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The Regularity of Solution for Weakly Coupled System Derived by Microwave Heating Model

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Abstract: In this paper, we study the regularity of the weak solution of the coupled system derived from the microwave heating model with frequency variable. We first show that the weak solution E of the system is Hölder continuous near the boundary of $S = \partial\Omega$. The main idea of the proof is based on the estimation of linear degenerate system in Campanato space. Then we show that the solution u of the heat conduction equation is Hölder continuous with exponent $\frac{\alpha}{2}$. Finally, under the appropriate conditions we show that the coupled system with microwave heating has a weak solution. Moreover the regularity of the weak solution is studied.

Keywords: time-harmonic Maxwell's equations; microwave heating model; regularity of weak solutions

1. Introduction

Microwave heating is a kind of heating method that can be converted into heat energy by absorbing microwave energy from an object. As a new type of energy carrier, it has been widely used in many fields such as food heating and thawing, biological sterilization, chemical synthesis and metal smelting (see [1–5]). Compared with the traditional heating mode, microwave heating has the characteristics of short heating time, high efficiency and small thermal inertia. Hence, the automatic control skill is able to be used in microwave heating process. However, due to the internal heat energy generated by the microwave, the heating of some heated stuff may be uneven or even “runaway heating” (see [6–10]). Therefore, it is a meaningful problem to realize uniform heating and minimum energy dissipation in microwave heating process.

The heat produced by microwaves causes the temperature of the heated object to be unevenly distributed. This is partly because various physical parameters such as electric permittivity, electric conductivity and magnetic permeability strongly depend on the temperature. Understanding the complicated dynamic interactions between electric field, magnetic field and temperature is of great important to the system modeling and design process.

If the variable coefficients of Maxwell's equations are not smooth, it will be difficult to obtain the regularity of the solution for Maxwell's equations. This also leads to the heat source of heat conduction equation is only belong to $L^1(Q_T)$. Therefore, it is difficult to prove the existence and regularity of solutions. Furthermore, the necessary optimality conditions for optimal control cannot be derived. Under the condition of the coefficient has better smoothness, Yin [9] proved the existence of the solution of the coupled system. But the smoothness of the coefficient is unattainable in the practical problems, even is discontinuity. The study of weak solution to the corresponding coupling system has become a significant problem. At the same time, in order to study necessary optimality

conditions of the optimal control problem with frequency variable, we need to study the regularity of the weak solution.

In this paper, we prove that the weak solution of system (4) is continuous. When $\frac{\gamma}{\omega}$ is bounded, the real part of $\frac{\gamma}{\omega}$ has a positive lower bound and ζ is Lipschitz continuous. Assume that the regularity of the coefficients $\frac{\gamma}{\omega}$ and ζ are minimal, then this regularity is optimal. The basic idea is to use Campanato type of estimates(see [11,12]).

We have derived weak coupling mathematical model of microwave heating system with frequency variation in [13]. In paper [13], the existence of global weak solutions is established by assuming suitable conditions. By applying Lax-Milgram theorem and monotone operator theory, the existence and uniqueness of solutions for weakly coupled systems are proved in [13]. However, the regularity of the weak solution has not been studied for the coupled system in [13]. Particularly, we consider the uniformity and boundedness of temperature. In this paper. We study that the temperature is Hölder continuous by appropriate assumptions. The regularity results provide the basis about the derivation of optimal conditions for uniform microwave heating

This paper is organized as follows. In Section 2, for completeness we introduce the mathematical model of microwave heating with frequency variable from [13]. In Section 3, we impose some basic assumptions which ensure the well-posedness of the underlying system. Existence of a weak solution is also established from [13]. In Section 4, some important estimates are derived. In Section 5, regularity of the weak solution is investigated.

2. Weak Coupling Mathematical Model of Microwave Heating System

We consider that the electromagnetic field does not change with temperature during microwave heating. In particular, the distribution of dielectric, permeability, and electric conductivity functions are temperature independent. The frequency of microwave is dependent only on its position x , i.e., $\omega = \omega(x)$, and has nothing to do with time.

In this paper, we assume that Ω is a bounded simply-connected domain in three-dimensional space R^3 . The boundary of Ω is C^1 continuous. Let $Q_T = \Omega \times (0, T]$ and $S_T = \partial\Omega \times (0, T]$. Hereafter, a bold letter represents a vector in R^3 .

