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The Generalized Viscosity Implicit Midpoint Rule for Nonexpansive Mappings in Banach Space

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Abstract: This paper constructs the generalized viscosity implicit midpoint rule for nonexpansive mappings in Banach space. It obtains strong convergence conclusions for the proposed algorithm and promotes the related results in this field. Moreover, this paper gives some applications. Finally, the paper gives six numerical examples to support the main results.

Keywords: generalized viscosity implicit midpoint rule; nonexpansive mapping; Banach space; strong convergence

1. Introduction

Let E be a Banach Space and E^* the dual space. J denotes the normalized duality mapping from E to 2^{E^*} and is defined by

$$J(x) = \left\{ f \in E^* :< x, f >= \|x\|^2 = \|f\|^2 \right\},\ x \in E.$$

It is well known that if E is a Hilbert space, then J is the identity mapping; if E is a smooth Banach space, then J is single-valued and denoted by j. More information on the normalized duality mapping can be found, for example, in [1,2].

Let C be a nonempty set of E. Mapping of $f : C \to C$ is contractive if $\|f(x) - f(y)\| \le \alpha\|x - y\|$, $\forall x, y \in C$, and $\alpha \in [0, 1)$. Mapping of $T : C \to C$ is nonexpansive if $\|T(x) - T(y)\| \le \|x - y\|$ and $\forall x, y \in C$. Let $F(T)$ denote the fixed point set of T. More information on nonexpansive mappings and their fixed points can be found, for example, in [3].

The implicit midpoint rule can effectively solve ordinary differential equations (see [4–9] and the references therein). Meanwhile, many authors have used viscosity iterative algorithms for finding common fixed points for nonlinear operators and solutions of variational inequality problems (see [10–17] and the references therein).

In 2004, Xu [10] proposed the explicit viscosity method for nonexpansive mappings in Hilbert space or uniformly smooth Banach space:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n,\ n \ge 0.$$

$\{x_n\}$ was generated by the above iterative algorithm and strongly converged to $q \in F(T)$, where q was the solution of variational inequality $\langle (I - f)q, x - q\rangle \ge 0$ and $x \in F(T)$.

In 2015, Xu et al. [18] constructed the viscosity implicit midpoint rule for nonexpansive mapping in Hilbert space:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right),\ n \ge 0.$$

$\{x_n\}$ was generated by the above. Under many conditions of $\{\alpha_n\}$, $\{x_n\}$ strongly converged to $q \in F(T)$, where q was the solution of variational inequality $\langle (I-f)q, x-q\rangle \geq 0$ and $x \in F(T)$.

In 2015, Ke et al. [19] improved the results of Xu et al. [18] from the viscosity implicit midpoint rule to generalized viscosity implicit rules for nonexpansive mappings in Hilbert spaces:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)x_{n+1}),\ n \geq 0 \tag{1}$$

and

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}),\ n \geq 0. \tag{2}$$

Under some conditions of $\{\alpha_n\}$, they proved that $\{x_n\}$ generated by (1) and (2) all strongly converged to $q \in F(T)$, where q was the solution of variational inequality $\langle (I-f)q, x-q\rangle \geq 0$ and $x \in F(T)$.

In 2017, Luo et al. [20] generalized the conclusions of Xu et al. [18] from Hilbert space to uniformly smooth Banach space:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\Big(\frac{x_n + x_{n+1}}{2}\Big),\ n \geq 0.$$

$\{x_n\}$ was generated by the above. Under many conditions, $\{x_n\}$ strongly converged to $q \in F(T)$, where q was the solution of variational inequality $\langle (I-f)q, j(x-q)\rangle \geq 0$ and $x \in F(T)$.

Motivated and inspired by Xu et al. [18], Ke et al. [19], and Luo et al. [20], this paper proposes the generalized viscosity implicit rules for nonexpansive mappings in Banach space and proves strong convergence results. Next, this paper applies the results to a general system of variational inequality problems in Banach space and fixed-point problems of strict pseudocontractive mappings. Finally, this paper gives numerical examples to support the main results.

2. Preliminaries

Let E be a Banach space. A mapping $\rho_E : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(t) = \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x \in S(E), \|y\| \leq t\Big\},$$

where $S(E)$ is the modulus of smoothness of E. E is uniformly smooth if $\lim\limits_{t\to 0}\frac{\rho_E(t)}{t} = 0$. For example, $L^p(p>1)$ is uniformly smooth Banach space. E is q-uniformly smooth if $c > 0$ such that $\rho_E(t) \leq ct^q$.

We needed the following lemmas to prove our main results.

Lemma 1. ([10,21]) *Assume $\{a_n\}$ is a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

$$\text{(i)} \sum_{n=0}^{\infty}\alpha_n = \infty;\ \text{(ii)} \limsup_{n\to\infty}\frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty}|\delta_n| < \infty$$

Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2. ([10,22]) *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \varnothing$, and $f : C \to C$ be a contractive mapping. Then the sequence $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ converges strongly to a point in $F(T)$. If we define a mapping $Q : \Pi_C \to F(T)$ by $Q(f) := \lim\limits_{t\to 0} x_t$, where Π_C is the set of contractive mapping from C to itself, then $Q(f)$ solves the following variational inequality:*

$$\langle (I-f)Q(f), j(Q(f)-p)\rangle \leq 0,\ \forall f \in \Pi_C, p \in F(T).$$

Lemma 3. ([12]) *Let C be a nonempty, closed convex subset of a real Banach space E, which has a uniformly Gâteaux differentiable norm, and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \varnothing$. Assume that $\{z_t\}$ strongly converges to a fixed point z of T as $t \to 0$, where $\{z_t\}$ is defined by $z_t = tf(z_t) + (1-t)Tz_t$. Suppose $\{x_n\} \subset C$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $\limsup_{n\to\infty} \langle f(z) - z, j(x_{n+1} - z)\rangle \leq 0$.*

3. Main Results

Theorem 1. *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $f : C \to C$ be a contractive mapping with $\alpha \in [0, 1)$, and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \varnothing$. $\{x_n\}$ is generated by the generalized viscosity implicit midpoint rule*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)x_{n+1}),\ n \geq 0, \tag{3}$$

where $\{\alpha_n\}, \{s_n\} \subset (0, 1)$ and satisfies the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ *or* $\sum_{n=1}^{\infty} \left|\alpha_{n+1} - \alpha_n\right| < \infty$;

(iv) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1,\ \forall n \geq 0$.

Then $\{x_n\}$ converges strongly to $q \in F(T)$, which is also the unique solution of the variational inequality $\langle (I - f)q, j(x - q)\rangle \geq 0$, and $x \in F(T)$.

Proof. The proof is split into five steps.

