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Approximation of Fixed Points for Suzuki's Generalized Non-Expansive Mappings

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Abstract: In this paper, we study a three step iterative scheme to approximate fixed points of Suzuki's generalized non-expansive mappings. We establish some weak and strong convergence results for such mappings in uniformly convex Banach spaces. Further, we show numerically that the considered iterative scheme converges faster than some other known iterations for Suzuki's generalized non-expansive mappings. To support our claim, we give an illustrative numerical example and approximate fixed points of such mappings using Matlab program. Our results are new and generalize several relevant results in the literature.

Keywords: Suzuki's generalized non-expansive mappings; iterative schemes; fixed points; weak and strong convergence results; uniformly convex Banach space

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1. Introduction

Throughout this paper, we assume that $\mathbb{N}$ is the set of all positive integers. We consider that C is a nonempty subset of a Banach space X and $F(T)$, the set of all fixed points of the mapping T on C. A mapping $T : C \to C$ is said to be non-expansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. It is called quasi non-expansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and for all $p \in F(T)$.

We know that $F(T)$ is nonempty when X is uniformly convex, C is bounded closed convex subset of X and T is non-expansive mapping, (cf. [1]).

In 2008, Suzuki [2] introduced the concept of generalized non-expansive mappings which is also called condition (C) and defined as:

A self mapping T on C is said to satisfy condition (C) if,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|, \ \forall\, x, y \in C.$$

Suzuki obtained existence of fixed points and convergence results for such mappings. Suzuki also showed that the notion of mappings satisfying condition (C) is more general than the notion of non-expansive mappings.

The following example supports the above claim.

Example 1 ([2]). *Define a self mapping T on $[0, 3]$ by*

$$T(x) = \begin{cases} 0, \ if\ x \neq 3, \\ 1, \ if\ x = 3. \end{cases}$$

Here T satisfies Suzuki's condition (C), but T is not a non-expansive mapping.

On the other hand, Banach contraction principle states that fixed point of contraction mappings can be approximated by Picard iterative scheme. In this scheme the sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = Tx_n, \ n \in \mathbb{N}. \end{cases} \tag{1}$$

It is well known that Picard iterative scheme does not converge to a fixed point of non-expansive mappings.

Therefore, in 1953, Mann [3] introduced a new iterative scheme to approximate the fixed points of non-expansive mappings. In this iterative scheme the sequence $\{x_n\}$ is defined in the following manner:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1 - a_n)x_n + a_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{a_n\}$ is a sequence in $(0, 1)$, satisfying appropriate conditions. It is also known that Mann iterative scheme fails to converge to fixed points of pseudo-contractive mappings.

So in 1974, Ishikawa [4] introduced a two step Mann iterative scheme to approximate fixed points of pseudo-contractive mappings, where the sequence $\{x_n\}$ is defined by

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1 - a_n)x_n + a_n Ty_n, \\ y_n = (1 - b_n)x_n + b_n Tx_n \ , n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1)$, satisfying appropriate conditions.

Rhoades [5] made a remark on the rate of convergence of Mann and Ishikawa iterative schemes: Mann iterative scheme for decreasing functions converges faster than Ishikawa scheme. For increasing functions Ishikawa iterative scheme is better than Mann iterative scheme, also Mann iterative scheme appears to be independent of the initial guess (see also [6]).

In 2000, Noor [7] introduced the following iterative scheme for general variational inequalities. In this scheme $\{x_n\}$ is defined by

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1 - a_n)x_n + a_n Ty_n, \\ y_n = (1 - b_n)x_n + b_n Tz_n, \\ z_n = (1 - c_n)x_n + c_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0, 1)$. He also studied the convergence criteria of this scheme. After that, in 2007, Agrawal et al. [8] introduced the following two step iterative scheme for nearly asymptotically non-expansive mappings. In this scheme $\{x_n\}$ is defined as follows:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1 - a_n)Tx_n + a_n Ty_n, \\ y_n = (1 - b_n)x_n + b_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{5}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1)$. They claimed that newly defined iterative scheme converges to a fixed point of contraction mappings same rate of convergence as Picard scheme but converges faster than Mann iterative scheme.

