$

which implies that

$$d_G(gu_G, fu_G) - d_G(gu_G, fx_{Gn(k)}) \leq d_G(fx_{Gn(k)}, fu_G)$$

Since ψ is non-decreasing, we get

$$\begin{aligned} \psi(d_G(gu_G, fu_G) - d_G(gu_G, fx_{Gn(k)})) &\leq \psi(d_G(fx_{Gn(k)}, fu_G)) \\ &\leq F(\psi(M(gx_{Gn(k)}, gu_G)), M(gx_{Gn(k)}, gu_G)) \end{aligned} \tag{20}$$

where

$$M(gx_{Gn(k)}, gu_G) = \max\Big\{d_G(gx_{Gn(k)}, gu_G), d_G(gx_{Gn(k)}, fx_{Gn(k)}), d_G(gu_G, fu_G), \frac{d_G(gx_{Gn(k)}, fu_G) + d_G(gu_G, fx_{Gn(k)}))}{2}\Big\}$$

Using (19), we obtain

$$\lim_{k\to\infty} M(gx_{Gn(k)}, gu_G) = d_G(gu_G, fu_G) > 0.$$

Thus, taking $k \to \infty$ in (20), we get $\psi(d_G(gu_G, fu_G)) < \psi(d_G(gu_G, fu_G))$, a contradiction. Therefore, $fu_G = gu_G$ and so $C(f, g) \neq \phi$. □

Theorem 4. *Suppose* f *and* g *satisfy condition (2.1)–(2.6), condition (2.6) and the following:*

(4.1) *If* $x_G, y_G \in C(f, g)$ *and* $gx_G \neq gy_G$*, then* $(gx_G, gy_G) \in E(G)$*.*

If $X_G(f, g) \neq \phi$*, then* $C_m(f, g) \neq \phi$*.*

Proof. By Theorem 2 $C(f,g) \neq \phi$. Let $x_G, y_G \in C(f,g)$ and suppose $gx_G \neq gy_G$ so that $M(gx_G, gy_G) \neq 0$. By assumption (K), $(gx_G, gy_G) \in E(G)$, and we have

$$\begin{aligned} \psi(d_G(fx_G, fy_G)) &\leq F(\psi(M(gx_G, gy_G)), M(gx_G, gy_G)) \\ &< \psi(M(gx_G, gy_G)) \\ &= \psi(d_G(fx_G, fy_G)) \end{aligned} \tag{21}$$

which is a contradiction. Therefore, $gx_G = gy_G$. Now by Lemma 1, $C_m(f,g) \neq \phi$. □

Corollary 6. *Let d'_G and d_G be any two metrics defined on X_G, and (X_G, d'_G) is complete. Suppose $f, g : X_G \to X_G$ satisfy conditions (5.1) and Theorem (2.1) to Theorem (2.5), and the following: for some $\psi \in \Psi^*$, $\phi \in \Phi^*$ and all $x_G, y_G \in X_G$ with $(gx_G, gy_G) \in E(G)$*

$$\psi(d_G(fx_G, fy_G)) \leq \psi(M(gx_G, gy_G)) - \phi(M(gx_G, gy_G)) \tag{22}$$

Then,

$$X_G(f,g) \neq \phi \text{ iff } C(f,g) \neq \phi$$

Corollary 7. *Let d'_G and d_G be any two metrics defined on X_G and (X_G, d'_G) is complete. Suppose $f, g : X_G \to X_G$ satisfy conditions (5.1) and condition (2.1) to condition (2.5) and the following: for some $\psi \in \Psi^*$, $\theta \in \Theta^*$ and all $x_G, y_G \in X_G$ with $(gx_G, gy_G) \in E(G)$*

$$\psi(d_G(fx_G, fy_G)) \leq \theta(M(gx_G, gy_G))\psi(M(gx_G, gy_G)) \tag{23}$$

Then,

$$X_G(f,g) \neq \phi \text{ iff } C(f,g) \neq \phi$$

Let $X_G = [0, \infty)$ and $d_G, d'_G : X_G \times X_G \to [0, \infty)$ be defined by

$$d_G(x_G, y_G) = \begin{cases} 0 & \text{if } x_G = y_G \\ \max\left\{x_G, y_G\right\} & \text{otherwise} \end{cases} \quad d'_G(x_G, y_G) = |x_G - y_G| \tag{24}$$

Then clearly, $d_G \geq d'_G$. We define

$$E(G) = \{(x_G, y_G) : x_G = y_G \text{ or } x_G, y_G \in [0, \frac{1}{2}] \text{ with } x_G \leq y_G\}$$

Consider the mappings $f : X_G \to X_G$ and $g : X_G \to X_G$, defined by

$$fx_G = \begin{cases} 4x_G^4, & \text{if } 0 \leq x_G \leq \frac{1}{2} \\ 2x_G^2, & \text{if } x_G > \frac{1}{2} \end{cases} \quad gx_G = \begin{cases} 2x_G^2, & \text{if } 0 \leq x_G \leq \frac{1}{2} \\ 8x_G^4, & \text{if } x_G > \frac{1}{2} \end{cases} \tag{25}$$

for all $x_G \in X_G$.

