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The Mixed Scalar Curvature of a Twisted Product Riemannian Manifolds and Projective Submersions

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Abstract: In the present paper, we study twisted and warped products of Riemannian manifolds. As an application, we consider projective submersions of Riemannian manifolds, since any Riemannian manifold admitting a projective submersion is necessarily a twisted product of some two Riemannian manifolds.

Keywords: complete Riemannian manifold; twisted and warped products; projective submersion

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1. Introduction

Recall that a Riemannian manifold is a real, smooth n-dimensional manifold M equipped with an inner product g_x on the tangent space T_xM at each point $x \in M$ that varies smoothly from point to point in the sense that if X and Y are differentiable vector fields on M, then $x \to g_x(X,Y)$ is a smooth function on M. The family $g = (g_x)$ of inner products is called a Riemannian metric. We can also regard a Riemannian metric g as a symmetric (0,2)-tensor field that is positive-definite at every point (i.e., $g_x(X_x, X_x) > 0$, whenever $X_x \neq 0$). Therefore, a Riemannian metric g is known as a Riemannian metric tensor. In a system of local coordinates on the manifold M given by n real-valued functions $(x^1, \dots, x^n)$, the vector fields $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ form a basis of tangent vectors T_xM at each point $x \in M$. In this coordinate system, we can define the components of g by the following equalities: $g_{ij}(x) = g_x(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. Equivalently, the Riemannian metric tensor g can be written in terms of the dual basis $dx^1, \dots, dx^n$ of the cotangent bundle T^*M as $g = \sum_{i,j=1}^n g_{ij}(x) dx^i \otimes dx^j$.

In [1], totally umbilical complementary foliations on a Riemannian manifold (M, g) were studied and some necessary and sufficient conditions for their existence and nonexistence were given. In particular, this manifold M is a topological product $M_1 \times M_2$ of some real smooth manifolds M_1 and M_2 equipped with the Riemannian metric tensor g of the following form:

$$\sum_{a,b=1}^{m} \mu^2(x) g_{ab}(x^1, \dots x^m) dx^a \otimes dx^b + \sum_{\alpha,\beta=m+1}^{n} \lambda^2(x) g_{\alpha\beta}(x^{m+1}, \dots x^n) dx^\alpha \otimes dx^\beta.$$

In this case, the Riemannian manifold $(M_1 \times M_2, g)$ is denoted by ${}_\mu M_1 \times_\lambda M_2$ and called a double twisted product of the Riemannian manifolds (M_1, g_1) and (M_2, g_2) with the smooth twisted functions λ and μ and dimensions m and $n-m$, respectively. In particular, if μ and λ satisfy the conditions $\mu = \mu(x^{m+1}, \dots, x^n)$ and $\lambda = \lambda(x^1, \dots, x^m)$, respectively, then ${}_\mu M_1 \times_\lambda M_2$ and called a double warped product of the Riemannian manifolds (M_1, g_1) and (M_2, g_2). In [1], they used the Green divergence theorem to prove the following proposition: If a double twisted product manifold is compact and has non-negative sectional curvature, then it is isometric to the Riemannian product $M_1 \times M_2$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2).

In another paper [2], the global geometry of Riemannian manifolds with two orthogonal complementary (not necessarily integrable) totally umbilical distributions have been studied. In particular, they used a generalized divergence theorem (see [3,4]) to prove the main statement about two orthogonal complementary totally umbilical distributions on a complete non-compact and oriented Riemannian manifold. In addition, we used well-known Liouville type theorems on harmonic, subharmonic, and superharmonic functions on complete, non-compact Riemannian manifolds (see, for example, [5]) for studying special types of doubly twisted and warped products of Riemannian manifolds.

In the paper, we apply results of the above-mentioned papers to study twisted and warped products of Riemannian manifolds. In particular, we consider the geometry of projective submersions of Riemannian manifolds, since any Riemannian manifold admitting a projective submersion is necessarily a twisted product of some two Riemannian manifolds. Moreover, we generalize results in [6,7] using the notion of the mixed scalar curvature of a Riemannian manifold endowed with two complementary orthogonal distributions (see [8] (p. 117)).