According to literature [13], the microwave heating process can be described by the weakly coupled system of Maxwell equations and heat conduction equations and the corresponding initial boundary value conditions as follows:

$$\begin{cases} \nabla \times (\frac{\gamma(x)}{\omega(x)} \nabla \times \mathbf{E}) + \zeta(x, \omega) \mathbf{E} = 0, & (x, t) \in Q_T, \\ u_t - \nabla [k(x, u) \nabla u] = \frac{1}{2} a_2(x, \omega) |\mathbf{E}|^2, & (x, t) \in Q_T, \\ \mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{G}, & (x, t) \in S_T, \\ u_n(x, t) = 0, & (x, t) \in S_T, \\ u(x, 0) = u_0, & x \in \Omega, \end{cases} \tag{1}$$

where the unknown functions $\mathbf{E}(x, t)$ and $u(x, t)$ respectively represent the electric field intensity and temperature at time t at $x \in \Omega$, and \mathbf{n} is the outward unit normal vector on $S = \partial\Omega$ and $u_n = \nabla u \cdot \mathbf{n}$ is the outward the normal derivative on $S = \partial\Omega$. The $\mathbf{G}(\cdot)$ is the time-harmonic electromagnetic field generated by the external photoelectric device.

$$\gamma(x) = \frac{1}{\mu_0(x)} (\frac{\mu'}{(\mu')^2 + (\mu'')^2} + i \frac{\mu''}{(\mu')^2 + (\mu'')^2}) \triangleq \gamma_1(x) + i\gamma_2(x), \tag{2}$$

$$\zeta(x, \omega) = -\varepsilon_0 \varepsilon' \omega + i(\varepsilon_0 \varepsilon'' \omega + \sigma) \triangleq -a_1(x, \omega) + ia_2(x, \omega), \tag{3}$$

where functions ε , μ and σ represent the dielectric, permeability and electric conductivity respectively.

3. The Well-Posedness of Solution of Weak Coupling System

$V(\cdot)$ is the vector valued function in three dimensions.

Definition 1. (see [14]) $H(curl, \Omega)$, $H(div, \Omega)$ and $H(curl, div, \Omega)$ space

$$\begin{aligned} H(curl, \Omega) &= \{V(\cdot) \in (L^2(\Omega))^3 : \nabla \times V \in (L^2(\Omega))^3\}, \\ H(div, \Omega) &= \{V(\cdot) \in (L^2(\Omega))^3 : \nabla \cdot V \in (L^2(\Omega))^3\}, \\ H(curl, div, \Omega) &= H(curl, \Omega) \cap H(div, \Omega). \end{aligned}$$

$H(curl, \Omega)$ is the Hilbert space with the following inner product:

$$\langle V, F \rangle = \int_{\Omega} [(\nabla \times V) \cdot (\nabla \times F^*) + V \cdot F^*] dx,$$

where F^* is the complex conjugate of F (see [9]).

The norm is (derived from inner product)

$$\|V\|_{H(curl, \Omega)} = \sqrt{(V, V)_{H(curl, \Omega)}}.$$

We impose some basic assumptions which ensure the well-posedness of the underlying system and proof of regularity.

H(1). The functions $u_0(\cdot)$ and $u_T(\cdot)$ are nonnegative with $u_0(\cdot)$ and $u_T(\cdot) \in L^2(\Omega)$.

H(2). The function $k : \Omega \times R \rightarrow R$ is given. Suppose that $x \mapsto k(x, u)$ is measurable on Ω for all $u \in R$ and $u \mapsto k(x, u)$ is uniformly Lipschitz continuous on R for almost all $x \in \Omega$. The following inequality conditions are satisfied:

$$0 < k_0 \leq k(x, u) \leq k_1 \leq 1,$$

for all $x \in \Omega, u \in R$ with positive constants k_0 and k_1 .

H(3).

- (a) The functions a_1, a_2 are bounded. Assume that a_1, a_2 are Lipschitz continuous with respect to ω -variables. At the same time there is

$$0 < a_0 \leq a_1(x, \omega), a_2(x, \omega) \leq b_0,$$

for some constants $a_0 > 0, b_0 > 0$.

- (b) Assumed that the function $\gamma = \frac{1}{\mu} \triangleq \gamma_1 + i\gamma_2$ is a bounded complex function. The real functions

$$\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0,$$

for all $x \in \Omega$ and some constant $\gamma_0 > 0$.