Step 1: Show that $\{x_n\}$ is bounded.
Take $p \in F(T)$, then we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|T(s_n x_n + (1-s_n)x_{n+1}) - p\| \\
&\leq \alpha\alpha_n \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)s_n \|x_n - p\| \\
&\quad + (1-\alpha_n)(1-s_n)\|x_{n+1} - p\| \\
&= [(1-\alpha_n)s_n + \alpha\alpha_n]\|x_n - p\| + (1-\alpha_n)(1-s_n)\|x_{n+1} - p\| \\
&\quad + \alpha_n \|f(p) - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \tfrac{(1-\alpha_n)s_n + \alpha\alpha_n}{1-(1-\alpha_n)(1-s_n)} \|x_n - p\| + \tfrac{\alpha_n}{1-(1-\alpha_n)(1-s_n)} \|f(p) - p\| \\
&= \left[1 - \tfrac{(1-\alpha)\alpha_n}{1-(1-\alpha_n)(1-s_n)}\right]\|x_n - p\| + \tfrac{(1-\alpha)\alpha_n}{1-(1-\alpha_n)(1-s_n)} \tfrac{\|f(p)-p\|}{(1-\alpha)}.
\end{aligned}$$

Then we get $\|x_{n+1} - p\| \leq \max\left\{\|x_n - p\|, \frac{\|f(p)-p\|}{(1-\alpha)}\right\}$. By induction, we get

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{(1-\alpha)}\right\},\ \forall n \geq 0.$$

So, $\{x_n\}$ is bounded. Then $\{f(x_n)\}$ and $\{T(s_n x_n + (1-s_n)x_{n+1})\}$ are also bounded.

Step 2: Show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

By (3), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n f(x_n)+(1-\alpha_n)T(s_nx_n+(1-s_n)x_{n+1})-\alpha_{n-1}f(x_{n-1})\\ &\quad -(1-\alpha_{n-1})T(s_{n-1}x_{n-1}+(1-s_{n-1})x_n)\|\\ &\le \alpha_n\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\cdot\|f(x_{n-1})-T(s_{n-1}x_{n-1}+(1-s_{n-1})x_n)\|\\ &\quad +(1-\alpha_n)\|T(s_nx_n+(1-s_n)x_{n+1})-T(s_{n-1}x_{n-1}+(1-s_{n-1})x_n)\|\\ &\le \alpha\alpha_n\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M+(1-\alpha_n)\|[s_nx_n+(1-s_n)x_{n+1}]\\ &\quad -[s_{n-1}x_{n-1}+(1-s_{n-1})x_n]\|\\ &\le \alpha\alpha_n\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M+(1-\alpha_n)(1-s_n)\|x_{n+1}-x_n\|\\ &\quad +(1-\alpha_n)s_{n-1}\|x_n-x_{n-1}\|\\ &= (1-\alpha_n)(1-s_n)\|x_{n+1}-x_n\|+[\alpha\alpha_n+(1-\alpha_n)s_{n-1}]\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M,\end{aligned}$$

where $M\ge\sup_{n\ge0}\|f(x_n)-T(s_nx_n+(1-s_n)x_{n+1})\|$.

It follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \tfrac{\alpha\alpha_n+(1-\alpha_n)s_{n-1}}{1-(1-\alpha_n)(1-s_n)}\|x_n-x_{n-1}\|+\tfrac{M}{1-(1-\alpha_n)(1-s_n)}|\alpha_n-\alpha_{n-1}|\\ &= \left[1-\tfrac{(1-\alpha)\alpha_n+(1-\alpha_n)(s_n-s_{n-1})}{1-(1-\alpha_n)(1-s_n)}\right]\|x_n-x_{n-1}\|+\tfrac{M}{1-(1-\alpha_n)(1-s_n)}|\alpha_n-\alpha_{n-1}|.\end{aligned}$$

From $0<\varepsilon\le s_n\le s_{n+1}<1$, we have $0<\varepsilon\le s_n\le 1-(1-\alpha_n)(1-s_n)<1$ and $\frac{(1-\alpha)\alpha_n+(1-\alpha_n)(s_n-s_{n-1})}{1-(1-\alpha_n)(1-s_n)}\ge(1-\alpha)\alpha_n$. Then we get

$$\|x_{n+1}-x_n\|\le[1-(1-\alpha)\alpha_n]\|x_n-x_{n-1}\|+\frac{M}{\varepsilon}|\alpha_n-\alpha_{n-1}|.$$

From $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\sum_{n=1}^{\infty}\left|\alpha_{n+1}-\alpha_n\right|<\infty$, and Lemma 1, we get $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Step 3: Show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

$$\begin{aligned}\|x_n-Tx_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T(s_nx_n+(1-s_n)x_{n+1})\|+\|T(s_nx_n+(1-s_n)x_{n+1})-Tx_n\|\\ &\le \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-T(s_nx_n+(1-s_n)x_{n+1})\|+(1-s_n)\|x_n-x_{n+1}\|\\ &\le (2-s_n)\|x_n-x_{n+1}\|+\alpha_nM\\ &\le 2\|x_n-x_{n+1}\|+\alpha_nM.\end{aligned}$$

From $\lim_{n\to\infty}\alpha_n=0$ and Step 2, we get $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Step 4: Show that $\limsup_{n\to\infty}\langle f(q)-q,j(x_{n+1}-q)\rangle\le0$.

Let $\{x_t\}$ be defined by $x_t=tf(x_t)+(1-t)Tx_t$. Then, from Lemma 2, $\{x_t\}$ converges strongly to $q\in F(T)$, which is also the unique solution of the variational inequality $\langle(I-f)q,j(x-q)\rangle\ge0$, $x\in F(T)$. From Steps 1–3 and Lemma 3, we get $\limsup_{n\to\infty}\langle f(q)-q,j(x_{n+1}-q)\rangle\le0$.

Step 5: Show that $\lim_{n\to\infty} x_n = q$.