Let $\psi : [0, \infty) \to [0, \infty)$ be defined by

$$\psi(t) = 2t$$

and $\theta : [0, \infty) \to [0, 1)$ be defined by

$$\theta(t) = \begin{cases} t, & \text{if } 0 \le t \le \frac{1}{2} \\ \frac{1}{2t+1}, & t > \frac{1}{2} \end{cases}$$

and $F \in \mathcal{C}$ be given by

$$F(r, t) = \theta(t).r$$

We will show that the pair (f, g) is a (F, ψ, G) contraction.

Let $(gx_G, gy_G) \in E(G)$. If $gx_G = gy_G$ (which is possible only if $x_G = y_G = 0$), then $fx_G = gy_G = 0$ and so $(fx_G, fy_G) \in E(G)$. If $(gx_G, gy_G) \in E(G)$ with $gx_G, gy_G \in [0, \frac{1}{2}]$ and $gx_G \le gy_G$, then we obtain $x_G, y_G \in [0, \frac{1}{2}]$, $x_G \le y_G$ and then $x_G^4, y_G^4 \in [0, \frac{1}{2}]$, $fx_G = 4x_G^4 \le 4y_G^4 = fy_G$, and thus $(fx_G, fy) \in E(G)$. Thus, the pair (f, g) is g-edge preserving in G.
Now, let $x_G, y_G \in X_G$ and $(gx_G, gy_G) \in E(G)$. From the argument given above, if $gx_G = gy_G$, then $\psi(d_G(fx_G, fy_G)) = 0$ and (f, g) satisfy the condition condition (5.2). If $(gx_G, gy_G) \in E(G)$ with $gx_G, gy_G \in [0, \frac{1}{2}]$ and $gx_G \le gy_G$, then

$$\begin{aligned} \psi(d_G(fx_G, fy_G) &= 8{y_G}^4 \\ &\le 2{y_G}^2 \\ &\le y_G.2y_G \\ &\le \max\{x_G, y_G\}.2\max\{x_G, y_G\} \\ &= \theta(d_G(x_G, y_G))\psi(d_G(x_G, y_G)) \\ &\le \theta(M(gx_G, gy_G))\psi(M(gx_G, gy_G)) \end{aligned}$$

and so (f, g) satisfy the condition condition (5.1).

We will show that g and f are d' - compatible. Let $\{x_{Gn}\}$ be a sequence in X_G, such that

$$\lim_{n\to\infty} gx_{Gn} = \lim_{n\to\infty} fx_{Gn} = u$$

Note that

$$d'(gfx_{Gn}, fgx_{Gn}) = 32x_{GN}^8 \to 0 \ \ as\ n \to \infty.$$

Clearly, $f : (X_G, d) \to (X_G, d')$ is G-continuous, and consequently, all the conditions of Theorem 2 are satisfied. Also note that $0, \frac{1}{2} \in C(f, g)$, $g0 \ne g(\frac{1}{2})$, and $(g0, g(\frac{1}{2}) \in E(G)$, and thus condition Theorem (4.1) of Theorem 4 is satisfied. Consequently, $\{0, \frac{1}{2}\} \subset X_G(f, g) \bigcap C(f, g) \bigcap C_m(f, g)$.

3. Applications

In this section, as an application of our results, we will give some fixed point results for a pair of set valued α-admissible contraction mappings in a metric space.
Throughout this section, (X, d) is any metric space, $S, T : X \to CB(X)$ two given mappings, and $\alpha : X \times X \to [0, \infty)$.