- (c) The function G is given and defined on S_T with the extended function $\bar{G} \in H(curl, \Omega)$. \bar{G} is Hölder continuous on Ω , and we have $\|\bar{G}(\cdot, t)\|_{H(curl, \Omega)} \leq c_0 \|G(\cdot, t)\|_{L^2(S)}$ for all t , where c_0 is a constant that depends only on Ω .

- (d) Assumed that the function $\omega : \Omega \rightarrow R$ is measurable. Let

$$U_{ad} = \{\omega : \Omega \rightarrow R \text{ is measurable} : 0 < \omega_0 \leq \omega(x) \leq \omega_1\},$$

where ω_0 and ω_1 are positive constants.

Since the controlled system we are considering is a weakly coupled system. Its heating process can be described by Maxwell equations determining the distribution of its electric field intensity.

The intensity of the electric field produces an internal heat source in the heating process, causing the temperature of the heated material to rise. In this way, the material can be heated.

Therefore, in mathematical analysis, we can first study the existence and uniqueness of Maxwell equations. Then we consider the existence and uniqueness of the solution for the initial boundary value problem of heat conduction equation under a given electric field intensity distribution.

We consider the system:

$$\begin{cases} \nabla \times (\frac{\gamma(x)}{\omega(x)} \nabla \times \mathbf{E}) + \zeta(x, \omega(x))\mathbf{E} = 0, & (x, t) \in Q_T, \\ \mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{G}, & (x, t) \in S_T. \end{cases} \tag{4}$$

From Theorem 3.2. of literature [13], the system (4) exists unique solution $\tilde{\mathbf{E}} \in L^\infty(0, T; H_0(\text{curl}, \Omega))$ for given $\omega(\cdot) \in U_{ad}$.

Next, we study the existence and uniqueness of the solution to the heat conduction equation under a given electric field intensity distribution. Consider the following system:

$$\begin{cases} u_t - \nabla [k(x, u)\nabla u] = \frac{1}{2}a_2(x, \omega(x))|\mathbf{E}|^2, & (x, t) \in Q_T, \\ u_n(x, t) = 0, & (x, t) \in S_T, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{5}$$

Similarly, from Theorem 3.5. of literature [13], there exists a unique solution $u \in W[0, T]$ to the problem (5) for any given $\omega(\cdot) \in U_{ad}$ and its corresponding solution $\mathbf{E} \in L^\infty(0, T; H(\text{curl}, \Omega))$ to (4).

In conclusion, the existence and uniqueness theorem (see [13] Theorem 3.6) of solutions for the coupled system (1).

4. Estimates of Solution for the Underlying System

Definition 2. (A weighted Sobolev Space) Let $\zeta \in L^\infty(\Omega)$ with $|\zeta(x)| \leq \zeta_0$ for some constant $\zeta_0 > 0$. Set

$$H_0(\text{curl}, \text{div}_{\zeta}, \Omega) = \{\mathbf{V} \in H_0(\text{curl}, \Omega) : \text{div}(\zeta\mathbf{V}) \in (L^2(\Omega))^3\},$$

with the norm

$$\|\mathbf{V}\|_{H_0(\text{curl}, \text{div}_{\zeta}, \Omega)}^2 = \int_{\Omega} \left[|\nabla \times \mathbf{V}|^2 + |\nabla(\zeta\mathbf{V})|^2 + |\mathbf{V}|^2 \right] dx.$$

The space $H_0(\text{curl}, \text{div}_{\zeta}, \Omega)$ is a Banach space. Moreover, the embedding operator from $H_0(\text{curl}, \text{div}_{\zeta}, \Omega)$ into $L^2(\Omega)$ is compact (see [15]).

For all $x_0 \in \Omega$ and $\rho > 0$, we set

$$B(x_0, \rho) = \{x \in \Omega : |x - x_0| < \rho\}.$$

Definition 3. (Campanato space $L^{2,\mu}(\Omega)$) Let function f in $L^2(\Omega)$ and a nonengative constant $\mu \geq 0$. Set

$$L^{2,\mu}(\Omega) = \{f(x) \in L^2(\Omega) : \|f\|_{L^{2,\mu}(\Omega)} = \|f\|_{L^2(\Omega)} + [f]_{2,\mu} < \infty\},$$

with the semi-norm

$$[f]_{2,\mu} = \sup_{\rho>0, x_0 \in \Omega} \rho^{-\mu} \int_{B(x_0, \rho)} |f - (f)_{x_0}|^2 dx,$$

where

$$(f)_{x_0} = \frac{1}{B(x_0, \rho)} \int_{B(x_0, \rho)} f(x) dx.$$

The space $L^{2,\mu}(\Omega)$ is a Banach space (see [11]). It has the following properties (see [11,12]).