2. The Mixed Scalar Curvature of Complete Twisted and Warped Products Riemannian Manifolds

Let (M, g) be an n-dimensional $(n \geq 2)$ Riemannian manifold with the Levi-Civita connection ∇ and let $TM = \mathrm{V} \oplus \mathrm{H}$ be a fixed orthogonal decomposition of the tangent bundle TM into vertical V and horizontal H distributions of dimensions $n - m$ and m, respectively. Next, we define the mixed scalar curvature of (M, g) as the following scalar function on M:

$$s_{\mathrm{mix}} = \sum\nolimits_{a=1}^{m} \sum\nolimits_{b=m+1}^{n} \sec(E_a,\, E_b),$$

where $\sec(E_a, E_b)$ is the sectional curvature of the mixed plane $\pi = \mathrm{span}\{E_a, E_b\}$ spanned by E_a and E_b for the local orthonormal frames $\{\, E_1, \ldots, E_m\}$ and $\{\, E_{m+1}, \ldots, E_n\}$ on TM adapted to V and H, respectively (see also [8] (p. 117); [9] (p. 23) and [10–13]). It is easy to see that this expression is independent of the chosen adapted frames. If V or H is a codimension-one distribution spanned by a unit normal vector field ξ then from (1) we obtain the following equality: $s_{\mathrm{mix}} = \mathrm{Ric}(\xi, \xi)$ with the Ricci tensor Ric. and a unit normal vector field ξ of the distribution V or H, respectively. Next, assume that V and H are totally geodesic and umbilical distributions, respectively. In this case, their second fundamental forms Q_V and Q_H satisfy the following equations: $Q_V = 0$ and $Q_H = g(\mathrm{H}, \mathrm{H}) \otimes \xi_H$ for the mean curvature vectors $\xi_H = (1/m)\mathrm{trace}_g Q_H$ of H (see [14] (pp. 148–151); [10]). Now consider an example to illustrate the above concepts. The twisted product $M_1 \times_\lambda M_2$ of Riemannian manifolds (M_1, g_1) and (M_2, g_2) is the manifold $M = M_1 \times M_2$ equipped with the Riemannian metric

$$g = \pi_1^* g_1 \oplus \lambda^2 \cdot \pi_2^* g_2, \tag{1}$$

where $\lambda : M_1 \times M_2 \to \mathbb{R}$ is a positive smooth function, called a twisted function, and $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_1$ are natural projections (see [15] and ([16] p. 15)).A twisted warped product $M_1 \times_\lambda M_2$ carries two canonical orthogonal complementary integrable distributions V and H given by vectors, which are tangent to the leaves of the product $M_1 \times M_2$. In addition, maximal integral manifolds of V and H are two canonical totally geodesic (vertical) and umbilical (horizontal) foliations, respectively (see [16] (p. 8) and [1,17]). In this case, we have the following.

Proposition 1. *Let an n-dimensional simply connected complete Riemannian manifold (M, g) be a twisted product $M_1 \times_\lambda M_2$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2) such that* $\dim M_1 > 1$ *and* $\dim M_2 > 1$. *Moreover, if (M, g) has non-negative sectional curvature, then it is isometric to the Riemannian* *product* $(M_1 \times M_2,\, g_1 \oplus g_2)$. *On the other hand, if* $\dim M_1 = 1$ *and the Ricci curvature of* (M,g) *is non-negative, then* $M_1 \times_\lambda M_2$ *is also isometric to the Riemannian product* $(M_1 \times M_2,\, g_1 \oplus g_2)$.

Proof. Let (M,g) be an n-dimensional complete Riemannian manifold equipped with a pair of orthogonal integrable distributions of complementary dimension V and H such that the distribution V is integrable with totally geodesic leaves. Assume that the sectional curvature of (M,g) is non-negative. Then H is also totally geodesic (see [18]). Now we fix a point $x \in M$ and let M_1 and M_2 be the maximal integral manifolds of distributions through x, respectively. Then, by the *de Rham decomposition theorem* (see [19] (p. 187)), we conclude that if (M,g) is a simply connected Riemannian manifold then it is isometric to the *Riemannian product* or, in other words, to the *direct product* $(M_1 \times M_2,\, g_1 \oplus g_2)$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2) for the Riemannian metrics g_1 and g_2 induced by g on M_1 and M_2, respectively.

