Article

Study on Non-Commutativity Measure of Quantum Discord

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Abstract: In this paper, we are concerned with the non-commutativity measure of quantum discord. We first present an explicit expression of the non-commutativity measure of quantum discord in the two-qubit case. Then we compare the geometric quantum discords for two dynamic models with their non-commutativity measure of quantum discords. Furthermore, we show that the results conducted by the non-commutativity measure of quantum discord are different from those conducted by both or one of the Hilbert-Schmidt distance discord and trace distance discord. These intrinsic differences indicate that the non-commutativity measure of quantum discord is incompatible with at least one of the well-known geometric quantum discords in the quantitative and qualitative representation of quantum correlations.

1. Introduction

In recent years, because quantum information processing is superior to classical information processing, quantum information theory and technology have developed dramatically (cf., e.g., [1–6]). As an important resource in quantum computation, quantum correlations have been investigated extensively in the last decades. So far, many forms of quantum correlations have been proposed; for example, quantum discord [7], quantum deficit [8], quantum correlation derived from the distance between the reduced states [9]. Among various quantum correlations, quantum discord and its derived measures are important (cf., e.g., [3,5–8,10–13]). Most of them are not so hard to calculate and are more robust against the effects of decoherence [10,14]. Quantum discord was initially introduced by Ollivier and Zurek and by Henderson and Vedral [7,8]. In 2010, Dakic, Vedral and Brukner [12] find a "Necessary and Sufficient Condition for Nonzero Quantum Discord" (geometric quantum discord for the Hilbert-Schmidt norm). Since then, several equivalent measures have been introduced. Recently, the non-commutativity measure of quantum discord has been discussed in [13]. In this work, we first study the problem "how to give an explicit expression of the non-commutativity measure of quantum discord in the two-qubit case?" Then we compare the geometric quantum discords (the Hilbert-Schmidt distance discord and trace distance discord) for two dynamic models of open quantum systems with their non-commutativity measure of quantum discords. We select open quantum systems as our resource quantum systems since they are significant quantum systems and they can induce occurrence of decoherence which can cause decreasing of quantum correlations and may induce failure of the algorithms.

2. Non-Commutativity Measure of Quantum Discord and Geometric Quantum Discords

Consider a composite quantum system $\mathcal{H}_{AB}$, which consists of two subsystems A and B. Quantum discord is the difference of two natural quantum extensions of the classical mutual information. In [8], the authors pointed out that the quantum discord reaches zero for and only for the classical-quantum

state. So we can look at the quantum discord as the 'distance' between the state ρ and the set of classical-quantum states. The state $\rho \in \mathcal{H}_{AB}$ is classical-quantum state if and only if ρ can be written the following form (cf. [10]):

$$\rho = \sum_i |i><i|_A \otimes \rho_B^i,$$

where $\{|i>_A\}$ is any orthonormal basis of subsystem A and ρ_B is a quantum state of subsystem B.

It is known that if $\rho \in \mathcal{H}_{AB}$ is a quantum state $\rho \in \mathcal{H}_{AB}$, then (cf. [13])

$$\rho = \sum_{ij} E_{ij} \otimes B_{ij},$$

where $E_{ij} = |i><j|_A$ and $B_{ij} =< i_A|\rho|j_A >$. In [13], two non-commutativity measures are presented by:

$$D'_{N1}(\rho) = \sum_\Omega ||[B_{ij}, B_{i'j'}]||_2, \qquad D_{N1}(\rho) = \sum_\Omega ||[B_{ij}, B_{i'j'}]||_{Tr}, \qquad (1)$$

where Ω is the set of all the possible pairs (regardless of the order), $[\cdot,\cdot]$ denotes the commutator, $||\cdot||_2$ is the Hilbert-Schmidt norm (i.e., $\|A\|_2 = \sqrt{Tr(A^\dagger A)}$) and $||\cdot||_{Tr}$ is the trace norm (i.e., $\|A\|_{Tr} = Tr(\sqrt{A^\dagger A})$).