Definition 6. *We say that the pair (S, T) is α-admissible if, and only if for all $x, y \in X$ with $\alpha(x, y) \ge 1$, the following conditions hold:*

(6.1) *For $u \in Sx$, there exists $v \in Ty$, such that $\alpha(u,v) \geq 1$.*
(6.2) *For $u \in Tx$, there exists $v \in Sy$, such that $\alpha(u,v) \geq 1$.*

Theorem 5. *Suppose the following conditions hold:*

(5.1) *There exists $x_0, x_1 \in X$ such that $x_1 \in Tx_0 \bigcup Sx_0$ and $\alpha(x_0, x_1) \geq 1$,*
(5.2) *α is a triangular function,*
(5.3) *The pair (S,T) is α-admissible,*
(5.4) *there exists $F \in \mathcal{C}$, $\psi \in \Psi^*$, such that for all $x, y \in X$ with $\alpha(x,y) \geq 1$*

$$\psi(H(Sx,Ty)) \leq F(\psi(M(x,y)), M(x,y)) \text{ and}$$
$$\psi(H(Tx,Sy)) \leq F(\psi(M(x,y)), M(x,y))$$

where

$$M(x,y) = \max\left\{d(x,y), d(Sx,x), d_G(Ty,y), \frac{d(y,Sx)+d(x,Ty)}{2}\right\}$$

Then $COFIX\{S,T\}$ is a singleton set.

Proof. Consider the graph G on (X,d), defined by $V(G) = X$ and $E(G) = (x,y) \in X \times X : \alpha(x,y) \geq 1)$. It is easy to see that the functions S and T satisfy all conditions of Theorem 1, and hence, $COFIX\{S,T\}$ is a singleton set. □

Similarly, we have the following results:

Theorem 6. *Suppose conditions Theorem (5.1), Theorem (6.1), Theorem (6.3), and the following hold:*

(6.1) *for all $x, y \in X$ with $\alpha(x,y) \geq 1$*
$$\psi(H(Sx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)) \text{ and}$$
$$\psi(H(Tx,Sy)) \leq \psi(M(x,y)) - \phi(M(x,y))$$

where $\psi \in \Psi^*$, $\phi \in \Phi^*$ and $M(x,y)$ is as in Theorem 5. Then $COFIX\{S,T\} \neq \phi$.

Theorem 7. *Suppose conditions Theorem (5.1), Theorem (6.1), Theorem (5.3), and the following hold:*

(7.1) *for all $x, y \in X$ with $\alpha(x,y) \geq 1$*
$$\psi(H(Sx,Ty)) \leq \theta(M(x,y))\psi(M(x,y)) \text{ and}$$
$$\psi(H(Tx,Sy)) \leq \theta(M(x,y))\psi(M(x,y))$$

where $\psi \in \Psi^*$, $\phi \in \Phi^*$ and $M(x,y)$ is as in Theorem 5. Then, $COFIX\{S,T\} \neq \phi$.

Example 2. *Let $X_G = [0,\infty) \subseteq \mathbb{R}$, and let the metrics $d_G, d_G^{'} : X_G \times X_G \to [0,\infty)$ be defined by*

$$\begin{aligned} d_G(x_G, y_G) &= \max\left\{x_G, y_G\right\} \\ d^{'}(x_G, y_G) &= L|x_G - y_G| \end{aligned} \tag{26}$$

for all $x_G, y_G \in X_G$, respectively, where L is a constant real number, such that $L \in (1,\infty)$. It is easy to see that $d < d^{'}$. Now, $E(G)$ is given by

$$E(G) = \{(x_G, y_G) : x_G = y_G \text{ or } x_G, y_G \in [0,1] \text{ with } x_G \leq y_G\} \tag{27}$$

Consider the mappings $f : X_G \to X_G$ and $g : X_G \to X_G$ defined by

$$fx = \ln\left(1 + \frac{x_G^2}{2}\right), \;\; gx = x_G^2 \tag{28}$$

for all $x_G \in X_G$, respectively.

Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be defined by

$$\psi(t) = \begin{cases} t, & \text{if } 0 \le t \le 1 \\ t^2, & \text{if } t > 1 \end{cases} \tag{29}$$

$$\phi(t) = \begin{cases} \frac{t^2}{2}, & \text{if } 0 \le t \le 2 \\ \frac{1}{2}, & \text{otherwiset} > 1 \end{cases} \tag{30}$$

Next, let $(gx, gy) \in E(G)$ if $x_G = y_G$. Then, $(fx, fy) \in E(G)$, and if $(gx, gy) \in E(G)$ with $gx \le gy$, then we obtain $gx = x_G^2, gy = y_G^2 \in [0, 1]$ and $x_G^2 = gx \le gy = y_G^2$, and we have $fx = \ln(1 + \frac{x_G^2}{2}) \le \ln(1 + \frac{y_G^2}{2}) = fy$ and $fx_G, fy \in [0, 1]$. This implies that $(fx_G, fy) \in E(G)$.

