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Restricted Gompertz-Type Diffusion Processes with Periodic Regulation Functions

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Abstract: We consider two different time-inhomogeneous diffusion processes useful to model the evolution of a population in a random environment. The first is a Gompertz-type diffusion process with time-dependent growth intensity, carrying capacity and noise intensity, whose conditional median coincides with the deterministic solution. The second is a shifted-restricted Gompertz-type diffusion process with a reflecting condition in zero state and with time-dependent regulation functions. For both processes, we analyze the transient and the asymptotic behavior of the transition probability density functions and their conditional moments. Particular attention is dedicated to the first-passage time, by deriving some closed form for its density through special boundaries. Finally, special cases of periodic regulation functions are discussed.

Keywords: diffusion processes; first-passage time; population dynamics

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1. Introduction

Deterministic and stochastic growth models are been widely used in the literature to study dynamics of a population. Some such models, as the logistic and Gompertz ones, are characterized by an intrinsic rate of growth and by a horizontal asymptote; both can be time-dependent. This limit, called carrying capacity function, can be caused by many environmental factors as living space, food availability and water supply. In particular, Coleman et al. [1] analyzes the behavior of the deterministic logistic model in the periodic environment by assuming that the intrinsic growth coefficient and the carrying capacity are periodic time-dependent function. In Mir [2] and in Mir and Dubeau [3], the authors study the effect of different time-dependent carrying capacities in deterministic Richards models, including logistic and Gompertz growths. Tjørve and Tjørve [4,5] propose a unified approach to the Richards-model family to describe the growth of animals and plants, as well as the number of bacteria and the volume of cancer cells.

The deterministic approach presents some limitations in mathematical biology since it is always arduous to predict the evolution of the system accurately. Indeed, the biological systems are subject to random fluctuations, due partially to environment factors, such as epidemics and nature disasters. To take into account the environmental fluctuations, often responsible for the discrepancies between experimental data and theoretical predictions, the idea of growth in random environment has been considered by various authors (cf., for instance, Goel and Richter-Dyn [6], Ricciardi [7,8] and Ricciardi et al. [9]). Some diffusion growth models in random environment are considered in Capocelli and Ricciardi [10], Tuckwell [11], Nobile and Ricciardi [12,13], Skiadas [14], and Kink [15]. Analytical properties of logistic and other growth models are taken into account in Di Crescenzo and Spina [16] and Di Crescenzo and Paraggio [17].

The Gompertz model is widely used in many aspects of biology because it can fit well the experimental date related to the evolution of same populations of organisms and of certain solid tumors (cf., for instance, [5] and references therein). In the classical Gompertz model, the intrinsic growth coefficient and the carrying capacity are chosen as constant parameters. However, sometimes the description of certain phenomena requires the use of time-dependent functions; in these cases, the Gompertz model turns out to be particularly effective to describe the population dynamics. For this reason, many efforts have recently been made to investigate the analytical and statistical properties of time-inhomogeneous Gompertz models. Specifically, the studies have focused on the following topics: *(i)* analysis of probabilistic characteristics, as the transition probability densities, the first-passage time density through a particular time dependent threshold (cf. Albano et al. [18,19], Ghost and Prajneshu [20]); *(ii)* discovery of appropriate procedures to estimate the unknown parameters and the time dependent functions characterizing the infinitesimal moments of the diffusion process (see, Gutiérrez et al. [21], Moummou et al. [22,23], Albano et al. [24,25], Román-Román et al. [26]); *(iii)* study of stochastic Gompertz process with jumps that occur at random time and reset the process to a fixed value; after each jump, the process restarts with modified grows parameters (see Spina et al. [27], Giorno et al. [28]). The Gompertz growth model plays a special role also in nonlinear models of interacting populations (cf., for instance, Goel et al. [29]) and in a non-autonomous predator–prey system (cf. Buonocore et al. [30]).

In the present paper, we consider two different time-inhomogeneous diffusion processes useful to model the evolution of a population in a random environment. They are obtained as approximations of the solution of deterministic Gompertz-type growth models. The first considered process is an unrestricted time-inhomogeneous Gompertz diffusion process, whereas the second one is a restricted diffusion process, obtained by including a reflecting condition. Indeed, sometimes in growth models, the population is not isolated, so that immigration effects may be occur. In these cases, it is necessary take into account diffusion processes with a reflecting condition in the zero state. A special diffusion process in the presence of reflecting boundaries are taken into account in Linetsky [31], Giorno et al. [32,33], and Buonocore et al. [34,35].