If, in addition, (M,g) is a simply connected complete Riemannian manifold of non-negative Ricci curvature and $\dim \mathrm{V} = 1$, then (M,g) is also the Riemannian product $(M_1 \times M_2,\, g_1 \oplus g_2)$ (see also [18]). □

Remark that an arbitrary point of $M_1 \times_\lambda M_2$ admits a neighborhood with local adapted coordinate system $x^1, \ldots, x^m, x^{m+1}, \ldots, x^n$ such that its metric $\sum_{i,j=1}^{n} g_{ij}\left(x^1, \ldots, x^n\right) d\,x^i \otimes d\,x^j$ has the form (see [1])

$$\begin{aligned} &\textstyle\sum_{a,b=1}^{m} g_{ab}\left(x^1, \ldots, x^m\right) d\,x^a \otimes d\,x^b \\ &+ e^{u(x^1,\ldots,x^n)} \textstyle\sum_{\alpha,\beta=m+1}^{n} g_{\alpha\beta}(x^{m+1}, \ldots, x^m)\, d\,x^\alpha \otimes d\,x^\beta. \end{aligned} \tag{2}$$

In addition, the mean curvature vector ξ_V of V has the local coordinates $\xi_a = -2^{-1}\partial\, u / \partial\, x^a$ in the case where $x^{m+1} = c^{m+1}, \ldots, x^n = c^n$ for arbitrary constants $c^{m+1}, \ldots, c^n$, and the mean curvature vector ξ_H of H has the local coordinates $\xi_\alpha = 0$ in the case where $x^1 = c^1, \ldots, x^m = c^m$ for arbitrary constants $\mathrm{c}^1, \ldots, c^m$ (see [18]). Thus, $\xi_V = -\pi_{1*}(\operatorname{grad}\log\lambda)$ and $\xi_H = 0$ (see also [20]; ([16] p. 8) and [17]). Therefore, we can formulate a corollary from [2] (Theorem 1).

Proposition 2. *Let an n-dimensional simply connected complete Riemannian manifold* (M,g) *be a twisted product* $M_1 \times_\lambda M_2$ *of some Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *such that* $m = \dim M_1 > 1$ *and* $n - m = \dim M_2 > 1$. *If the mixed scalar curvature of* $M_1 \times_\lambda M_2$ *is nonpositive and the twisted function* λ *satisfies the condition* $\|\pi_{1*}(\operatorname{grad}\,\log\,\lambda)\| \in L^1(M,g)$ *for the canonical projections* $\pi_{1*} : TM_1 \times TM_2 \to TM_1$, *then* $M_1 \times_\lambda M_2$ *is isometric to a Riemannian product* $(M_1 \times M_2, g_1 \oplus g_2)$ *of Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2).

If λ depends only on the first factor, i.e., $\lambda : M_2 \to \mathbb{R}$, then $M_1 \times M_2$ with metric $g = \pi_1^* g_1 \oplus \lambda^2 \cdot \pi_2^* g_2$ is called a warped product and the positive function λ is regarded as a warped function (see [20,21]). In the case of a warped product $M_1 \times_\lambda M_2$ the mean curvature vectors of V and H are defined by the identities $\xi_H = 0$ and $\xi_V = -\pi_{1*}(\operatorname{grad}\,\log\,\lambda) = -\operatorname{grad}\,\log\,\lambda$, because λ depends on M_2 (see [16] (p. 48)). In this case, the condition $\|\,\xi_V\| \in L^1(M,g)$ has the form $\|\operatorname{grad}\,\log\,\lambda\,\| \in L^1(M,g)$. Therefore, we can formulate a corollary from [2] (Theorem 1).

Proposition 3. *Let an n-dimensional simply connected complete Riemannian manifold* (M,g) *be a warped product* $M_1 \times_\lambda M_2$ *of some Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *such that* $m = \dim M_1 > 1$ *and* $(n - m) = \dim M_2 > 1$. *If the mixed scalar curvature of* $M_1 \times_\lambda M_2$ *is nonpositive and the warped function* λ *satisfies* $\|\operatorname{grad}\,\log\,\lambda\| \in L^1(M,g)$, *then* $M_1 \times_\lambda M_2$ *is isometric to a Riemannian product* $(M_1 \times M_2,\, g_1 \oplus g_2)$ *of Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2).