Since $d < d'$, we need to prove that the function $f : (X_G, d) \to (X_G, d')$ is a g-Cauchy sequence in X_G. Let $\epsilon > 0$, and let $\{x_{Gn}\}$ be a sequence in X_G such that $\{gx_{Gn}\}$ is a Cauchy sequence in (X_G, d). Then, there exists $k \in \mathbb{N}$, such that for all $n, m \ge k, d_G(gx_{Gn}, gx_m) < \frac{\epsilon}{L}$. Then, we have

$$\begin{aligned}
d'(fx_{Gn}, fx_m) &= L|fx_{Gn} - fx_m| \\
&= L\left|\ln\left(1 + \frac{(x_{Gn})^2}{2}\right) - \ln\left(1 + \frac{(x_m)^2}{2}\right)\right| \\
&= L\left|\ln \frac{1 + \frac{(x_m)^2}{2}}{1 + \frac{(x_{Gn})^2}{2}}\right| \\
&= L\left|\ln\left(1 + \frac{\frac{(x_m)^2}{2} - \frac{(x_{Gn})^2}{2})}{1 + \frac{(x_{Gn})^2}{2}}\right)\right| \\
&\le L\left[\ln\left(1 + |\frac{(x_{Gn})^2}{2} - \frac{(x_M)^2}{2}|\right)\right] \\
&\le L\left[\frac{2\ln(1 + \frac{1}{2}|(x_{Gn})^2 - (x_m)^2|)}{|(x_{Gn})^2 - (x_m)^2|}|(x_{Gn})^2 - (x_m)^2|\right] \\
&< L|(x_{Gn})^2 - (x_m)^2| \\
&= Ld_G(gx_{Gn}, gx_m) \\
&< L \cdot \frac{\epsilon}{L} = \epsilon.
\end{aligned}$$

This implies that $f : (X_G, d) \to (X_G, d')$ is a g- Cauchy on X_G.

It can easily be shown that $f : (X_G, d) \to (X_G, d')$ is G-continuous. As a result, we will only need to prove that g and f are d'-compatible. Let $\{x_{Gn}\}$ be a sequence in X_G, such that

$$\lim_{n\to\infty} gx_{Gn} = \lim_{n\to\infty} fx_{Gn} = u \tag{31}$$

Then, we have $\ln(1 + u/2) = a$ and so it follows that $u = 0$. Now, we have

$$d'(gfx_{Gn}, fgx_{Gn}) = L\left|\ln\left(1 + \frac{(x_{Gn})^2}{2}\right)^2 - \ln\left(1 + \frac{(x_{Gn})^4}{2}\right)^2\right| \to 0 \tag{32}$$

as $n \to \infty$. It is easy to see that there exists a point $u \in X_G$, such that $(gu, fu) \in E(G)$, and thus $X_G(f, g) \neq \phi$.

Consequently, all the conditions of Theorem 2 are satisfied.

Example 3. *Let $X_G = [0, \infty) \subseteq \mathbb{R}$ and let the metrics $d_G, d'_G : X_G \times X_G \to [0, \infty)$ be defined by*

$$\begin{aligned} d_G(x_G, y_G) &= \max\left\{x_G, y_G\right\} \\ d'_G(x_G, y_G) &= L.\max\left\{x_G, y_G\right\} \end{aligned} \tag{33}$$

for all $x_G, y_G \in X_G$ and some $L \in (1, \infty)$. Then clearly, $d_G < d'_G$. We define

$$E(G) = \{(x_G, y_G) : x_G = y_G \text{ or } x_G, y_G \in [0,1] with\; x_G \leq y_G\}$$

Consider the mappings $f : X_G \to X_G$ and $g : X_G \to X_G$ defined by

$$fx_G = \begin{cases} x_G^4, & \text{if } 0 \leq x_G \leq 1 \\ x_G^2, & \text{if } x_G > 1 \end{cases} \quad gx_G = \begin{cases} 2x_G^2, & \text{if } 0 \leq x_G \leq 1 \\ 2x_G^4, & \text{if } x_G > 1 \end{cases} \tag{34}$$

for all $x_G \in X_G$.