Clearly:

(1) $D_{N1}(\rho) = 0$ and $D'_{N1}(\rho) = 0$ if and only if ρ is a quantum-classical state.
(2) D_{N1} and D'_{N1} are invariant under local unitary operations.

Moreover, we see that if $\rho = \sum_{ij} A_{ij} \otimes E'_{ij}$. According to Equation (1), we have:

$$D'_N(\rho) = \sum_{\Omega'} ||[A_{ij}, A_{i'j'}]||_2, \qquad D_N(\rho) = \sum_{\Omega'} ||[A_{ij}, A_{i'j'}]||_{Tr}, \qquad (2)$$

where $E'_{ij} = |i><j|_B$, $A_{ij} =< i_B|\rho|j_B >$ and $\{|i>_B\}$ is any orthonormal basis of subsystem B, and $D_N(\rho)$ ($D'_N(\rho)$) equals zero if and only if ρ is a zero-discord state.

The Hilbert-Schmidt distance discord (cf., e.g., [12]) is defined by:

$$D_{Hs}(\rho) = \min_{\chi \in CQ} d_{Hs}(\rho, \chi), \qquad d_{Hs}(\rho, \chi) = \|\rho - \chi\|_2^2,$$

where CQ is the set of classical-quantum states. The trace distance discord (cf., e.g., [15]) is defined by:

$$D_{Tr}(\rho) = \min_{\chi \in CQ} d_{Tr}(\rho, \chi), \qquad d_{Tr}(\rho, \chi) = \|\rho - \chi\|_{Tr},$$

Both the Hilbert-Schmidt distance discord and the trace distance discord are geometric quantum discords.

3. An Explicit Expression of The Non-Commutativity Measure of Quantum Discord in the Two-Qubit Case

Since the non-commutativity measures of quantum discord are invariant under local unitary operations, every state ρ is locally unitary equivalent to:

$$\rho = \frac{1}{4}(I \otimes I + X\sigma \otimes I + I \otimes Y\sigma + \sum_i c_i \sigma_i \otimes \sigma_i), \qquad (3)$$

where $X = (x_1, x_2.x_3)$, $Y = (y_1, y_2, y_3)$, $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$ in two qubits, and σ_i $(i = 1, 2, 3)$ are the three Pauli matrices (cf. [5]), which is called Bloch's representation. Therefore, if the state ρ satisfies Equation (3), we deduce that:

$$A_{00} = \frac{1}{4} \begin{pmatrix} 1 + x_3 + y_y3 + c_3 & x_1 - ix_2 \\ x_1 + ix_2 & 1 - x_3 + y_y3 - c_3 \end{pmatrix}, \quad A_{01} = \frac{1}{4} \begin{pmatrix} y_1 - iy_2 & c_1 - c_2 \\ c_1 + c_2 & y_1 - iy_2 \end{pmatrix},$$

$$A_{10} = \frac{1}{4} \begin{pmatrix} y_1 + iy_2 & c_1 + c_2 \\ c_1 - c_2 & y_1 + iy_2 \end{pmatrix}, \quad A_{11} = \frac{1}{4} \begin{pmatrix} 1 + x_3 - y_y3 - c_3 & x_1 - ix_2 \\ x_1 + ix_2 & 1 - x_3 - y_y3 + c_3 \end{pmatrix}.$$

$$[A_{00}, A_{01}]^{\dagger}[A_{00}, A_{01}] = \frac{1}{64} \begin{pmatrix} u^2 + K_+^2 (c_1 + c_2)^2 & 2c_1c_2K_+(x_1 - ix_2) \\ 2c_1c_2K_+(x_1 + ix_2) & c_1^2y_2^2 + c_2^2y_1^2 + K_+^2(c_1 - c_2)^2 \end{pmatrix},$$

$$[A_{00}, A_{10}]^{\dagger}[A_{00}, A_{10}] = \frac{1}{64} \begin{pmatrix} u^2 + K_+^2 (c_1 - c_2)^2 & -2c_1c_2K_+(x_1 - ix_2) \\ -2c_1c_2K_+(x_1 + ix_2) & c_1^2y_2^2 + c_2^2y_1^2 + K_+^2(c_1 + c_2)^2 \end{pmatrix},$$