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n(f(x_n)-q)+(1-\alpha_n)(T(s_nx_n+(1-s_n)x_{n+1})-q)\|^2 \\
&= (1-\alpha_n)\langle T(s_nx_n+(1-s_n)x_{n+1})-q, j(x_{n+1}-q)\rangle + \alpha_n\langle f(x_n)-q, j(x_{n+1}-q)\rangle \\
&\le (1-\alpha_n)\|s_n(x_n-q)+(1-s_n)(x_{n+1}-q)\|\cdot\|x_{n+1}-q\| + \alpha\alpha_n\|x_n-q\|\cdot\|x_{n+1}-q\| \\
&\quad +\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&\le (1-\alpha_n)s_n\|x_n-q\|\cdot\|x_{n+1}-q\| + (1-\alpha_n)(1-s_n)\|x_{n+1}-q\|^2 + \alpha\alpha_n\|x_n-q\|\cdot\|x_{n+1}-q\| \\
&\quad +\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&= [(1-\alpha_n)s_n+\alpha\alpha_n]\|x_n-q\|\cdot\|x_{n+1}-q\| + (1-\alpha_n)(1-s_n)\|x_{n+1}-q\|^2 \\
&\quad +\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&\le \tfrac{(1-\alpha_n)s_n+\alpha\alpha_n}{2}\left(\|x_n-q\|^2+\|x_{n+1}-q\|^2\right) + (1-\alpha_n)(1-s_n)\|x_{n+1}-q\|^2 \\
&\quad +\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\le \tfrac{(1-\alpha_n)s_n+\alpha\alpha_n}{2-[(1-\alpha_n)s_n+\alpha\alpha_n]-2(1-\alpha_n)(1-s_n)}\|x_n-q\|^2 \\
&\quad + \tfrac{2\alpha_n}{2-[(1-\alpha_n)s_n+\alpha\alpha_n]-2(1-\alpha_n)(1-s_n)}\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&= \left[1-\tfrac{2(1-\alpha)\alpha_n}{2-[(1-\alpha_n)s_n+\alpha\alpha_n]-2(1-\alpha_n)(1-s_n)}\right]\|x_n-q\|^2 \\
&\quad + \tfrac{2\alpha_n(1-\alpha)}{2-[(1-\alpha_n)s_n+\alpha\alpha_n]-2(1-\alpha_n)(1-s_n)}\,\tfrac{\langle f(q)-q, j(x_{n+1}-q)\rangle}{1-\alpha}.
\end{aligned}
$$

Because

$$
\frac{2\alpha_n(1-\alpha)}{2-[(1-\alpha_n)s_n+\alpha\alpha_n]-2(1-\alpha_n)(1-s_n)} = \frac{2\alpha_n(1-\alpha)}{\alpha_n(2-\alpha)+(1-\alpha_n)s_n} > \frac{2-2\alpha}{3-\alpha}\alpha_n,
$$

so, from $\sum_{n=0}^{\infty}\alpha_n = \infty$, Step 4, and Lemma 1, we get $\lim_{n\to\infty} x_n = q$. This completes the proof. □

It is well known that Hilbert space is uniformly smooth Banach space. So, we can get the main results of [19].

Corollary 1. ([19]) *Let C be a nonempty, closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$
x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_nx_n+(1-s_n)x_{n+1}),
$$

where $\{\alpha_n\}, \{s_n\} \subset (0,1)$ that satisfies the following conditions:

(1) $\lim_{n\to\infty}\alpha_n = 0$;

(2) $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(3) $\sum_{n=1}^{\infty}\left|\alpha_{n+1}-\alpha_n\right| < \infty$;

(4) $0 < \varepsilon \le s_n \le s_{n+1} < 1, \forall n \ge 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the unique solution of the variational inequality $\langle (I-f)x, y-x\rangle \ge 0, \forall y \in F(T)$.

If we let $s_n = \frac{1}{2}$, we can get the main results of [20].

Corollary 2. ([20]) *Let C be a closed convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and $f : C \to C$ a contraction with coefficient $\alpha \in [0, 1)$. Let $\{x_n\}$ be a sequence generated by the following viscosity implicit midpoint rule:*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T\Big(\frac{x_n + x_{n+1}}{2}\Big),\ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *either* $\sum_{n=1}^{\infty} \big|\alpha_{n+1} - \alpha_n\big| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then $\{x_n\}$ converges strongly to a fixed point of T, which also solves the following variational inequality: $\langle (I - f)q, j(x - q)\rangle \geq 0$, $x \in F(T)$.

Theorem 2. *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $f : C \to C$ be a contractive mapping with $\alpha \in [0, 1)$ and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. $\{x_n\}$ is generated by the generalized viscosity implicit midpoint rule*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}) + e_n,\ n \geq 0, \tag{4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0, 1)$, $\{e_n\} \subset E$ and satisfies the conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty} \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\sum_{n=1}^{\infty} \big|\alpha_{n+1} - \alpha_n\big| < \infty$ *and* $\sum_{n=1}^{\infty} \big|\beta_{n+1} - \beta_n\big| < \infty$;

(iv) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$, $\forall n \geq 0$;

(v) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\|e_n\| = o(\beta_n)$.

Then $\{x_n\}$ converges strongly to $q \in F(T)$, which is also the unique solution of the variational inequality $\langle (I - f)q, j(x - q)\rangle \geq 0$, and $x \in F(T)$.

Proof. The proof is split into five steps.

Step 1: Show that $\{x_n\}$ is bounded.
Take $p \in F(T)$, then we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}) + e_n - p\| \\ &\leq \alpha_n \|x_n - p\| + \alpha\beta_n \|x_n - p\| + \beta_n \|f(p) - p\| + \gamma_n s_n \|x_n - p\| \\ &\quad + \gamma_n (1 - s_n)\|x_{n+1} - p\| + \|e_n\| \\ &= (\alpha_n + \alpha\beta_n + \gamma_n s_n)\|x_n - p\| + \gamma_n (1 - s_n)\|x_{n+1} - p\| + \beta_n \|f(p) - p\| + \|e_n\|. \end{aligned}$$

It follows that

$$[1 - \gamma_n (1 - s_n)]\|x_{n+1} - p\| \leq (\alpha_n + \alpha\beta_n + \gamma_n s_n)\|x_n - p\| + \beta_n \|f(p) - p\| + \|e_n\|.$$

From $\alpha_n + \beta_n + \gamma_n = 1$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\le \tfrac{\alpha_n + \alpha\beta_n + \gamma_n s_n}{1-\gamma_n(1-s_n)}\|x_n - p\| + \tfrac{\beta_n}{1-\gamma_n(1-s_n)}\|f(p) - p\| + \tfrac{\|e_n\|}{1-\gamma_n(1-s_n)} \\ &\le \left[1 - \tfrac{1-\alpha_n-\gamma_n-\alpha\beta_n}{1-\gamma_n(1-s_n)}\right]\|x_n - p\| + \tfrac{\beta_n}{1-\gamma_n(1-s_n)}\|f(p) - p\| + \tfrac{\|e_n\|}{\varepsilon} \\ &= \left[1 - \tfrac{\beta_n(1-\alpha)}{1-\gamma_n(1-s_n)}\right]\|x_n - p\| + \tfrac{\beta_n(1-\alpha)}{1-\gamma_n(1-s_n)}\tfrac{\|f(p)-p\|}{(1-\alpha)} + \tfrac{\|e_n\|}{\varepsilon} \\ &\le \max\left\{\|x_0 - p\|, \tfrac{\|f(p)-p\|}{(1-\alpha)} + \tfrac{\|e_n\|}{\varepsilon}\right\}. \end{aligned}$$