Lemma 4. For $\mu \in (n, n + 2)$, the space $L^{2,\mu}(\Omega)$ is equivalent to $C^\alpha(\bar{\Omega})$, where $\alpha = \frac{\mu-n}{2}$.

For weakly coupled systems, we first consider the regularity of the solution to the linear Maxwell Equations (4). Let

$$\mathbf{F} = -\left(\nabla \times \left(\frac{\gamma(x)}{\omega(x)} \nabla \times \mathbf{G}\right) + \zeta(x, \omega(x))\mathbf{G}\right).$$

Lemma 5. Under the assumption H(3), let $q(x) = \frac{\gamma(x)}{\omega(x)} = q_1(x) + iq_2(x)$, $\zeta(x, \omega(x)) = \zeta_1(x) + i\zeta_2(x)$ for any $\omega \in U_{ad}$. There exists positive constant α_0 and β_0 such that

$$q_1(x), q_2(x), \gamma_1(x), \gamma_2(x) \geq \alpha_0 > 0, \tag{6}$$

$$\|q\|_{L^\infty(\Omega)} + \|\zeta\|_{L^\infty(\Omega)} \leq \beta_0, \tag{7}$$

and there exists a positive constant B_0 such that

$$\|\mathbf{F}\|_{L^2(\Omega)} \leq B_0, \tag{8}$$

for $F \in L^2(\Omega)$.

The System (4) is degenerate. The classical regularity theory for elliptic system is not valid here. We will prove Hölder continuous of the solution for System (4). The following Lemma plays a key role in the following proof.

Lemma 6. (see [16] Lemma 3.3) Let $F \in H(\text{div}, \Omega)$ with $\text{div}F = 0$. There exist vector field $\mathbf{V}(x)$ such that

$$\begin{cases} \nabla \times \mathbf{V} = F, & \text{in } \Omega, \\ \nabla \cdot \mathbf{V} = 0, & \text{in } \Omega, \\ \mathbf{n} \cdot \mathbf{V} = 0, & \text{on } S, \end{cases} \tag{9}$$

and

$$\|\mathbf{V}\|_{H^1(\Omega)} \leq C_3 \|F\|_{L^2(\Omega)}, \tag{10}$$

$$\|\mathbf{V}\|_{L^2(\Omega)} \leq C_3 \|F\|_{L^2(\Omega)}, \tag{11}$$

where C_3 depends only on Ω .

5. Regularity of Weak Solution

Theorem 7. Under the assumption H(3), then the weak solution \mathbf{E} of the System (4) is Hölder continuous in $\bar{\Omega}$ for any $\omega \in U_{ad}$, and there exists a constant $C_4 > 0$ such that

$$\|\mathbf{E}\|_{C^\alpha(\bar{\Omega})} \leq C_4,$$

where C_4 depends only on known data.

Proof. Let $\tilde{\mathbf{E}} = \mathbf{E} - \mathbf{G}$. From (4), we know that $\tilde{\mathbf{E}}$ satisfies the following equations

$$\begin{cases} \nabla \times \left(\frac{\gamma(x)}{\omega(x)} \nabla \times \tilde{\mathbf{E}}\right) + \zeta(x, \omega(x))\tilde{\mathbf{E}} = \mathbf{F}, & x \in \Omega, \\ \mathbf{n} \times \tilde{\mathbf{E}} = 0, & x \in S. \end{cases} \tag{12}$$

From the existence theorem (see [13] Theorem 3.2.) of solutions, the System (12) has a unique solution $\tilde{\mathbf{E}} \in H(\text{curl}, \Omega)$.

We prove that there is $\mu \in (1, 2)$ such that

$$\nabla \times \tilde{\mathbf{E}} \in L^{2,\mu}(\Omega),$$

$$\nabla \cdot \tilde{\mathbf{E}} \in L^{2,\mu}(\Omega),$$

and

$$\|\nabla \cdot \tilde{\mathbf{E}}\|_{L^{2,\mu}(\Omega)} + \|\nabla \times \tilde{\mathbf{E}}\|_{L^{2,\mu}(\Omega)} \leq C.$$

where C depends only on known data.