In 2014, Abbas and Nazir [9] introduced a new iterative scheme to approximate fixed points of non-expansive mappings in uniformly convex Banach space. In this scheme the sequence $\{x_n\}$ starting at initial guess $x_1 \in C$ is defined as:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1-a_n)Ty_n + a_nTz_n, \\ y_n = (1-b_n)Tx_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \ n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$. Authors showed numerically that this scheme converges to a fixed point of contraction mapping, faster than all of Picard, Mann and Agarwal iterative schemes.

In 2014, Thakur et al. [10] introduced the following iterative scheme for non-expansive mappings. In this scheme the sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1-a_n)Tx_n + a_nTy_n, \\ y_n = (1-b_n)z_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \ n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$. Authors proved that this process converges to a fixed point of contraction mapping, faster than all of Picard, Mann, Ishikawa, Noor, Agarwal and Abbas and Nazir iterative schemes in the sense of Berinde [11].

Recently, Sahu et al. [12] and Thakur et al. [13] introduced the following same iterative scheme for non-expansive mappings in uniformly convex Banach space:

$$\begin{cases} x = x_1 \in C, \\ x_{n+1} = (1-a_n)Tz_n + a_nTy_n, \\ y_n = (1-b_n)z_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \ n \in \mathbb{N}, \end{cases} \tag{8}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$. Authors proved that this scheme converges to a fixed point of contraction mapping, faster than all the known iterative schemes. Also, the authors provided an example to support their claim.

In 2011, Phuengrattana [14] proved convergence results for Suzuki's generalized non-expansive mappings by using Ishikawa iterative scheme in uniformly convex Banach spaces. Recently, fixed point theorems for Suzuki's generalized non-expansive mappings and nonlinear mappings have been studied by a large number of researchers, e.g., see [15–20].

Motivated by the above, we prove some weak and strong convergence results using iterative scheme (8) for Suzuki's generalized non-expansive mappings in uniformly convex Banach spaces. Our results generalize and extend the corresponding results of Sahu et al. [12], Thakur et al. [13] and many others in the literature.

2. Preliminaries

In this section, we recall following definitions, propositions and lemmas which will be used in our main results.

Definition 1. *Let C be a nonempty, closed and convex subset of a Banach space X. A mapping $T : C \to X$ is called demiclosed with respect to $y \in X$, if for each sequence $\{x_n\}$ in C and each $x \in C$, $\{x_n\}$ converges weakly at x and $\{Tx_n\}$ converges strongly at y imply that $Tx = y$.*

Definition 2. *A Banach space X is said to satisfy Opial's condition [21] if for each weakly convergent sequence $\{x_n\}$ to $x \in X$,*

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$$

holds, for all $y \in X$, with $y \neq x$.

Definition 3. *Let $\{x_n\}$ be a bounded sequence in a Banach space X. For $x \in C \subset X$, we set*

$$r(x, \{x_n\}) = \lim_{n\to\infty} \sup \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ relative to C is defined by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is well known that, $A(C, \{x_n\})$ consists exactly one point, in the case when X is uniformly convex Banach space.

Proposition 1 ([2])**.** *Let T be a self mapping on a nonempty subset C of a Banach space X.*

- *(i)* *If T is non-expansive then T satisfies the condition (C).*
- *(ii)* *Every mapping satisfying condition (C) with a fixed point is quasi non-expansive.*
- *(iii)* *If T satisfies condition (C), then*

$$\|x - Ty\| \leq 3\|Tx - y\| + \|x - y\|, \ \forall\, x, y \in C.$$

Lemma 1 ([2])**.** *Let T be a self mapping on a subset C of a Banach space X with the Opial's property. Assume that T satisfies condition (C). If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero.*

Lemma 2 ([2])**.** *Let T be a self map on a weakly compact convex subset C of a uniformly convex Banach space X. Assume that T satisfies condition (C), then T has a fixed point.*

Lemma 3 ([22])**.** *Suppose X is uniformly convex Banach space and $0 < a \leq s_n \leq b < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} \sup \|x_n\| \leq d$, $\lim_{n\to\infty} \sup \|y_n\| \leq d$ and $\lim_{n\to\infty} \sup \|s_n x_n + (1 - s_n)y_n\| = d$ holds, for some $d \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main Results

In this section, we prove some weak and strong convergence theorems using iterative scheme (8) for Suzuki's generalized non-expansive mappings in uniformly convex Banach spaces. First, we obtain following useful lemmas to be use in next theorems.