So, $\{x_n\}$ is bounded. Then $\{f(x_n)\}$ and $\{T(s_n x_n + (1-s_n)x_{n+1})\}$ are also bounded.
Step 2: Show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.
By (4), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}) + e_n - [\alpha_{n-1}x_{n-1} + \beta_{n-1}f(x_{n-1}) \\ &\quad + \gamma_{n-1}T(s_{n-1}x_{n-1} + (1-s_{n-1})x_n) + e_{n-1}]\| \\ &= \|\alpha_n(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})x_{n-1} + \beta_n[f(x_n) - f(x_{n-1})] + (\beta_n - \beta_{n-1})f(x_{n-1}) \\ &\quad + \gamma_n[T(s_n x_n + (1-s_n)x_{n+1}) - T(s_{n-1}x_{n-1} + (1-s_{n-1})x_n)] \\ &\quad - [(\alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1})]T(s_{n-1}x_{n-1} + (1-s_{n-1})x_n) + e_n - e_{n-1}\| \\ &\le \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|x_{n-1} - T(s_{n-1}x_{n-1} + (1-s_{n-1})x_n)\| \\ &\quad + \beta_n\|f(x_n) - f(x_{n-1})\| + \left|\beta_n - \beta_{n-1}\right| \cdot \|f(x_{n-1}) - T(s_{n-1}x_{n-1} + (1-s_{n-1})x_n)\| \\ &\quad + \gamma_n\|T(s_n x_n + (1-s_n)x_{n+1}) - T(s_{n-1}x_{n-1} + (1-s_{n-1})x_n)\| + \|e_n - e_{n-1}\| \\ &\le \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1 + \alpha\beta_n\|x_n - x_{n-1}\| + \left|\beta_n - \beta_{n-1}\right|M_1 \\ &\quad + \gamma_n\|(1-s_n)(x_{n+1} - x_n) + s_{n-1}(x_n - x_{n-1})\| + \|e_n - e_{n-1}\| \\ &\le \gamma_n(1-s_n)\|x_{n+1} - x_n\| + (\alpha_n + \alpha\beta_n + \gamma_n s_{n-1})\|x_n - x_{n-1}\| \\ &\quad + (|\alpha_n - \alpha_{n-1}| + \left|\beta_n - \beta_{n-1}\right|)M_1 + 2\|e_n\|, \end{aligned}$$

where $M_1 \ge \max\left\{\sup_{n\ge0}\|x_n - T(s_n x_n + (1-s_n)x_{n+1})\|, \sup_{n\ge0}\|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\|\right\}$.

It follows that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \tfrac{\alpha_n + \alpha\beta_n + \gamma_n s_{n-1}}{1-\gamma_n(1-s_n)}\|x_n - x_{n-1}\| + \tfrac{M_1}{1-\gamma_n(1-s_n)}\left(|\alpha_n - \alpha_{n-1}| + \left|\beta_n - \beta_{n-1}\right|\right) + 2\|e_n\| \\ &= \left[1 - \tfrac{(1-\alpha)\beta_n + \gamma_n(s_n - s_{n-1})}{1-\gamma_n(1-s_n)}\right]\|x_n - x_{n-1}\| + \tfrac{M_1}{1-\gamma_n(1-s_n)}\left(|\alpha_n - \alpha_{n-1}| + \left|\beta_n - \beta_{n-1}\right|\right) + 2\|e_n\| \end{aligned}$$

From $0 < \varepsilon \le s_n \le s_{n+1} < 1$, we have $0 < \varepsilon \le s_n \le 1 - (1-\alpha_n)(1-s_n) < 1$ and $\frac{(1-\alpha)\beta_n + \gamma_n(s_n - s_{n-1})}{1-\gamma_n(1-s_n)} \ge (1-\alpha)\beta_n$. Then we get

$$\|x_{n+1} - x_n\| \le [1 - (1-\alpha)\beta_n]\|x_n - x_{n-1}\| + \frac{M_1}{\varepsilon}\left(|\alpha_n - \alpha_{n-1}| + \left|\beta_n - \beta_{n-1}\right|\right) + 2\|e_n\|.$$

From $\sum_{n=0}^{\infty}\beta_n = \infty$, $\sum_{n=1}^{\infty}\left|\alpha_{n+1} - \alpha_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\beta_{n+1} - \beta_n\right| < \infty$, $\sum_{n=1}^{\infty}\|e_n\| < \infty$, and Lemma 1 we get $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3: Show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T(s_nx_n + (1-s_n)x_{n+1})\| + \|T(s_nx_n + (1-s_n)x_{n+1}) - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|x_n - T(s_nx_n + (1-s_n)x_{n+1})\| + (1-s_n)\|x_n - x_{n+1}\| \\ &\quad +\beta_n\|f(x_n) - T(s_nx_n + (1-s_n)x_{n+1})\| + \|e_n\| \\ &\le (2-s_n)\|x_n - x_{n+1}\| + (\alpha_n + \beta_n)M_1 + \|e_n\| \\ &\le 2\|x_n - x_{n+1}\| + (1-\gamma_n)M_1 + \|e_n\|.\end{aligned}$$

From $\lim_{n\to\infty}\gamma_n = 1$, $\sum_{n=1}^{\infty}\|e_n\| < \infty$ and Step 2, we get $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Step 4: Show that $\limsup_{n\to\infty}\langle f(q) - q, j(x_{n+1} - q)\rangle \le 0$.

Let $\{x_t\}$ be defined by $x_t = tf(x_t) + (1-t)Tx_t$. Then, from Lemma 2, $\{x_t\}$ converges strongly to $q \in F(T)$, which is also the unique solution of the variational inequality $\langle (I-f)q, j(x-q)\rangle \ge 0$ and $x \in F(T)$. From Steps 1–3 and Lemma 3, we get $\limsup_{n\to\infty}\langle f(q) - q, j(x_{n+1} - q)\rangle \le 0$.

We have

$$\begin{aligned}\|x_t - x_n\|^2 &= (1-t)\langle Sx_t - Sx_n + Sx_n - x_n, J(x_t - x_n)\rangle + t\langle fx_t - x_t + x_t - x_n, J(x_t - x_n)\rangle \\ &\le (1-t)\|x_t - x_n\|^2 + (1-t)\|Sx_n - x_n\|\cdot\|x_t - x_n\| \\ &\quad +t\|x_t - x_n\|^2 + t\langle fx_t - x_t, J(x_t - x_n)\rangle \\ &= \|x_t - x_n\|^2 + (1-t)\|Sx_n - x_n\|\cdot\|x_t - x_n\| + t\|x_t - x_n\|^2 + t\langle fx_t - x_t, J(x_t - x_n)\rangle.\end{aligned}$$

It follows that $\langle fx_t - x_t, J(x_t - x_n)\rangle \le \frac{1-t}{t}\|Sx_n - x_n\|\cdot\|x_t - x_n\|$. From Step 3, we get $\limsup_{n\to\infty}\langle f(q) - q, j(x_{n+1} - q)\rangle \le 0$.