Let $\psi : [0, \infty) \to [0, \infty)$ be defined by

$$\psi(t) = 2t$$

and, $\theta : [0, \infty) \to [0, 1)$ be defined by

$$\theta(t) = \begin{cases} t, & \text{if } 0 \leq t < 1 \\ \dfrac{1}{t+1}, & t \geq 1 \end{cases}$$

Next, if $(gx_G, gy_G) \in E(G)$ and $x_G = y_G$ then $(fx_G, fy_G) \in E(G)$ and if $(gx_G, gy_G) \in E(G)$ with $gx_G \leq gy_G$, then we obtain $x_G, y_G \in [0, \frac{1}{\sqrt{2}}$, $x_G \leq y_G$ and then $x_G^4, y_G^4 \in [0, \frac{1}{4}$, $fx_G = x_G^4 \leq y_G^4 = fy_G$, and thus $(fx_G, fy) \in E(G)$.

Since $d_G < d'_G$, we need to prove that the function $f : (X_G, d) \to (X_G, d')$ is a g-Cauchy sequence in X_G. Let $\epsilon > 0$, and let $\{x_{Gn}\}$ be a sequence in X_G such that $\{gx_{Gn}\}$ is a Cauchy sequence in (X_G, d). Then, there exists $k \in \mathbb{N}$ such that, for all $n, m \geq k$, $d_G(gx_{Gn}, gx_m) < \frac{\epsilon}{L}$. Then, we have

$$d'_G(fx_{Gn}, fx_{Gm}) = L\max\left\{fx_{Gn}, fx_{Gm}\right\}$$

We will consider the following cases:

Case 1 : $x_{Gn}, x_{Gm} \in [0, 1]$. Then, we have

$$\begin{aligned} d'_G(fx_{Gn}, fx_m) &= L\max\left\{x_{Gn}^4, x_{Gm}^4\right\} \leq L\max\left\{2x_{Gn}^2, 2x_{Gm}^2\right\} \\ &= L\max\left\{gx_{Gn}, gx_{Gm}\right\} = L.d_G(gx_{Gn}, gx_{Gm}) < L.\frac{\epsilon}{L} = \epsilon \end{aligned}$$

Case 2 : $x_{Gn}, x_{Gm} \in (0, \infty$. Then, we have

$$\begin{aligned} d'_G(fx_{Gn}, fx_m) &= L\max\left\{x_{Gn}^2, x_{Gm}^2\right\} \leq L\max\left\{2x_{Gn}^4, 2x_{Gm}^4\right\} \\ &= L\max\left\{gx_{Gn}, gx_{Gm}\right\} = L.d_G(gx_{Gn}, gx_{Gm}) < L.\frac{\epsilon}{L} = \epsilon \end{aligned}$$

Case 3 : $x_{Gn} \in [0,1]$, $x_{Gm} \in (1,\infty)$. *Then, we have*

$$d'_G(fx_{Gn}, fx_m) = L\max\left\{x_{Gn}^4, x_{Gm}^2\right\} \le L\max\left\{2x_{Gn}^2, 2x_{Gm}^4\right\}$$

$$= L\max\left\{gx_{Gn}, gx_{Gm}\right\} = L.d_G(gx_{Gn}, gx_{Gm}) < L.\frac{\epsilon}{L} = \epsilon$$

Case 1 : $x_{Gn} \in (0,\infty)$, $x_{Gn} \in [0,1]$. *Then, we have*

$$d'_G(fx_{Gn}, fx_m) = L\max\left\{x_{Gn}^2, x_{Gm}^4\right\} \le L\max\left\{2x_{Gn}^4, 2x_{Gm}^2\right\}$$

$$= L\max\left\{gx_{Gn}, gx_{Gm}\right\} = L.d_G(gx_{Gn}, gx_{Gm}) < L.\frac{\epsilon}{L} = \epsilon$$

Thus, $f : (X_G, d) \to (X_G, d')$ *is a g-Cauchy on* X_G.

It can be easily shown that $f : (X_G, d) \to (X_G, d')$ *is G-continuous. We will show that* g *and* f *are* d' *compatible. Let* $\{x_{Gn}\}$ *be a sequence in* X_G, *such that*

$$\lim_{n\to\infty} gx_{Gn} = \lim_{n\to\infty} fx_{Gn} = u$$

Then, we have $\ln(1 + u/2) = a$, *and so it follows that* $u = 0$. *Now, we have*

$$d'(gfx_{Gn}, fgx_{Gn}) = L\left|\ln\left(1 + \frac{(x_{Gn})^2}{2}\right)^2 - \ln\left(1 + \frac{(x_{Gn})^4}{2}\right)^2\right| \to 0 \tag{35}$$

as $n \to \infty$. *It is easy to see that there exists a point* $u \in X_G$ *such that* $(gu, fu) \in E(G)$, *and thus* $X_G(f, g) \neq \phi$.
Consequently, all the conditions of Theorem 2 are satisfied.

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