$$[A_{00}, A_{11}]^{\dagger}[A_{00}, A_{11}] = \frac{1}{16} \begin{pmatrix} (x_1^2 + x_2^2)c_3^2 & 0 \\ 0 & (x_1^2 + x_2^2)c_3^2 \end{pmatrix},$$

$$[A_{01}, A_{10}]^{\dagger}[A_{01}, A_{10}] = \frac{1}{16} \begin{pmatrix} c_1^2c_2^2 & 0 \\ 0 & c_1^2c_2^2 \end{pmatrix},$$

$$[A_{01}, A_{11}]^{\dagger}[A_{01}, A_{11}] = \frac{1}{64} \begin{pmatrix} u^2 + K_-^2 (c_1 + c_2)^2 & -2c_1c_2K_-(x_1 - ix_2) \\ -2c_1c_2K_-(x_1 + ix_2) & c_1^2y_2^2 + c_2^2y_1^2 + K_-^2(c_1 - c_2)^2 \end{pmatrix},$$

$$[A_{10}, A_{11}]^{\dagger}[A_{10}, A_{11}] = \frac{1}{64} \begin{pmatrix} u^2 + K_-^2 (c_1 - c_2)^2 & 2c_1c_2K_-(x_1 - ix_2) \\ 2c_1c_2K_-(x_1 + ix_2) & c_1^2y_2^2 + c_2^2y_1^2 + K_-^2(c_1 + c_2)^2 \end{pmatrix}.$$

Finally, we obtain:

$$D_N(\rho) = \frac{\sum_{i=1}^4 \mu_i + 4|c_1c_2| + |c_3|\sqrt{x_1^2 + x_2^2}}{8},$$

$$D_N'(\rho) = \frac{\sqrt{u^2 + K_-^2(c_1^2 + c_2^2)} + \sqrt{u^2 + K_+^2(c_1^2 + c_2^2)} + |c_1c_2| + |c_3|\sqrt{x_1^2 + x_2^2}}{2\sqrt{2}}$$

in Equation (2), where:

$$K_{\pm} = c3 \pm x3, u = \sqrt{c_1^2y_2^2 + c_2^2y_1^2},$$

$$\mu_{1,2} = \sqrt{|u^2 + K_+^2 \pm 2c_1c_2K_+\sqrt{K_+^2 + (x_1^2 + x_2^2)}|},$$

$$\mu_{3,4} = \sqrt{|u^2 + K_-^2 \pm 2c_1c_2K_-\sqrt{K_-^2 + (x_1^2 + x_2^2)}|}.$$

4. Comparisons between Two Geometric Quantum Discords for Two Dynamic Models of Open Quantum Systems and Corresponding Non-Commutativity Measure of Quantum Discords

It is known that a real physical system is never perfectly closed. Environment acting on the quantum system are regarded as noise. In mathematics, it means that there exist a series of operators (referred to as { E_u }) acting on the density matrix. In this way, quantum operations can be represented in an elegant form known as the operator-sum (Kraus) representation, and by [16] we see that:

$$\sum_{u,v} E_{u,v}^{\dagger} E_{u,v} = I, \qquad E_{u,v} = E_{u_A} \otimes E_{v_B}.$$

Thus, the related density matrix can be written in the following form:

$$\rho(t) = \sum_{u,v} E_{u,v} \rho(0) E_{u,v}^{\dagger}. \tag{4}$$

Dephasing channel. It is known that the dephasing channel is the only channel that has possibility of having no energy penalty when quantum information loses. If two qubits pass through the dephasing channel respectively, the Hamiltonian ([16]):

$$H = \hbar q a^{\dagger} a (b^{\dagger} + b)$$

where a and $a^{\dagger}$, and b and $b^{\dagger}$ are the annihilation operator and creation operator of A and B respectively, and the Kraus operators are:

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\hbar q} \end{pmatrix} \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\hbar q,} \end{pmatrix} \tag{5}$$

$$E_0 = E_{0_A} = E_{0_B}, E_1 = E_{1_A} = E_{1_B}$$

where $q = 1 - e^{-2\gamma t}$ is photon scattering rate for the system, and γ is phase damping dissipation rate. In this paper, we suppose $\hbar = 1$.