Step 5: Show that $\lim_{n\to\infty} x_n = q$.

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\alpha_n(x_n - q) + \beta_n(f(x_n) - q) + \gamma_n(T(s_nx_n + (1-s_n)x_{n+1}) - q) + e_n\|^2 \\ &= \alpha_n\langle x_n - q, j(x_{n+1} - q)\rangle + \beta_n\langle f(x_n) - q, j(x_{n+1} - q)\rangle \\ &\quad +\gamma_n\langle T(s_nx_n + (1-s_n)x_{n+1}) - q, j(x_{n+1} - q)\rangle + \langle e_n, j(x_{n+1} - q)\rangle \\ &\le \alpha_n\|x_n - q\|\cdot\|x_{n+1} - q\| + \alpha\beta_n\|x_n - q\|\cdot\|x_{n+1} - q\| + \beta_n\langle f(q) - q, j(x_{n+1} - q)\rangle \\ &\quad +\gamma_n\|s_n(x_n - q) + (1-s_n)(x_{n+1} - q)\|\cdot\|x_{n+1} - q\| + \|e_n\|\cdot\|x_{n+1} - q\| \\ &\le (\alpha_n + \alpha\beta_n + \gamma_ns_n)\|x_n - q\|\cdot\|x_{n+1} - q\| + (1-s_n)\gamma_n\|x_{n+1} - q\|^2 \\ &\quad +\beta_n\langle f(q) - q, j(x_{n+1} - q)\rangle + \|e_n\|\cdot\|x_{n+1} - q\| \\ &\le \frac{\alpha_n + \alpha\beta_n + \gamma_ns_n}{2}\|x_n - q\|^2 + \left[\frac{\alpha_n + \alpha\beta_n + \gamma_ns_n}{2} + (1-s_n)\gamma_n\right]\|x_{n+1} - q\|^2 \\ &\quad +\beta_n\langle f(q) - q, j(x_{n+1} - q)\rangle + \|e_n\|\cdot\|x_{n+1} - q\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le \frac{\alpha_n + \alpha\beta_n + \gamma_ns_n}{2 - (\alpha_n + \alpha\beta_n + \gamma_ns_n) - 2\gamma_n(1-s_n)}\|x_n - q\|^2 \\ &\quad + \frac{2\beta_n}{2 - (\alpha_n + \alpha\beta_n + \gamma_ns_n) - 2\gamma_n(1-s_n)}\langle f(q) - q, j(x_{n+1} - q)\rangle + \frac{2M_2\|e_n\|}{2 - (\alpha_n + \alpha\beta_n + \gamma_ns_n) - 2\gamma_n(1-s_n)} \\ &= \left[1 - \frac{2(1-\alpha)\beta_n}{2 - (\alpha_n + \alpha\beta_n + \gamma_ns_n) - 2\gamma_n(1-s_n)}\right]\|x_n - q\|^2 \\ &\quad + \frac{2(1-\alpha)\beta_n}{2 - (\alpha_n + \alpha\beta_n + \gamma_ns_n) - 2\gamma_n(1-s_n)}\frac{\langle f(q) - q, j(x_{n+1} - q)\rangle}{1-\alpha} + \frac{2M_2\|e_n\|}{2 - (\alpha_n + \alpha\beta_n + \gamma_ns_n) - 2\gamma_n(1-s_n)}\end{aligned}$$

where $M_2 \ge \max\left\{\sup_{n\ge 0}\|x_{n+1} - q\|\right\}$

Because

$$\begin{aligned}\frac{2\beta_n(1-\alpha)}{2-(\alpha_n+\alpha\beta_n+\gamma_n s_n)-2\gamma_n(1-s_n)} &= \frac{2\beta_n(1-\alpha)}{2-\alpha_n-\alpha\beta_n-2\gamma_n+\gamma_n s_n} \\ &= \frac{2\beta_n(1-\alpha)}{1+1-\alpha_n-\alpha\beta_n-\gamma_n-\gamma_n+\gamma_n s_n} \\ &= \frac{2\beta_n(1-\alpha)}{1+(1-\alpha)\beta_n-(1-s_n)\gamma_n} > \frac{2-2\alpha}{2-\alpha}\beta_n,\end{aligned}$$

so from $\sum_{n=0}^{\infty}\beta_n = \infty$, $\|e_n\| = o(\beta_n)$, Step 4, and Lemma 1, we get $\lim_{n\to\infty} x_n = q$. This completes the proof. □

If $\alpha_n = 0$, we can get Theorem 1. So, Theorem 2 is a generalization of Theorem 1. And, the computational efficiency of Theorem 2 is better than Theorem 1.

It is well known that Hilbert space is uniformly smooth Banach space. So, we can get the main results of [19].

Corollary 3. ([19]) *Let C be a nonempty, closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \varnothing$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}),$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0,1)$ that satisfies the following conditions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty}\gamma_n = 1$;

(2) $\sum_{n=0}^{\infty}\beta_n = \infty$;

(3) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$;

(4) $0 < \varepsilon \le s_n \le s_{n+1} < 1$, $\forall n \ge 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the unique solution of the variational inequality $\langle (I-f)x, y-x\rangle \ge 0$ and $\forall y \in F(T)$.

If we let $s_n = \frac{1}{2}$ and $\alpha_n = 0$, we can also get the main results of [20]. The results of Theorem 2 generalize the relevant results of [23].

Corollary 4. ([23]) *Let E be a uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \varnothing$ and $f : C \to C$ a generalized contraction mapping. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}),$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty}\beta_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$;

(iii) $\lim_{n\to\infty}|\alpha_{n+1}-\alpha_n| = 0$ *and* $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$;

(iv) $0 < \varepsilon \le s_n \le s_{n+1} < 1$ *for all* $n \ge 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the solution of the variational inequality $\langle (I-f)x^*, j(y-x^*)\rangle \ge 0$ for all $y \in F(T)$.

4. Applications

(1) A fixed point problem for strict pseudocontractive mapping.

If there exists $\lambda \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2, \ \forall x, y \in C,$$

then $T : C \to C$ is called λ-strict pseudocontractive mapping.

Zhou [24] obtained the relationship between nonexpansive mapping and λ-strict pseudocontractive mapping.