Lemma 4. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by iterative scheme (8), then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in C$. Since T satisfies condition (C), therefore by Proposition 1, T is quasi non-expansive mapping. That is,

$$\|Tx - p\| \leq \|x - p\|, \text{ for all } x \in C \text{ and for all } p \in F(T).$$

Now from iterative scheme (8), we get

$$\begin{aligned} \|z_n - p\| &= \|(1-c_n)x_n + c_nTx_n - p\| \\ &\leq (1-c_n)\|x_n - p\| + c_n\|Tx_n - p\| \\ &\leq (1-c_n)\|x_n - p\| + c_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{9}$$

And

$$\begin{aligned} \|y_n - p\| &= \|(1-b_n)z_n + b_nTz_n - p\| \\ &\leq (1-b_n)\|z_n - p\| + b_n\|Tz_n - p\| \\ &\leq (1-b_n)\|z_n - p\| + b_n\|z_n - p\| \\ &= \|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{10}$$

Using (9) and (10), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-a_n)Tz_n + a_nTy_n - p\| \\ &\leq (1-a_n)\|Tz_n - p\| + a_n\|Ty_n - p\| \\ &\leq (1-a_n)\|x_n - p\| + a_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{11}$$

This shows that the sequence $\{\|x_n - p\|\}$ is non-increasing and bounded below for all $p \in F(T)$. Hence, $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Lemma 5. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping. Let $\{x_n\}$ be a sequence defined by iterative scheme (8). Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. By Lemma 4, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} \|x_n - p\| = \alpha. \tag{12}$$

From (9), (10) and (12), we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq \alpha. \tag{13}$$

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq \alpha. \tag{14}$$

Since T satisfies condition (C), we have

$$\|Tx_n - p\| = \|Tx_n - Tp\| \leq \|x_n - p\|$$

$$\implies \quad \limsup_{n\to\infty} \|Tx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq \alpha. \tag{15}$$

Similarly,

$$\lim_{n\to\infty} \sup \|Ty_n - p\| \le \lim_{n\to\infty} \sup \|y_n - p\| \le \alpha. \tag{16}$$
$$\lim_{n\to\infty} \sup \|Tz_n - p\| \le \lim_{n\to\infty} \sup \|z_n - p\| \le \alpha. \tag{17}$$

Again,

$$\begin{aligned} \alpha &= \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|(1-a_n)Tz_n + a_nTy_n - p\| \\ &= \lim_{n\to\infty} \|(1-a_n)(Tz_n - p) + a_n(Ty_n - p)\|. \end{aligned} \tag{18}$$

From (16)–(18) and using Lemma 3, we have

$$\lim_{n\to\infty} \|Tz_n - Ty_n\| = 0. \tag{19}$$

Now,

$$\|x_{n+1} - p\| = \|(1-a_n)Tz_n + a_nTy_n - p\| \le \|Tz_n - p\| + a_n\|Ty_n - Tz_n\|.$$

Taking the lim inf on both sides, we get

$$\begin{aligned} \alpha &= \lim_{n\to\infty} \inf \|x_{n+1} - p\| \le \lim_{n\to\infty} \inf \|Tz_n - p\| \\ \implies \alpha &\le \lim_{n\to\infty} \inf \|z_n - p\|. \end{aligned} \tag{20}$$

So that, (13) and (20) give,

$$\lim_{n\to\infty} \|z_n - p\| = \alpha.$$

Thus,

$$\begin{aligned} \alpha &= \lim_{n\to\infty} \|z_n - p\| = \lim_{n\to\infty} \|(1-c_n)x_n + c_nTx_n - p\| \\ &= \lim_{n\to\infty} \|(1-c_n)(x_n - p) + c_n(Tx_n - p)\|. \end{aligned} \tag{21}$$