In the case of a warped product $M_1 \times_\lambda M_2$, formula (7) from [2] can be rewritten in the form

$$(n-m)\Delta \log \lambda = -s_{\rm mix} + (n-m)(n-m-1)\|\text{grad} \log \lambda\|^2, \tag{3}$$

where Δ is the Laplace-Beltrami operator, and the norm of the vector field grad $\log \lambda$ is defined by g. The Christoffel symbols Γ^i_{jk} of the Levi-Civita connection associated with the metric (1) of $M_1 \times_\lambda M_2$ are well known (see the formulas from the proof of Theorem 2). Then (by Riemannian calculations) we can obtain the following relations:

$$\Delta \log \lambda = \Delta_1 \log \lambda + (n-m)\,\|\,\text{grad} \log \lambda\|^2,$$

where Δ_1 is the Laplace-Beltrami operator defined by g_1. In addition, we have the following obvious equalities:

$$\Delta_1 \log \lambda = \Delta_1 \lambda - \|\text{grad} \log \lambda\|^2, \tag{4}$$

where the norm of the vector field grad $\log \lambda$ is defined by g. As a result, we obtain from (3) to (4) the differential equation for the mixed scalar curvature $s_{\rm mix}$ of a doubly warped product $M_1 \times_\lambda M_2$:

$$s_{\rm mix} = -(n-m)\,\Delta_1 \lambda/\lambda. \tag{5}$$

If λ is a subharmonic function on (M_1, g_1), then from (5) we conclude that $s_{\rm mix} \le 0$. Therefore, we can formulate a corollary from [2] (Theorem 1).

Proposition 4. *Let an n-dimensional simply connected complete non-compact Riemannian manifold (M, g) be a warped product $M_1 \times_\lambda M_2$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2) such that the warped function λ is subharmonic and satisfies the condition $\|$ grad $\log \lambda \| \in L^1(M, g)$. Then $M_1 \times_\lambda M_2$ is isometric to a Riemannian product $(M_1 \times M_2,\, g_1 \oplus g_2)$.*

Moreover, $s_{\rm mix}$ is nonpositive everywhere on (M_1, g_1) if and only if λ is a positive subharmonic function defined on (M_1, g_1). If, in addition, we assume that (M_1, g_1) is complete and $\lambda \in L^p(M_1, g_1)$ for some $p > 1$ then λ must be identically constant (see [5] (p. 663)). On the other hand, $s_{\rm mix}$ is non-negative everywhere on (M_1, g_1) if and only if the warped function λ is a positive superharmonic function defined on (M_1, g_1). If we assume, in addition, that (M_1, g_1) is a complete manifold and $\|\text{grad}\,\lambda\| \in L^1(M_1, g_1)$, then λ is a harmonic function (see Section 4). If we assume, in addition, that $\lambda \in L^p(M_1, g_1)$ for some $p > 1$, then λ is constant (see [5] (p. 663)). In both cases $M_1 \times_\lambda M_2$ is isometric to the product $(M_1 \times M_2,\, g_1 \oplus g_2)$. Then we have the following.

Theorem 1. *Let an n-dimensional simply connected complete Riemannian manifold (M, g) be a warped product $M_1 \times_\lambda M_2$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2) such that (M_1, g_1) is complete and $\lambda \in L^p(M_1, g_1)$ for some $p > 1$. If the mixed scalar curvature $s_{\rm mix}$ of $M_1 \times_\lambda M_2$ is nonpositive everywhere on (M_1, g_1) then $M_1 \times_\lambda M_2$ is isometric to the Riemannian product $(M_1 \times M_2,\, g_1 \oplus g_2)$. On the other hand, if $s_{\rm mix}$ is non-negative everywhere on (M_1, g_1) and $\|$ grad $\lambda \| \in L^1(M_1, g_1)$, then $M_1 \times_\lambda M_2$ is isometric to the Riemannian product $(M_1 \times M_2,\, g_1 \oplus g_2)$ as well.*

3. Projective Submersions

A submersion of an n-dimensional Riemannian manifold (M, g) onto another n'-dimensional $(n > n')$ Riemannian manifold (M', g') is a surjective C^∞-map $f : (M, g) \to (M', g')$ such that the induced map $f_{*x} : T_x M \to T_{\pi(x)} M'$ has a maximum rank at each point $x \in M$. The inverse image $f^{-1}(y)$ of a point $y \in M'$ is called a fiber of f. In this case, we can define the almost product structure $TM = \mathrm{H} \oplus \mathrm{V}$ on (M, g), where $\mathrm{H} = \mathrm{Ker}\, f_*$ and $\mathrm{V} = \mathrm{H}^\perp$.