Lemma 4. ([24]) *Let C be a nonempty, closed convex subset of a real 2-uniformly smooth Banach space E and $T : C \to C$ be a λ-strict pseudocontractive mapping. For $\alpha \in (0,1)$, we define $T_\alpha x := (1-\alpha)x + \alpha Tx$. Then, $\alpha \in (0, \frac{\lambda}{K^2}]$, where K is the 2-uniformly smooth constant. Then, $T_\alpha : C \to C$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

So $T_\alpha : C \to C$ is nonexpansive, and then we can get the following results.

Theorem 3. *Let E be a 2-uniformly smooth Banach space, C be a closed convex subset of E, $f : C \to C$ be a contractive mapping with $k \in [0,1)$, $T : C \to C$ be a λ-strict pseudocontractive mapping, and $T_\alpha : C \to C$ be defined by $T_\alpha x := (1-\alpha)x + \alpha Tx$ with $\alpha \in (0, \frac{\lambda}{K^2}]$. $\{x_n\}$ is generated by the generalized viscosity implicit midpoint rule*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_\alpha(s_n x_n + (1-s_n)x_{n+1}), \ n \geq 0, \quad (5)$$

where $\{\alpha_n\}, \{s_n\} \subset (0,1)$ and satisfies the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ *or* $\sum_{n=1}^{\infty} \left|\alpha_{n+1} - \alpha_n\right| < \infty$;

(iv) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1, \forall n \geq 0$.

Then $\{x_n\}$ converges strongly to $q \in F(T)$, which is also the unique solution of the variational inequality $\langle (I-f)q, j(x-q)\rangle \geq 0$, and $x \in F(T)$.

Theorem 4. *Let E be a 2-uniformly smooth Banach space, C be a closed convex subset of E, $f : C \to C$ be a contractive mapping with $k \in [0,1)$, $T : C \to C$ be a λ-strict pseudocontractive mapping, and $T_\alpha : C \to C$ be defined by $T_\alpha x := (1-\alpha)x + \alpha Tx$ with $\alpha \in (0, \frac{\lambda}{K^2}]$. $\{x_n\}$ is generated by the generalized viscosity implicit midpoint rule*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T_\alpha(s_n x_n + (1-s_n)x_{n+1}) + e_n, \ n \geq 0, \quad (6)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0,1), \{e_n\} \subset E$ and satisfies the conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1, \lim_{n\to\infty} \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\sum_{n=1}^{\infty} \left|\alpha_{n+1} - \alpha_n\right| < \infty$ *and* $\sum_{n=1}^{\infty} \left|\beta_{n+1} - \beta_n\right| < \infty$;

(iv) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1, \forall n \geq 0$;

(v) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\|e_n\| = o(\beta_n)$.

Then $\{x_n\}$ converges strongly to $q \in F(T)$, which is also the unique solution of the variational inequality $\langle (I-f)q, j(x-q)\rangle \geq 0$, and $x \in F(T)$.

(2) A general system of a variational inequality problem in Banach space.

The problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, j(x - x^*) \rangle \geq 0, \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, j(x - y^*) \rangle \geq 0, \forall x \in C, \end{cases}$$

is called the general system of variational inequalities in Banach space, where $\lambda, \mu > 0$ and A, B are two nonlinear mappings.

If there exists $j(x-y) \in J(x-y)$ satisfying $\langle Ax - Ay, j(x-y) \rangle \geq 0, \forall x, y \in C$, then $A : C \to E$ is called accretive. If there exists $j(x-y) \in J(x-y)$ and $\alpha > 0$ satisfying $\langle Ax - Ay, j(x-y) \rangle \geq \alpha \|Ax - Ay\|^2, \forall x, y \in C$, then $A : C \to E$ is called α-inverse-strongly accretive.

Lemma 5. ([25]) *Let C be a nonempty, closed convex subset of a real 2-uniformly smooth Banach space E. Let Q_C be the sunny, nonexpansive retraction from E onto C. Let $A, B : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $G : C \to C$ be a mapping defined by $G(x) = Q_C[Q_C(x - \mu Bx) - \lambda A Q_C(x - \mu Bx)]$, $\forall x \in C$. If $0 < \lambda \leq \frac{\alpha}{K^2}$ and $0 < \mu \leq \frac{\alpha}{K^2}$, then $G : C \to C$ is nonexpansive.*

Thus, $G : C \to C$ is nonexpansive, and we can get the following results. More information on nonexpansive retracts and retractions can be found in [26,27].

Theorem 5. *Let E be a 2-uniformly smooth Banach space, C be a closed convex subset of E, $A, B : C \to E$ be, respectively, α-inverse-strongly accretive and β-inverse-strongly accretive, $f : C \to C$ be a contractive mapping with $k \in [0, 1)$, and $G : C \to C$ be defined by Lemma 5. $\{x_n\}$ is generated by the generalized viscosity implicit midpoint rule*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) G(s_n x_n + (1 - s_n) x_{n+1}), \ n \geq 0, \tag{7}$$

where $\{\alpha_n\}, \{s_n\} \subset (0, 1)$ and satisfies the conditions:

(i) $\lim_{n \to \infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ *or* $\sum_{n=1}^{\infty} \left| \alpha_{n+1} - \alpha_n \right| < \infty$;

(iv) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1, \forall n \geq 0$.

Then $\{x_n\}$ converges strongly to $q \in F(G)$, which is also the unique solution of the variational inequality $\langle (I - f) q, j(x - q) \rangle \geq 0$, and $x \in F(G)$.

Theorem 6. *Let E be a 2-uniformly smooth Banach space, C be a closed convex subset of E, $A, B : C \to E$ be, respectively, α-inverse-strongly accretive and β-inverse-strongly accretive, $f : C \to C$ be a contractive mapping with $k \in [0, 1)$, and $G : C \to C$ be defined by Lemma 5. $\{x_n\}$ is generated by the generalized viscosity implicit midpoint rule*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n G(s_n x_n + (1 - s_n) x_{n+1}) + e_n, \ n \geq 0, \tag{8}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0, 1), \{e_n\} \subset E$ and satisfies the conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1, \lim_{n \to \infty} \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\sum_{n=1}^{\infty} \left| \alpha_{n+1} - \alpha_n \right| < \infty$ *and* $\sum_{n=1}^{\infty} \left| \beta_{n+1} - \beta_n \right| < \infty$;

(iv) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1, \forall n \geq 0$;

(*v*) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\|e_n\| = o(\beta_n)$.

Then $\{x_n\}$ *converges strongly to* $q \in F(G)$, *which is also the unique solution of the variational inequality* $\langle (I-f)q, j(x-q)\rangle \geq 0$, *and* $x \in F(G)$.