From (12), (15), (21) and using Lemma 3, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Conversely, assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $p \in A(C, \{x_n\})$, by Proposition 1, we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \lim_{n\to\infty} \sup \|x_n - Tp\| \\ &\le \lim_{n\to\infty} \sup(3\|Tx_n - x_n)\| + \|x_n - p\|) \\ &= \lim_{n\to\infty} \sup \|x_n - p\| \\ &= r(p, \{x_n\}) = r(C, \{x_n\}). \end{aligned}$$

$\implies Tp \in A(C, \{x_n\})$. Since X is uniformly convex, $A(C, \{x_n\})$ is singleton, hence we have $Tp = p$. □

Theorem 1. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping. Let $\{x_n\}$ be a sequence defined by iterative scheme (8). Assume that X satisfies Opial's condition, then $\{x_n\}$ converges weakly to a point of $F(T)$.*

Proof. Let $p \in F(T)$, then $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 4. Now we prove that $\{x_n\}$ has unique weak sub-sequential limit in $F(T)$. Let x and y be weak limits of the subsequences $\{x_{n_j}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ respectively. From Lemma 5, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed at zero by Lemma 1. This implies that $(I-T)x = 0 \implies x = Tx$, similarly $Ty = y$.

Next we show uniqueness. If $x \neq y$, then by using Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - x\| &= \lim_{n_j\to\infty} \|x_{n_j} - x\| \\ &< \lim_{n_j\to\infty} \|x_{n_j} - y\| \\ &= \lim_{n\to\infty} \|x_n - y\| \\ &= \lim_{n_k\to\infty} \|x_{n_k} - y\| \\ &< \lim_{n_k\to\infty} \|x_{n_k} - x\| \\ &= \lim_{n\to\infty} \|x_n - x\|. \end{aligned}$$

This is a contradiction, so $x = y$. Consequently, $\{x_n\}$ converges weakly to a point of $F(T)$. □

Theorem 2. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping. Let $\{x_n\}$ be a sequence defined by iterative scheme (8). Then $\{x_n\}$ converges to a point of $F(T)$ if and only if $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$ or $\lim_{n\to\infty} \sup d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf\{\|x_n - p\| : p \in F(T)\}$.*

Proof. Necessity is obvious.

Conversely, assume that $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$. From Lemma 4, $\lim_{n\to\infty} \|x_n - p\|$ exists, for all $p \in F(T)$ therefore $\lim_{n\to\infty} d(x_n, F(T)) = 0$ by assumption. We show that $\{x_n\}$ is a Cauchy sequence in C. As $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for given $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $n \geq m_0$,

$$\begin{aligned} d(x_n, F(T)) &< \frac{\epsilon}{2} \\ \implies \inf\{\|x_n - p\| : p \in F(T)\} &< \frac{\epsilon}{2}. \end{aligned}$$

In particular, $\inf\{\|x_{m_0} - p\| : p \in F(T)\} < \frac{\epsilon}{2}$. Therefore there exists $p \in F(T)$ such that

$$\|x_{m_0} - p\| < \frac{\epsilon}{2}.$$

Now for $m, n \geq m_0$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &\leq \|x_{m_0} - p\| + \|x_{m_0} - p\| \\ &= 2\|x_{m_0} - p\| < \epsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in C. As C is closed subset of a Banach space X, so that there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$. Now $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0 \implies q \in F(T)$. □

Theorem 3. *Let C be a nonempty, compact and convex subset of a uniformly convex Banach space X, and let T and $\{x_n\}$ be as in Lemma 5, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2, $F(T) \neq \emptyset$, so by Lemma 5, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since C is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p$ strongly for some $p \in C$. By Proposition 1, we have

$$\|x_{n_j} - Tp\| \leq 3\|Tx_{n_j} - x_{n_j}\| + \|x_{n_j} - p\|, \ \forall\, j \geq 1.$$

Letting $j \to \infty$, we get that $x_{n_j} \to Tp$. This implies that $Tp = p$ i.e., $p \in F(T)$. Also, $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 4. Thus p is the strong limit of the sequence $\{x_n\}$ itself. □

Senter and Dotson [23] introduced the notion of mapping satisfying condition (I) which is defined as follows.