From the System (12), we know

$$\text{div}(\zeta \tilde{\mathbf{E}} - \mathbf{F}) = 0.$$

From Lemma 6, we know that there exists a potential vector field $\mathbf{V}(x)$ such that

$$\begin{cases} \zeta \tilde{\mathbf{E}} - \mathbf{F} = \nabla \times \mathbf{V}, & x \in \Omega, \\ \nabla \cdot \mathbf{V} = 0, & x \in \Omega, \\ \mathbf{n} \times \mathbf{V} = 0, & x \in S. \end{cases} \tag{13}$$

Moreover

$$\|\mathbf{V}\|_{H^1(\Omega)} + \|\mathbf{V}\|_{L^{2,2}(\Omega)} \leq C(\|\tilde{\mathbf{E}}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^2(\Omega)}),$$

where C depends only on Ω and the upper bound of ζ .

Substitute $\mathbf{V}(x)$ into Equations (12), and we have

$$\begin{cases} \nabla \times \left(\frac{\gamma(x)}{\omega(x)} \nabla \times \tilde{\mathbf{E}} - \mathbf{V}\right) = 0, & x \in \Omega, \\ \mathbf{n} \times \tilde{\mathbf{E}} = 0, & x \in S, \\ \mathbf{n} \cdot \mathbf{V} = 0, & x \in S. \end{cases} \tag{14}$$

From the boundary condition $\mathbf{n} \times \tilde{\mathbf{E}} = 0$, we have

$$\mathbf{n} \cdot (\nabla \times \tilde{\mathbf{E}}) = 0, \quad x \in S.$$

For any smooth function $\Psi \in C^\infty(\Omega)$, we can use Gauss's divergence theorem to get

$$\begin{aligned} \int_{\Omega} \mathbf{n} \cdot (\nabla \times \tilde{\mathbf{E}}) \Psi ds &= \int_{\Omega} \nabla \cdot ((\nabla \times \tilde{\mathbf{E}}) \Psi) dx \\ &= \int_{\Omega} (\nabla \times \tilde{\mathbf{E}}) \cdot \nabla \Psi dx = \int_S (\mathbf{n} \times \tilde{\mathbf{E}}) \cdot \nabla \Psi ds \\ &= 0. \end{aligned}$$

Hence,

$$\mathbf{n} \cdot (\nabla \times \tilde{\mathbf{E}}) = 0, \quad x \in \partial\Omega.$$

Since Ω is simply-connected, there exists the potential function $\Psi(x)$. The following equation holds

$$\frac{\gamma(x)}{\omega(x)} \nabla \times \tilde{\mathbf{E}} - \mathbf{V} = \nabla \Psi, \quad x \in \Omega. \tag{15}$$

Therefore,

$$\nabla \times \tilde{\mathbf{E}} = \frac{\omega(x)}{\gamma(x)} [\nabla \Psi + \mathbf{V}], \quad x \in \Omega. \tag{16}$$

By calculation

$$\begin{cases} \nabla \cdot [\frac{\omega(x)}{\gamma(x)}(\nabla\Psi + \mathbf{V})] = 0, & x \in \Omega, \\ \mathbf{n} \cdot [\frac{\gamma(x)}{\omega(x)}\nabla \times \tilde{\mathbf{E}}] = 0, & x \in S, \\ \mathbf{n} \cdot \mathbf{V} = 0, & x \in S, \end{cases} \tag{17}$$

that is

$$\mathbf{n} \cdot (\nabla\Psi) = \mathbf{n} \cdot (\frac{\omega(x)}{\gamma(x)}\nabla \times \tilde{\mathbf{E}} - \mathbf{V}) = 0, \quad x \in \Omega.$$

From $\frac{\gamma(x)}{\omega(x)}$ has a positive lower bound, we use $L^2(\Omega)$ -theory of elliptic equations to get

$$\|\Psi\|_{H^1(\Omega)} \leq C\|\mathbf{V}\|_{L^2(\Omega)}.$$

where C depends only on known data.