Recall that a curve γ in (M, g) is called a pregeodesic provided there is a reparameterization of γ such that γ is a geodesic. O'Neill uses the term "pregeodesic" to refer to such curves in his monograph [22]. Consider this concept in more details. A smooth map $\gamma : t \in J \to \gamma(t) \in M$ from an open interval $J \subset \mathbb{R}$ into a Riemannian manifold (M, g) is said to be a pregeodesic if it satisfies $\nabla_X X = \varphi(t)\ X$, where $X := \frac{d\gamma}{dt}$ is tangent to γ and ∇ is the Levi-Civita connection of (M, g). Let us reparametrize γ so that t becomes an affine parameter (see [23]). In this case, $\nabla_X X = 0$ and γ is called a geodesic. Examining the equation $\nabla_X X = 0$, we can infer that either γ is an immersion, i.e., $\frac{d\gamma}{dt} \neq 0$ for all $t \in J$, or $\gamma(J)$ is a point of the manifold M.

If an arbitrary pregeodesic in (M, g) is mapping by f into a pregeodesic in (M', g'), f is called a projective mapping (see the theory of projective mappings or, in other word, geodesic mappings in [24]). Moreover, let $f : (M, g) \to (M', g')$ be a *projective submersion* and $\dim M' < \dim M$ (see [6,7]). Under this assumption, we have $TM = \mathrm{H} \oplus \mathrm{V}$, where the distribution $\mathrm{H} = \mathrm{Ker}\, f_*$ is integrable with totally geodesic leaves and the distribution $\mathrm{V} = \mathrm{H}^{\perp}$ is integrable with totally umbilical leaves (see [6,7]). Moreover, in [25] they proved the following proposition: "If a complete simply connected Riemannian manifold (M, g) admits a projective submersion f, then it is isometric to some twisted product of two manifolds (M_1, g_1) and (M_2, g_2) such that the fibers of f and their orthogonal complements correspond to the canonical fibering of the product $M_1 \times M_2$." We can prove the converse statement for this proposition in the form of the following local theorem.

Theorem 2. *Let $M = M_1 \times_\lambda M_2$ be a twisted product of the Riemannian manifolds (M_1, g_1) and (M_2, g_2) with the Riemannian metric $g = \pi_1^* g_1 \oplus \lambda^2 \cdot \pi_2^* g_2$, where $\lambda : M_1 \times M_2 \to \mathbb{R}$ is a positive twisted function, $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are natural projections. Then the second natural projection from $M = M_1 \times_\lambda M_2$ onto $(M_2, \bar{g}_2)$ for $\bar{g}_2 = \lambda^2 \cdot \pi_2^* g_2$ is a projective submersion.*

Proof. Let $M = M_1 \times_\lambda M_2$ be a twisted product of the manifolds (M_1, g_1) and (M_2, g_2) and $x^1, \ldots, x^m, x^{m+1}, \ldots, x^n$ be a local coordinate system of $M = M_1 \times_\lambda M_2$ such that $x^1, \ldots, x^m$ and $x^{m+1}, \ldots, x^n$ are local coordinate systems of M_1 and M_2, respectively. If in addition, we denote by $g_{(1)\,ab}$ and $g_{(2)\,\alpha\beta}$ the components of the metric tensor g_1 and g_2, respectively, then with respect to the local coordinate system $x^1, \ldots, x^m, x^{m+1}, \ldots, x^n$ of $M = M_1 \times_\lambda M_2$ its Riemannian metric $g = \pi_1^* g_1 \oplus \lambda^2 \cdot \pi_2^* g_2$ has the local components

$$(g_{ij}) = \begin{pmatrix} g_{(1)\,ab} & 0 \\ 0 & \lambda^2\, g_{(2)\,\alpha\beta} \end{pmatrix}$$

for $i, j, k, l = 1, \ldots, n;\ a, b, c, d = 1, \ldots, m$ and $\alpha, \beta, \gamma, \varepsilon = m+1, \ldots, n$. In this case, the Christoffel symbols Γ_{ij}^k of g are given as follows (see [26]):

$$\Gamma_{ab}^c = \Gamma_{(1)\,ab}^c, \Gamma_{\alpha\,a}^\beta = (\,\partial_a \ln \lambda)\ \delta_\alpha^\beta, \Gamma_{\alpha\,\beta}^a = -\lambda\, g_{(1)}^{ab}\, (\,\partial_b \lambda)\ g_{(2)\,\alpha\beta},$$