5. Numerical Examples

We give six numerical examples to support the main results.

Example 1. *Let R be the real line with Euclidean norm,* $f : R \to R$ *be defined by* $f(x) = \frac{x}{4}$, $T : R \to R$ *be defined by* $T(x) = \frac{x}{2}$, $\alpha_n = \frac{1}{n}$, *and* $s_n = 1 - \frac{1}{n}$. *So,* $F(T) = \{0\}$. $\{x_n\}$ *is generated by (3). From Theorem 1,* $\{x_n\}$ *converges strongly to 0.*

Next, we simplify the form of (3) and get

$$x_{n+1} = \frac{2 - 4n + 2n^2 + n^3}{6n^2 - 2n^3} x_n. \tag{9}$$

Next, we take $x_0 = 1$ into (9). Finally, we get the following numerical results in Figure 1.

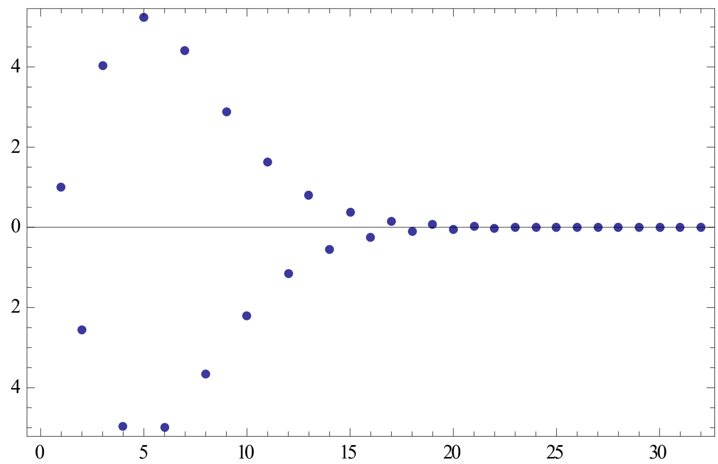

Figure 1. Numerical results.

Example 2. *Let R be the real line with Euclidean norm,* $f : R \to R$ *be defined by* $f(x) = \frac{x}{4}$, $T : R \to R$ *be defined by* $T(x) = \frac{x}{2}$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n}$, $\gamma_n = 1 - \frac{2}{n}$, $e_n = \frac{1}{n^2}$, *and* $s_n = 1 - \frac{1}{n}$. *So,* $F(T) = \{0\}$. $\{x_n\}$ *is generated by (4). From Theorem 2,* $\{x_n\}$ *converges strongly to 0.*

Next, we simplify the form of (4) and get

$$x_{n+1} = \frac{4 - n + 2n^2}{4 - 2n + 4n^2} x_n + \frac{2}{2 - n + 2n^2}. \tag{10}$$

Next, we take $x_0 = 1$ into (10). Finally, we get the following numerical results in Figure 2.

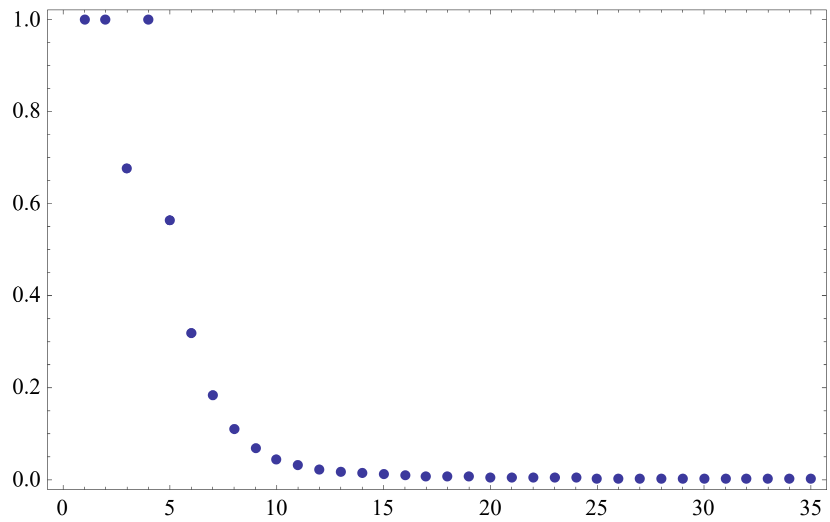

Figure 2. Numerical results.

Example 3. *Let* $\langle \cdot, \cdot \rangle : R^3 \times R^3 \to R$ *be the inner product and defined by*

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_2 y_3.$$

Let $\| \cdot \| : R^3 \to R$ *be the usual norm and defined by* $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ *for any* $x = (x_1, x_2, x_3)$. *For any* $x \in R^3$, *let* $f : R^3 \to R^3$ *be defined by* $f(x) = \frac{x}{6}$ *and* $T : R^3 \to R^3$ *be defined by* $T(x) = \frac{x}{4}$. *So,* $F(T) = \{0\}$. *Let* $\alpha_n = \frac{1}{n}$ *and* $s_n = 1 - \frac{1}{n}$, *then they satisfy the conditions of Theorem 1.*$\{x_n\}$ *is generated by (3). From Theorem 1,* $\{x_n\}$ *converges strongly to 0.*

Next, we simplify the form of (3) and get

$$x_{n+1} = \frac{3 - 4n + 3n^2}{3 - 3n + 12n^2} x_n. \tag{11}$$

Next, we take $x_1 = (1, 2, 3)$ into (11). Finally, we get the following numerical results in Figure 3.

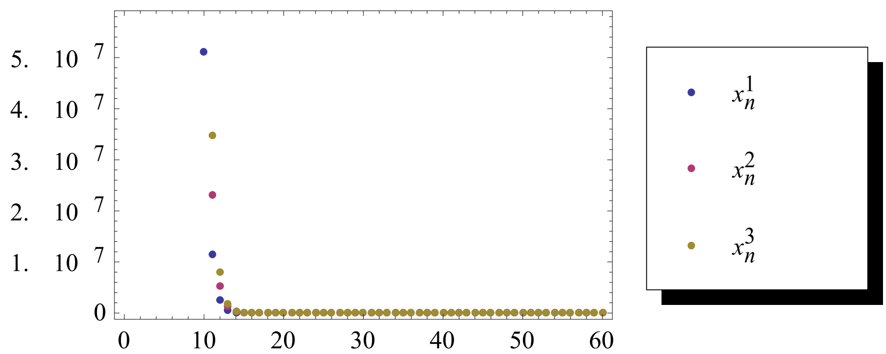

Figure 3. Numerical results.