According to the estimate of Campanato space $L^{2,\mu}(\Omega)$, we see that there exist $\mu \in (1, 2)$ such that

$$\|\nabla\Psi\|_{L^{2,\mu}(\Omega)} \leq C(\|\mathbf{V}\|_{L^{2,\mu}(\Omega)} + \|\Psi\|_{H^1(\Omega)}).$$

which is bounded from the estimate for $V(x)$.

From the Equations (15), one can see that

$$\|\nabla \times \tilde{\mathbf{E}}\|_{L^{2,\mu}(\Omega)} \leq C\|\nabla\Psi\|_{L^{2,\mu}(\Omega)}.$$

On the other hand, from (13) we know

$$\nabla(\zeta(x)\tilde{\mathbf{E}}) = 0.$$

From the boundedness of $\zeta(x)$, we have

$$\int_{B_\rho} |\nabla \cdot \tilde{\mathbf{E}}|^2 dx \leq C \int_{B_\rho} |\tilde{\mathbf{E}}|^2 dx,$$

for $\forall x \in \Omega$ and $\rho > 0$, where B_ρ is $B_\rho \cap \Omega$ and $\tilde{\mathbf{E}} \in H^1(\Omega)$.

From Lemma 4, we know that $\tilde{\mathbf{E}} \in L^{2,2}(\Omega)$ and

$$\sup_{\rho>0, x_0 \in \Omega} \rho^{-2} \int_{B_\rho} |\nabla \cdot \tilde{\mathbf{E}}| dx \leq C.$$

From what has been discussed above, we get

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{E}} &\in L^{2,\mu} \\ \nabla \times \tilde{\mathbf{E}} &\in L^{2,\mu} \end{aligned}$$

and

$$\|\nabla \cdot \tilde{\mathbf{E}}\|_{L^{2,\mu}(\Omega)} + \|\nabla \times \tilde{\mathbf{E}}\|_{L^{2,\mu}(\Omega)} \leq C\|\mathbf{F}\|_{L^{2,\mu}(\Omega)},$$

where C only depends on the known data constant, i.e.,

$$\|\tilde{\mathbf{E}}\|_{L^{2,2+\mu}(\Omega)} \leq C\|\mathbf{F}\|_{L^{2,\mu}(\Omega)}.$$

From Lemma 4, the space dimension $n = 3$ and $\mu_0 = 2 + \mu > 3$. We see that

$$\tilde{\mathbf{E}} \in C^\alpha(\bar{\Omega}),$$

where $\alpha = \frac{\mu-n}{2} > 0$. Hence $\tilde{\mathbf{E}}$ is Hölder continuous.

Let $\tilde{\mathbf{E}} = \mathbf{E} - \mathbf{G}$, we have $\mathbf{E} = \tilde{\mathbf{E}} + \mathbf{G}$. From the hypothesis of \mathbf{G} in assumption H(3), we know that

$$\mathbf{E} \in C^\alpha(\bar{\Omega}).$$

□

Next we study the regularity of the heat conduction equation. Let

$$Q(x_0, t_0; \rho) = B(x_0, \rho) \times (t_0 - \rho, t_0], \quad (x_0, t_0) \in Q_T,$$

for $B(x_0, \rho)$. One has a similar result to Lemma 4 with dimension n replaced by $n + 1$.

We consider the following parabolic equation

$$\begin{cases} u_t - L[u] = f_0(x, t) + \sum_{i=1}^n (f_i(x, t))_{x_i}, & (x, t) \in Q_T, \\ u_n(x, t) = 0, & (x, t) \in S_T, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

where

$$L[u] = (a_{ij}(x, t)u_{x_i})_{x_j} + b_i(x, t)u_{x_i} + c(x, t)u.$$

The following basic assumptions are needed.

H(4).

- (a) Let $a_{ij}(x, t)$ be measurable in Q_T and satisfy the following conditions for ellipticity:

$$a_1|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq a_2|\xi|^2, \quad \xi \in R^n, a_1, a_2 > 0.$$

- (b) Let $b_i(x, t)$ and $c_i(x, t)$ belong to $L^\infty(Q_T)$ with

$$\sum_{i=1}^n \|b_i\|_{L^\infty(Q_T)} + \|c\|_{L^\infty(Q_T)} \leq A_3.$$

- (c) Let

$$\begin{aligned} f_0(x, t) &\in L^{2,(\mu-2)^+}(Q_T), \\ f_i(x, t) &\in L^{2,\mu}(Q_T), \end{aligned}$$

for some $\mu > 0$.