For the Hilbert-Schmidt distance discord and trace distance discord, the dephasing channel has an important property: the freezing phenomenon (or the semi-freezing phenomenon). However, the non-commutativity measures of quantum discord is different. Actually, if we consider the initial state:

$$\rho_1(0) = \frac{1}{4}(I \otimes I + \sum_i c_i(0)\sigma_i \otimes \sigma_i)$$

with Equations (4) and (5), we have:

$$\rho_1(t) = \frac{1}{4}(I \otimes I + \sum_i c_i(t)\sigma_i \otimes \sigma_i)$$

where:

$$c_1(t) = c_1(0)e^{-2\gamma t}, \quad c_2(t) = c_2(0)e^{-2\gamma t}, \quad c_3(t) = c_3(0).$$

In Figure 1, we plot the time evolution of four different quantum discords for $c_1(0) = 0.6$, $c_2(0) = 0$, $c_3(0) = 0.2$. The plot clearly shows that the Hilbert-Schmidt distance discord and trace distance discord have the sharp transition when $t' = 0.5493/\gamma$. If $t < t'$, their values keep constant. If we change the initial condition (mainly about c_3), we can increase the time of the values keeping constant. However, the non-commutativity measures of quantum discord do not have this feature. They are strictly monotonous decreasing and strictly convex with time growing. Moreover, no matter what value of c_3 (except zero) could be, they still remain strictly monotonous decreasing.

Multimode vacuum field coupling the qubits. Consider a system consists of two qubits, with ω being the transition frequency and two-level energy separated by the energy gap $\hbar\omega$. The qubits are coupled to a multimode radiation field whose modes are initially in the vacuum state $|0>$. The evolution system in time is governed by the following master equation (cf., e.g., [17–19]):

$$\frac{\partial \rho}{\partial t} = -i\omega \sum_{i=1}^{2} [\sigma_3^i, \rho] - i \sum_{i \neq j}^{2} \xi_{ij} [\sigma_+^i \sigma_-^j, \rho] + \frac{1}{2} \sum_{i,j=1}^{2} \zeta_{ij}(2\sigma_-^j \rho \sigma_+^i - \{\sigma_+^i \sigma_-^j, \rho\}) \tag{6}$$

where $\sigma_{\pm}^i$ are the raising and lowering operators and σ_3^i is the energy operator (Pauli operator) of the ith qubit. The spontaneous decay rates of the qubits caused by the vacuum field $\gamma' = \xi_{ii}$ coupling

with the qubits. If $i \neq j$, ξ_{ij} and ζ_{ij} in Equation (6) are described by the collective damping and the "dipole-dipole" interaction, and take the forms:

$$\xi_{ij} = \frac{3}{2}\gamma'\left[\frac{sin(kr_{ij})}{kr_{ij}} + \frac{cos(kr_{ij})}{(kr_{ij})^2} - \frac{sin(kr_{ij})}{(kr_{ij})^3}\right]$$

$$\zeta_{ij} = \frac{3}{4}\gamma'\left[-\frac{cos(kr_{ij})}{kr_{ij}} + \frac{sin(kr_{ij})}{(kr_{ij})^2} + \frac{cos(kr_{ij})}{(kr_{ij})^3}\right]$$