Definition 4. *A self map T on C is said to satisfy condition (I), if there is a nondecreasing function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ and $h(r) > 0$, $\forall\, r > 0$ such that $d(x, Tx) \geq h(d(x, F(T)))$, for all $x \in C$, where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$.*

Now we also prove a strong convergence result using condition (I).

Theorem 4. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a Suzuki's generalized non-expansive mapping satisfying condition (I). Then the sequence $\{x_n\}$ defined by iterative scheme (8) converges strongly to a fixed point of T.*

Proof. We proved in Lemma 5 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{22}$$

From condition (I) and (22), we get

$$\begin{aligned} 0 \leq \lim_{n\to\infty} h(d(x_n, F(T))) &\leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0 \\ \Longrightarrow \lim_{n\to\infty} h(d(x_n, F(T))) &= 0. \end{aligned}$$

Since $h : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $h(0) = 0, h(r) > 0, \forall\, r > 0$, hence we have

$$\lim_{n\to\infty} (d(x_n, F(T))) = 0.$$

Now all the conditions of Theorem 2 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a fixed point of T. □

Remark 1. *All the results in this paper generalize the corresponding results of Sahu et al. [12], Thakur et al. [13] and many others because mappings here are generalized non-expansive and iterative scheme is more general than the others.*

Now, we furnish the following example to support our results.

Example 2. *Define a self mapping T on $[1,2]$ by*

$$T(x) = \begin{cases} 3 - x, \ if \ x \in [1, \frac{10}{9}), \\ \frac{x+16}{9}, \ if \ x \in [\frac{10}{9}, 2]. \end{cases}$$

Here T is a Suzuki's generalized non-expansive mapping, but T is not a non-expansive.

Verification. Take $x = \frac{111}{100}$ and $y = \frac{10}{9}$, then

$$\|x-y\| = \|\frac{111}{100} - \frac{10}{9}\| = \frac{1}{900}.$$

And

$$\begin{aligned} \|Tx - Ty\| &= \|3 - \frac{111}{100} - \frac{154}{81}\| \\ &= \frac{91}{8100} > \frac{1}{900} = \|x-y\|. \end{aligned}$$

Hence T is not a non-expansive mapping.

Now we verify that T is a Suzuki's generalized non-expansive mapping.

Here following cases arise:

Case I. If either $x, y \in [1, \frac{10}{9})$ or $x, y \in [\frac{10}{9}, 2]$. Then in both the cases T is non-expansive mapping and hence T is Suzuki's generalized non-expansive mapping.

Case II. Let $x \in [1, \frac{10}{9})$. Then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\|x - (3-x)\| = \frac{1}{2}\|2x - 3\| \in (\frac{7}{18}, \frac{1}{2}]$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have $\frac{3-2x}{2} \leq y - x \implies y \geq \frac{3}{2}$ and hence $y \in [\frac{3}{2}, 2]$. Now,

$$\|Tx - Ty\| = \|\frac{y+16}{9} - 3 + x\| = \|\frac{y + 9x - 11}{9}\| < \frac{1}{9}.$$

And

$$\|x-y\| = |x-y| > |\frac{10}{9} - \frac{3}{2}| = |\frac{20-27}{18}| = \frac{7}{18} > \frac{1}{9}.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x-y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

Case III. Let $x \in [\frac{10}{9}, 2]$. Then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\|\frac{x+16}{9} - x\| = \|\frac{16-8x}{18}\| \in [0, \frac{64}{162}]$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have $\frac{16-8x}{18} \leq |x-y|$, which gives two possibilities:

(a) Let $x < y$, then $\frac{16-8x}{18} \leq y - x$, i.e., $\frac{10x+16}{18} \leq y \implies y \in [\frac{244}{162}, 2] \subset [\frac{10}{9}, 2]$. So

$$\|Tx - Ty\| = \|\frac{x+16}{9} - \frac{y+16}{9}\| = \frac{1}{9}\|x-y\| \leq \|x-y\|.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x-y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