$$\Gamma_{\alpha\gamma}^\beta = \Gamma_{(2)\,\alpha\gamma}^\beta + (\partial_\alpha\ \log \lambda)\ \delta_\gamma^\beta + (\partial_\gamma\ \log \lambda\,)\ \delta_\alpha^\beta - g_{(2)}^{\beta\varepsilon}\, (\partial_\varepsilon\ \log \lambda)\ g_{(2)\,\alpha\gamma},$$

and the others are zero, where $\partial_a = \frac{\partial}{\partial x^a}$ and $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$. In particular, from these identities we obtain $\Gamma_{\alpha\gamma}^\beta = \bar{\Gamma}_{(2)\,\alpha\gamma}^\beta$ for the metric $\bar{g}_2 = \lambda^2 g_2$ and its Christoffel symbols $\bar{\Gamma}_{(2)\,\alpha\gamma}^\beta$. An arbitrary pregeodesic line $\gamma : t \in J \subset \mathbb{R} \to \gamma(t) \in M$ can be defined by the equations $\frac{d\,x^k}{d\,t^2} + \Gamma_{ij}^k \frac{d\,x^i}{d\,t} \frac{d\,x^j}{d\,t} = \varphi(t)\, \frac{d\,x^k}{d\,t}$. Let us consider a natural projection $\bar{\pi}_2$ from $M = M_1 \times_\lambda M_2$ onto $(M_2, \bar{g}_2)$, which is defined the condition $x^a = \mathrm{const}$. In this case, the natural projection $\bar{\pi}_2(\gamma)$ of a pregeodesic line γ of $M_1 \times_\lambda M_2$ has the following equations:

$$\frac{d\,x^\alpha}{d\,s^2} + \bar{\Gamma}_{(2)\beta\gamma}^\alpha \frac{d\,x^\beta}{d\,s} \frac{d\,x^\gamma}{d\,s} = \left(\varphi(t)\, + (\partial_a \lambda)\, \frac{d\,x^a}{d\,s}\right) \frac{d\,x^\alpha}{d\,s}.$$

Thus, we conclude that the natural projection $\bar{\pi}_2(\gamma)$ of a pregeodesic line γ is a pregeodesic line of $(M_2, \bar{g}_2)$. $\square$

Considering the above, we can formulate a corollary of our Proposition 1.

Corollary 1. *Let $f : (M, g) \to (M', g')$ be a projective submersion of a simply connected complete Riemannian manifold (M, g) onto another Riemannian manifold (M', g') such that $\dim M' < \dim M$. If the sectional curvature of (M, g) is non-negative, then (M, g) is a Riemannian product $(M_1 \times M_2,\, g_1 \oplus g_2)$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2) such that integral manifolds of $\operatorname{Ker} f_*$ and $(\operatorname{Ker} f_*)^\perp$ correspond to the canonical foliations of the product $M_1 \times M_2$. On the other hand, if the Ricci curvature of (M, g) is non-negative and $\dim \operatorname{Ker} f_* = 1$, then (M, g) is a Riemannian product of the leaves of distributions $\operatorname{Ker} f_*$ and $(\operatorname{Ker} f_*)^\perp$.*

Let $f : (M, g) \to (M', g')$ be a projective submersion of a complete Riemannian manifold (M, g) onto another Riemannian manifold (M', g') such that $n = \dim M > \dim M' = m > 1$. Then from (3) we obtain the following divergence formula:

$$\operatorname{div} \xi_V = -1/(n-m)\, s_{\mathrm{mix}} + (n-m)(n-m-1)\, \|\, \xi_V \|^2 . \tag{6}$$

If now we suppose that (M, g) is a complete, non-compact, oriented Riemannian manifold with nonpositive mixed scalar curvature s_{mix}, then from (6) we obtain the inequality $\operatorname{div} \xi_V \le 0$. Moreover, if $\|\, \xi_V \| \in L^1(M, g)$ then by the results from [3,4] we conclude that $\operatorname{div} \xi_V = 0$. In this case, if the distribution V has dimension more than one then from (6) we obtain the following equality: $\|\, \xi_V \|^2 = 0$. It means that V is integrable with maximal totally geodesic integral manifolds (i.e., a totally geodesic foliation). Then (M, g) is locally isometric to the Riemannian product $(M_1 \times M_2,\, g_1 \oplus g_2)$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2) for the Riemannian metric g_1 and g_2 induced by g on M_1 and M_2, respectively. In addition, recall that every simply connected manifold M is orientable. Summarizing, we formulate the following statement.