where $k = \frac{2\pi}{\lambda}$ is the wave vector with λ being the atomic resonant wavelength and $r_{ij} = |r_i - r_j|$ is the distance between the qubits, we assume that the atomic dipole moments are parallel to each other and are polarized in the direction perpendicular to the interatomic axis.
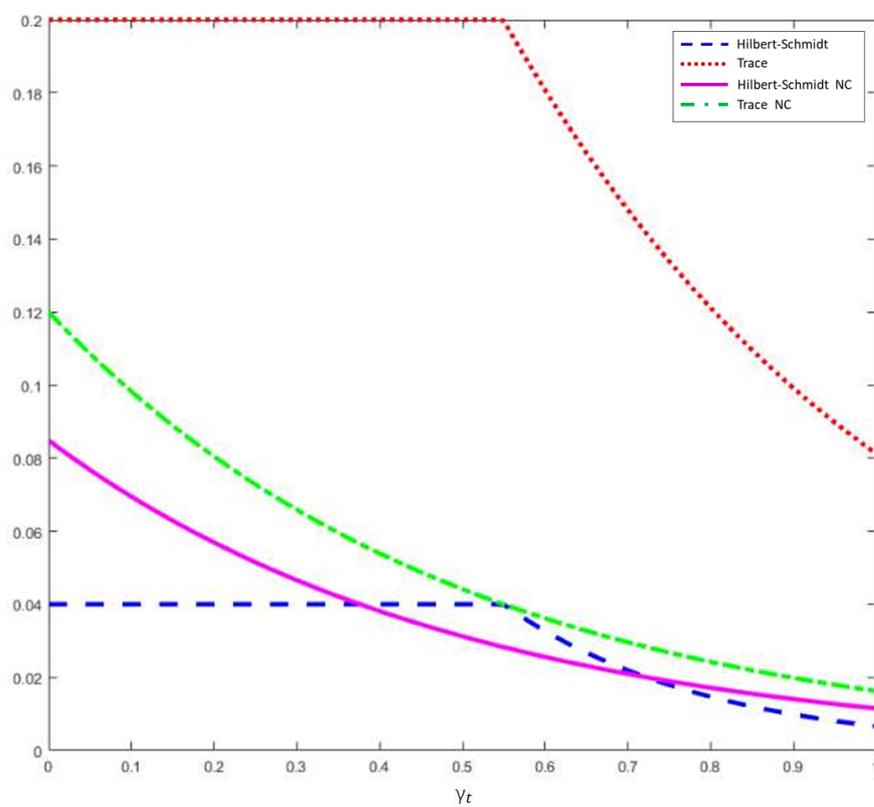
Figure 1. A two-qubit system under the operation of Dephasing channel where $c_1(0) = 0.6$, $c_2(0) = 0$, $c_3(0) = 0.2$, and NC means non-commutativity.
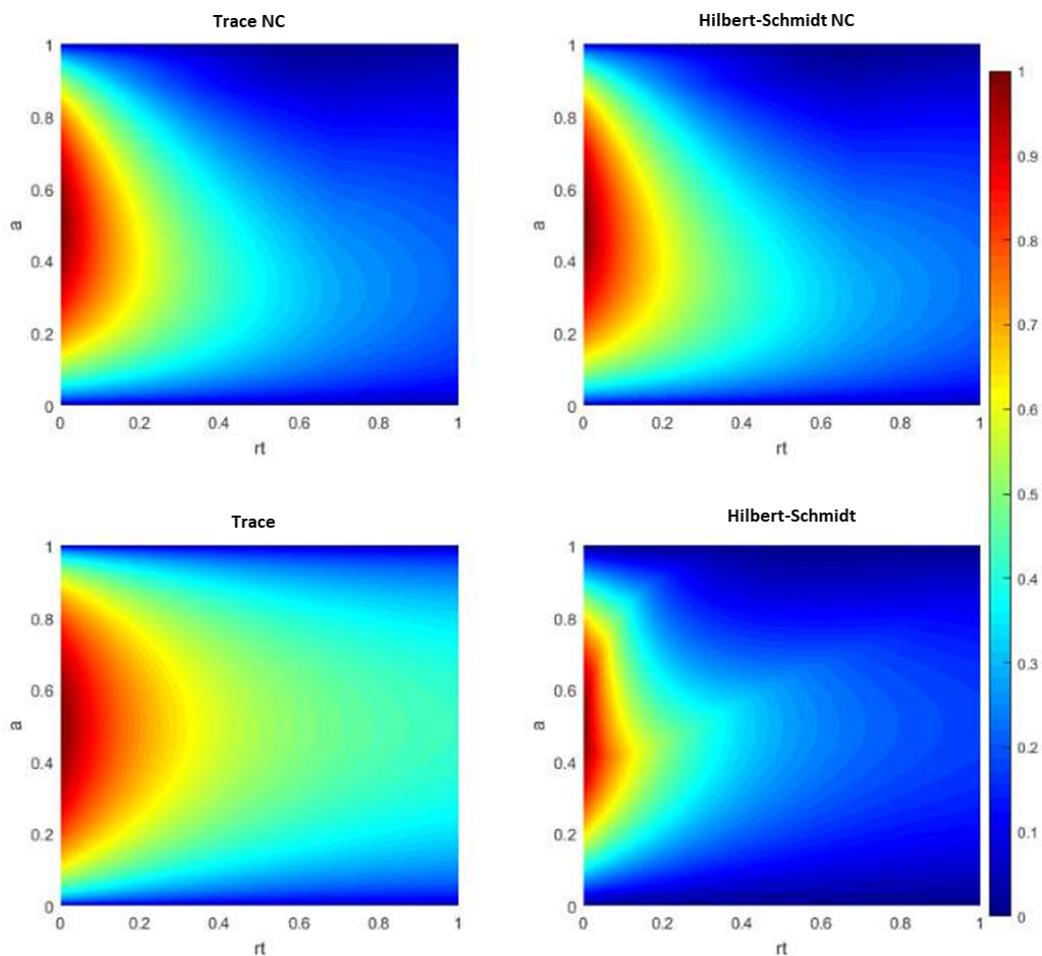
For this model, we consider initial state:

$$\rho_2(0) = |\Psi><\Psi|, |\Psi> = \sqrt{\alpha}|11> + \sqrt{1-\alpha}|00>.$$

Then $\rho_2(t)$ in the standard basis $\{|11>, |10>, |01>, |00>\}$ take the following forms:

$$\rho_{11}^{(2)}(t) = \alpha e^{-2\gamma' t}$$

$$\rho_{13,41}^{(2)}(t) = \sqrt{\alpha(1-\alpha)} e^{-\gamma' t}$$

$$\rho_{22,33}^{(2)}(t) = a_1 [e^{-\xi_{12}^+ t} - e^{-2\gamma' t}] + a_2 [e^{-\xi_{12}^- t} - e^{-2\gamma' t}]$$

$$\rho_{23,32}^{(2)}(t) = a_1 [e^{-\xi_{12}^+ t} - e^{-2\gamma' t}] + a_2 [e^{-\xi_{12}^- t} - e^{-2\gamma' t}]$$

$$\rho_{44}^{(2)}(t) = 1 - \rho_{11}^2(t) - \rho_{22}^2(t) - \rho_{33}^2(t)$$

where $\xi_{12}^\pm = \gamma' \pm \xi_{12}, a_{1,2} = \alpha \xi_{12}^\pm / \xi_{12}^\mp$. In Figure 2, we show the change of the discord with respect to γt and α as $\xi_{ij} = 0$, where the values are normalized.

Figure 2. Change of the initial state $|\Psi>$ with respect to γt and α as the interatomic distance is $\xi_{ij} = 0$.

For this model, the trace distance discord is symmetrical about $\alpha = 0.5$. Moreover, compared to the trace distance discord, we find that the non-commutativity measure of quantum discord and the Hilbert-Schmidt distance discord have more similar properties at different α values. On the other hand, the Hilbert-Schmidt distance discord will revive after α large enough. In this aspect, the non-commutativity measure of quantum discord are closer to the trace distance discord.

5. Conclusions

In conclusion, we present an explicit expression of the non-commutativity measure of quantum discord in the two-qubit case. We also compare the non-commutativity measure of quantum discord with the geometric quantum discord in the models of two qubits passing through the dephasing channel and the multimode vacuum field coupling the qubits, respectively. Our study shows that the non-commutativity measures of quantum discord lose some important features of the geometric quantum discords, such as the freezing phenomenon in the dephasing channel and the revival in the multimode vacuum field coupling the qubits.