Lemma 8. (see [17]) Under the assumption H(4), the solution $u \in C([0, T]; H^1(\Omega))$ of the general parabolic equation satisfies the following estimate:

$$\|\nabla u\|_{L^{2,\mu}(Q_T)} \leq C(\|f_0\|_{L^{2,(\mu-2)^+}(Q_T)} + \sum_{i=1}^n \|f_i\|_{L^{2,\mu}(Q_T)}),$$

where C is a constant depending only on known data a_1, a_2, A_3 and Q_T . In particular, there is

$$u \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q}_T),$$

for $\mu \in (n, n + 2)$, where $\alpha = \frac{\mu-n}{2}$.

Remark 9. The significance of Lemma 8 is that when $\mu \in (n + 2, n + 4)$, the condition f_0, f_i for the continuity of Hölder of the weak solution are weaker than that of the classical result. (see [18–20]).

Theorem 10. Under the assumptions of H(1), H(2), H(3) and H(4), There exists unique weak solution (\mathbf{E}, u) for the coupled System (1) with

$$\begin{aligned} \mathbf{E} - \mathbf{G} &\in L^\infty(0, T; H_0(\text{curl}, \Omega)), \\ u &\in C([0, T]; H^1(\Omega)). \end{aligned}$$

Moreover, the weak solution possesses the following regularity:

$$\begin{aligned} \nabla \times \mathbf{E} &\in L^\infty(0, T; L^{2,\mu_0}(\Omega)), \\ \nabla u &\in L^{2,\mu_0+2}(Q_T). \end{aligned}$$

In particular,

$$u \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q}_T),$$

where $\alpha = \frac{\mu_0 - 1}{2}$.

Proof. From Theorem 3.6. of literature [13], we know that there exists the solution (\mathbf{E}, u) for coupled system (1) with

$$\begin{aligned} \mathbf{E} - \mathbf{G} &\in L^\infty(0, T; H_0(\text{curl}, \Omega)), \\ u &\in C([0, T]; H^1(\Omega)). \end{aligned}$$

We assume that $\mathbf{G}(x, t) = 0$ on $\partial\Omega$. Otherwise, we define

$$\hat{\mathbf{E}} = \mathbf{E} - \mathbf{G}.$$

From Theorem 7, we have

$$\mathbf{E} \in C^\alpha(\bar{\Omega}),$$

for any given $t \in [0, T]$. Moreover

$$\begin{aligned} \|\mathbf{E}(\cdot, t)\|_{C^\alpha(\bar{\Omega})} &\leq C, \\ \|\nabla \times \mathbf{E}\|_{L^{2,2+\mu}(\bar{\Omega})} &\leq C. \end{aligned}$$

Now we think that $k(x, u)$ is independent of u . Let $k(x, u) = \tilde{k}(x)$, we use $L^{2,\mu}$ theory of parabolic equation and Lemma 8. We have

$$\|\nabla u\|_{L^{2,2+\mu_0}(Q_T)} \leq C \|\nabla \times \mathbf{E}\|_{L^{2,2+\mu}(Q_T)}.$$

From the standard embedding (see [19]), we have

$$u \in L^{2,\mu+4}(Q_T).$$

For $\mu \in (1, 2)$, we have $\mu + 4 \in (5, 7)$. Therefore

$$u \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q}_T),$$

where $\alpha = \frac{\mu - 1}{2}$.

In the case of $k = k(x, u)$, we have

$$u \in C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_T).$$

Since $k(x, u)$ is smooth, we can use Schauder's theorem to get the following result

$$u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q}_T).$$

□

Remark 11. *The necessary optimality conditions for the optimal control problem in the microwave heating process will be carried out in a separate paper.*

6. Conclusions

In this paper, we study the regularity of the weak solution of the coupled system derived from the microwave heating model with frequency variable. The distribution of dielectric, permeability, and electric conductivity functions are temperature independent. The frequency of microwave is dependent only on its position x , i.e., $\omega = \omega(x)$, and has nothing to do with time. We established that the temperature is Hölder continuous, even if electric conductivity has a jump discontinuity with respect to the temperature change. This paper presents a method that is based on the estimation of linear degenerate system in the Campanato space. We use this idea to deal with time harmonic Maxwell equations with rough coefficients. On the other hand, by using the regularity results of the coupled system, the necessary optimality conditions for uniform microwave heating in three directions are derived.

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