Theorem 3. *Let $f : (M, g) \to (M', g')$ be a projective submersion of a simply connected complete Riemannian manifold (M, g) onto another Riemannian manifold (M', g') such that $\dim M > \dim M' > 1$, and let the mean curvature vector field ξ_V of $(\operatorname{Ker} f_*)^\perp$ satisfies the condition $\|\, \xi_V \| \in L^1(M, g)$. If the mixed scalar curvature s_{mix} of (M, g) is nonpositive then (M, g) is locally isometric to a Riemannian product $(M_1 \times M_2,\, g_1 \oplus g_2)$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2) such that integral manifolds of $\operatorname{Ker} f_*$ and $(\operatorname{Ker} f_*)^\perp$ correspond to the canonical foliations of the product $M_1 \times M_2$.*

Remark 1. If (M, g) is a Riemannian manifold of nonpositive sectional curvature then its mixed scalar curvature s_{mix} is also nonpositive. Therefore, we can formulate an analogue of Corollary 1.

For the case $m = n - 1$, from (2) we obtain the divergence formula

$$\operatorname{div} \xi_V = -1(n-1)\, s_{\mathrm{mix}} + (n-2)\, \|\, \xi_V \|^2 \tag{7}$$

(see also [1,27]), where $s_{\mathrm{mix}} = \operatorname{Ric}(\xi, \xi)$ for a unit normal vector field ξ of H. From (7) we conclude that the following corollary from Theorem 3 is true.

Corollary 2. *Let $f : (M, g) \to (M', g')$ be a projective submersion of an n-dimensional simply connected complete Riemannian manifold (M, g) onto a Riemannian manifold (M', g') such that $\dim M' = n - 1$, and let the mean curvature vector field ξ_V of $(\operatorname{Ker} f_*)^\perp$ satisfies the condition $\|\, \xi_V \| \in L^1(M, g)$. If the Ricci curvature of (M, g) is nonpositive then (M, g) is locally isometric to a Riemannian product $(M_1 \times M_2,\, g_1 \oplus g_2)$ of some Riemannian manifolds (M_1, g_1) and (M_2, g_2) such that integral manifolds of $\operatorname{Ker} f_*$ and $(\operatorname{Ker} f_*)^\perp$ correspond to the canonical foliations of the product $M_1 \times M_2$.*

4. Appendix

Here, we prove the lemma analogous to the Yau's statement from [5] (p. 660), where he has argued that on a complete non-compact Riemannian manifold each subharmonic function, whose gradient has integrable norm, must be harmonic.

Lemma 1. *If* (M, g) *is a complete Riemannian manifold without boundary, then any superharmonic function* $f \in C^2(M)$ *with* $\|\mathrm{grad}\, f\| \in L^1(M, g)$ *is harmonic.*

Proof. If we assume that $\varphi = -f$ for any superharmonic function $f \in C^2 M$ then the conditions $\Delta f \leq 0$ and $\|\mathrm{grad}\, f\| \in L^1(M, g)$, which must be satisfied for the superharmonic function f, can be written in the form $\Delta\, \varphi \geq 0$ and $\|\,\mathrm{grad}\, \varphi\| \in L^1(M, g)$. In this case, using the Yau's result for subharmonic functions, we conclude that $\Delta\, \varphi = 0$ and hence $f = -\varphi$ is harmonic. □

5. Concluding Remarks

In this article, we investigated the geometry in the large of warped and twisted products manifolds and presented undoubtedly new results. Our studies are important not only for geometry, but also for theoretical physics, because many exact solutions (e.g., Schwarzschild solution and Robertson-Walker model) of the Einstein field equations and modified field equations are warped products, see e.g., [28]. For instance, the Schwarzschild solution and Robertson-Walker model are warped products. While the Robertson-Walker model describes a simply connected homogeneous isotropic expanding or contracting universe, the Schwarzschild solution is the best relativistic model of the outer space around a massive star. The Schwarzschild model lays the groundwork for the description of the final stages of gravitational collapse and the objects known today as black holes. Moreover, twisted products being natural extensions of warped products, also find applications in theoretical physics, see e.g., [29]. In conclusion, remark that the notion of warped product manifolds plays very important roles not only in geometry but also in mathematical physics.

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