Example 4. *Let* $\langle \cdot, \cdot \rangle : R^3 \times R^3 \to R$ *be the inner product and defined by*

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_2 y_3.$$

Let $\|\cdot\|: R^3 \to R$ be the usual norm and defined by $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for any $x = (x_1, x_2, x_3)$. For any $x \in R^3$, let $f: R^3 \to R^3$ be defined by $f(x) = \frac{x}{6}$ and $T: R^3 \to R^3$ be defined by $T(x) = \frac{x}{4}$. So, $F(T) = \{0\}$. Let $\alpha_n = \frac{1}{n}, \beta_n = \frac{1}{n}, \gamma_n = 1 - \frac{2}{n}, e_n = \frac{1}{n^2}$, and $s_n = 1 - \frac{1}{n}$, then they satisfy the conditions of Theorem 2. $\{x_n\}$ is generated by (4). From Theorem 2, $\{x_n\}$ converges strongly to 0.

Next, we simplify the form of (4) and get

$$x_{n+1} = \frac{6 + 5n + 3n^2}{6 - 3n + 12n^2} x_n + \frac{4}{2 - n + 4n^2}. \tag{12}$$

Next, we take $x_1 = (1, 2, 3)$ into (12). Finally, we get the following numerical results in Figure 4.

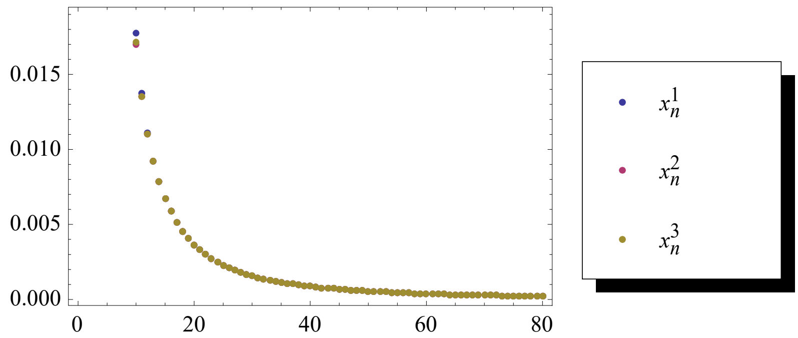

Figure 4. Numerical results.

Example 5. *Let R be the real line with Euclidean norm, $f: R \to R$ be defined by $f(x) = \frac{x}{4}$, $T: R \to R$ be defined by $T(x) = \frac{x}{2} + 1$, $\alpha_n = \frac{1}{n}$, and $s_n = 1 - \frac{1}{n}$. So, $F(T) = \{2\}$. $\{x_n\}$ is generated by (3). From Theorem 1, $\{x_n\}$ converges strongly to 2.*

Next, we simplify the form of (3) and get

$$x_{n+1} = \frac{2 - 3n + 2n^2}{2 - 2n + 4n^2} x_n + \frac{2n^2 - 2n}{1 - n + 2n^2}. \tag{13}$$

Next, we take $x_0 = 1$ into (13). Finally, we get the following numerical results in Figure 5.

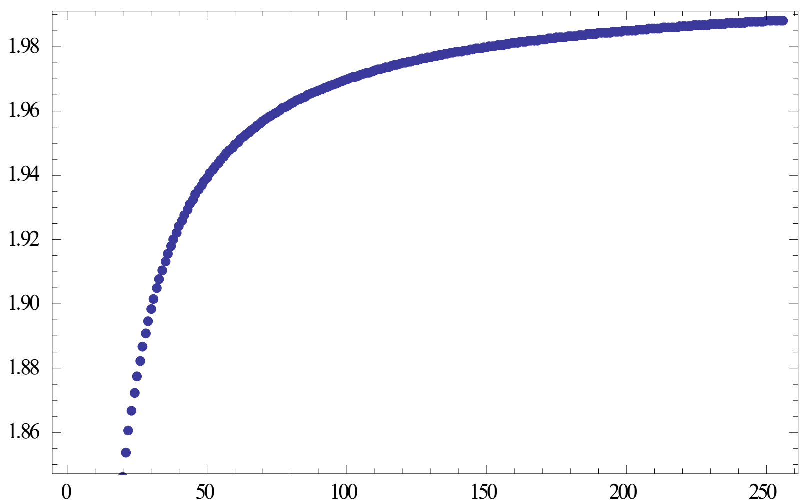

Figure 5. Numerical results.

Example 6. *Let R be the real line with Euclidean norm, $f : R \to R$ be defined by $f(x) = \frac{x}{4}$, $T : R \to R$ be defined by $T(x) = \frac{x}{2} + 1$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n}$, $\gamma_n = 1 - \frac{2}{n}$, $e_n = \frac{1}{n^2}$, and $s_n = 1 - \frac{1}{n}$. So, $F(T) = \{2\}$. $\{x_n\}$ is generated by (4). From Theorem 2, $\{x_n\}$ converges strongly to 2.*

Next, we simplify the form of (4) and get

$$x_{n+1} = \frac{4 - n + 2n^2}{4 - 2n + 4n^2} x_n + \frac{2(n-1)^2}{2 - n + 2n^2}. \tag{14}$$

Next, we take $x_0 = 1$ into (14). Finally, we get the following numerical results in Figure 6.

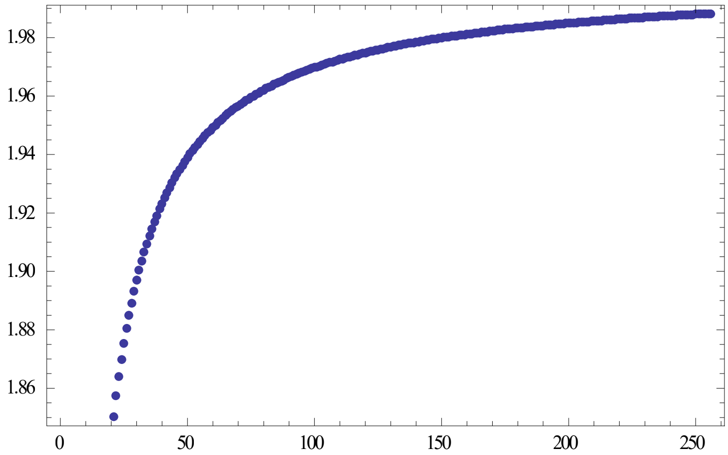

Figure 6. Numerical results.

6. Conclusions

This paper proposes the generalized viscosity implicit rules for nonexpansive mappings in Banach space and concretely constructs two iterative algorithms:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(s_n x_n + (1 - s_n)x_{n+1}),$$
$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}) + e_n.$$

This paper obtains strong convergence results. Results promote the work of Ke et al. [19], Luo et al. [20], and Yan et al. [23] from Hilbert spaces to a general Banach spaces and their iterative algorithms and relevant conclusions. In the end, this paper gives six numerical examples to support the main results.

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