Author Contributions: All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Das, D.; Bhattacharya, B.; Datta, C.; Roy, A.; Jebaratnam, C.; Majumdar, A.S.; Srikanth, R. Operational characterization of quantumness of unsteerable bipartite states. *Phys. Rev.* **2018**, *97*, 062335. [CrossRef]
2. Gharibian, S. Quantifying nonclassicality with local unitary operations. *Phys. Rev.* **2012**, *86*, 042106. [CrossRef]
3. Hosseini, S.; Rahimi-Keshari, S.; Haw, J.Y.; Assad1, S.M.; Chrzanowski, H.M.; Janousek, J.; Symul, T.; Ralph, T.C.; Lam, P.K.; Gu, M.; et al. Experimental verification of quantum discord in continuousvariable states and operational significance of discord consumption. In Proceedings of the Conference on Lasers and Electro-Optics Europe-Technical Digest 2014, San Jose, CA, USA, 3–4 January 2014.
4. Jebaratnam, C.; Aravinda, S.; Srikanth, R. Nonclassicality of local bipartite correlations. *Phys. Rev.* **2017**, *95*, 032120. [CrossRef]
5. Luo, S.L. Quantum discord for two-qubit systems. *Phys. Rev.* **2008**, *77*, 042303. [CrossRef]
6. Yune, J.; Hong, K.H.; Lim, H.T.; Lee, J.C.; Kwon, O.; Han, S.W.; Moon, S.; Kim, Y.S.; Kim, Y.H. Experimental verification of quantum discord in continuousvariable states and operational significance of discord consumption. In Proceedings of the 2015 11th Conference on Lasers and Electro-Optics Pacific Rim (CLEO-PR), Busan, Korea, 24–28 August 2015.
7. Ollivier, H.; Zurek, W.H. Quantum discord: A measure of the quantumness of correlations. *Phys. Rev. Lett.* **2002**, *88*, 017901. [CrossRef] [PubMed]
8. Henderson, L.; Vedral, V. Classical, quantum and total correlations. *J. Phys. Math. Gen.* **2001**, *34*, 6899–6905. [CrossRef]
9. Guo, Y.; Li, X.L.; Li, B.; Fan, H. Quantum correlation induced by the average distance between the reduced states. *Int. J. Theor. Phys.* **2015**, *54*, 2022–2030. [CrossRef]
10. Adesso, G.; Bromley, T.R.; Cianciaruso, M. Measures and applications of quantum correlations. *J. Phys. Math. Theor.* **2016**, *49*, 473001. [CrossRef]
11. Coecke, B.; Fritz, T.; Spekkens, R.W. A mathematical theory of resources. *Inf. Comput.* **2016**, *250*, 59–86. [CrossRef]
12. Dakic, B.; Vedral, V.; Brukner, C. Necessary and sufficient condition for nonzero quantum discord. *Phys. Rev. Lett.* **2010**, *105*, 190502. [CrossRef] [PubMed]
13. Guo, Y. Non-commutativity measure of quantum discord. *Sci. Rep.* **2016**, *6*, 25241. [CrossRef] [PubMed]
14. Mazzola, L.; Piilo, J.; Maniscalco, S. Sudden transition between classical and quantum decoherence. *Phys. Rev. Lett.* **2010**, *104*, 200401. [CrossRef] [PubMed]
15. Ciccarello, F.; Tufarelli, T.; Giovannetti, V. Toward computability of trace distance discord. *New J. Phys.* **2014**, *16*, 013038. [CrossRef]
16. Nielsen, M.A.; Chuang, I.L. *Quantum Computation and Quantum Information*; Cambridge: Cambridge University Press: London, UK, 2000.

17. Ficek, Z.; Tanas, R. Dark periods and revivals of entanglement in a two-qubit system. *Phys. Rev.* **2006**, *74*, 024304. [CrossRef]

18. Hu, M.L.; Fan, H. Robustness of quantum correlations against decoherence. *Ann. Phys.* **2012**, *327*, 851–860. [CrossRef]

19. Yu, T.; Eberly, J.H. Finite-time disentanglement via spontaneous emission. *Phys. Rev. Lett.* **2004**, *93*, 140404. [CrossRef] [PubMed]