(b) Let $x > y$, then $\frac{16-8x}{18} \leq x - y$, i.e., $y \leq \frac{26x-16}{18} \implies y \leq \frac{116}{162}$ and $y \leq 2$, so $y \in [1, 2]$. Since $y \in [1, 2]$ and $y \leq \frac{26x-16}{18} \implies \frac{18y+16}{26} \leq x$. Since $x \in [\frac{34}{26}, 2]$ and $y \in [\frac{10}{9}, 2]$ is already included in Case I. Therefore consider, $x \in [\frac{34}{26}, 2]$ and $y \in [1, \frac{10}{9})$. Then

$$\|Tx - Ty\| = \|\frac{x+16}{9} - 3 + y\| = \|\frac{x + 9y - 11}{9}\| < \frac{1}{9}.$$

And

$$\|x-y\| = |x-y| > |\frac{34}{6} - \frac{10}{9}| = |\frac{306-260}{234}| = \frac{46}{234} > \frac{1}{9}.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x-y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

Thus T is Suzuki's generalized non-expansive mapping.

With help of Matlab Program Software, we obtain the comparison Table 1 and Figure 1 for various iterative schemes with control sequences $a_n = 0.85$, $b_n = 0.65$, $c_n = 0.45$ and initial guess $x_1 = 1.2$.

Table 1. A comparison table of iterative schemes.

Item	Picard	Mann	Ishikawa	Noor	Agarwal	Abbas	Thakur	Sahu, Thakur
1	1.200000	1.200000	1.200000	1.200000	1.200000	1.200000	1.200000	1.200000
2	1.911111	1.804444	1.848099	1.850281	1.954765	1.953570	1.967526	1.972859
3	1.990123	1.952198	1.971158	1.971980	1.997442	1.997305	1.998682	1.999079
4	1.998903	1.988315	1.994523	1.994756	1.999855	1.999844	1.999946	1.999969
5	1.999878	1.997144	1.998960	1.999019	1.999992	1.999991	1.999998	1.999999
6	1.999986	1.999302	1.999803	1.999816	2.000000	1.999999	2.000000	2.000000
7	1.999998	1.999829	1.999963	1.999966	2.000000	2.000000	2.000000	2.000000
8	2.000000	1.999958	1.999993	1.999994	2.000000	2.000000	2.000000	2.000000
9	2.000000	1.999990	1.999999	1.999999	2.000000	2.000000	2.000000	2.000000
10	2.000000	1.999998	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000
11	2.000000	1.999999	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000
12	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000

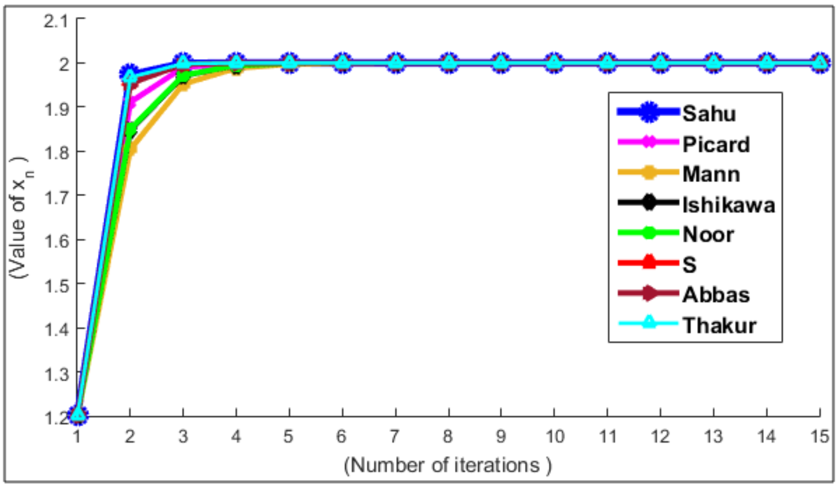

Figure 1. Convergence behavior of Sahu iterative scheme with other iterative schemes.

Remark 2. *The iterative scheme* (8) *converges faster than the Picard, Mann, Ishikawa, Noor, Agarwal, Abbas and Thakur iterative schemes for Suzuki's generalized non-expansive mappings as shown in the above table and figure. The class of Suzuki's generalized non-expansive mappings is bigger than the class of non-expansive mappings as shown in the Example* 2.

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