In Section 2, we consider the Gompertz diffusion process $X(t)$ with growth rate $\beta(t)$, carrying capacity $e^{\nu(t)}$ and noise intensity $\sigma^2(t)$, whose conditional median coincides with the deterministic Gompertz growth. Results concerning the transition probability density function (pdf) and its moments are derived. Particular attention is dedicated to analyzing the first-passage time (FPT) problem and the asymptotic behavior of the transition densities and of the FPT densities through a constant boundary S in two cases: *(a)* for asymptotically constant functions $\beta(t), \nu(t), \sigma^2(t)$ and for *(b)* for asymptotically constant functions $\beta(t), \sigma^2(t)$ with $\nu(t)$ periodic function. In Section 3, we assume that the carrying capacity is a periodic time-dependent function and specialize the results of Section 2 when $\beta(t) = \beta$ and $\nu(t) = \ln[a + b\sin(2\pi t/Q)]$; same comparisons are carried out for *(i)* $\sigma^2(t) = \sigma^2$ and for *(ii)* $\sigma^2(t) = \sigma^2\,(1 - e^{-2\beta\, t})^2$ in order to highlight both the role of $\sigma^2(t)$.

In Section 4, we perform a similar analysis for the Gompertz process $Y(t)$ in the presence of a reflecting boundary in zero state. Since, in the Gompertz process considered in Sections 2 and 3 the state zero is unattainable, in Section 4, we build the restricted process by shifting the original Gompertz process and then by introducing in this last one a reflecting condition in the zero state. For the restricted process $Y(t)$, the function $\beta(t)$ represents again the growth rate, $\nu(t)$ is a regulation function and $\sigma^2(t)$ gives the noise intensity. For this process, we analyze the transient and asymptotic behavior of the transition pdf and of FPT density through the constant boundary. Finally, in Section 5, the process $Y(t)$ is studied for the same cases of Section 3 with $\beta(t) = \beta$ and $\nu(t) = \ln[a + b\sin(2\pi t/Q)]$ in order to highlight both the role of $\sigma^2(t)$ and the effect of the reflecting condition on the population growth.

2. Time-Inhomogeneous Gompertz-Type Growth

In this section, we analyze the inhomogeneous deterministic and stochastic Gompertz model. For the stochastic process, we focus on the transition pdf and its moments, on the FPT density and on the asymptotic behavior of the process.

2.1. Deterministic Evolution

Let $\nu(t):\mathbb{R}^+\to\mathbb{R}$ and $\beta(t):\mathbb{R}^+\to R^+$ be continuous functions. We assume that the population density $x(t)$ evolves according to the Gompertz law:

$$x(t)=\exp\Big\{\nu(t)-\big[\nu(t_0)-\ln x_0\big]e^{-[\varphi(t)-\varphi(t_0)]}\Big\},\qquad t\geq t_0\geq 0, \tag{1}$$

where $x(t_0)=x_0$ and

$$\varphi(t)=\int_0^t\beta(u)\,du. \tag{2}$$

The function $\beta(t)$ represents the growth rate of the population and the function $e^{\nu(t)}$ describes the time-dependent carrying capacity. Indeed, if $0<x_0<e^{\nu(t_0)}$, then $x(t)<e^{\nu(t)}$ for all $t\geq t_0$, whereas, if $x_0>e^{\nu(t_0)}$, then $x(t)>e^{\nu(t)}$ for all $t\geq t_0$.

We remark that Label (1) is solution of the differential equation:

$$\frac{dx(t)}{dt}=x(t)\Big\{\nu'(t)+\beta(t)\,\big[\nu(t)-\ln x(t)\big]\Big\},\qquad x(t_0)=x_0.$$

From (1), the population density at time $t+\Delta t$ can be expressed as:

$$x(t+\Delta t)=x(t)\,\exp\Big\{e^{-\varphi(t+\Delta t)}\Big[\nu(t+\Delta t)\,e^{\varphi(t+\Delta t)}-\nu(t)\,e^{\varphi(t)}\Big]-e^{-\varphi(t+\Delta t)}\Big[e^{\varphi(t+\Delta t)}-e^{\varphi(t)}\Big]\ln x(t)\Big\}. \tag{3}$$

Starting from (3), a time-inhomogeneous stochastic diffusion process can be constructed.

2.2. Stochastic Evolution

Under the assumption of random environment, we interpret $\nu(t+\Delta t)\,e^{\varphi(t+\Delta t)}-\nu(t)\,e^{\varphi(t)}$ as the mean of the increment $Z(t+\Delta t)-Z(t)$ of a time-inhomogeneous Wiener process $\{Z(t),t\geq 0\}$ having mean function $\nu(t)e^{\varphi(t)}$ and covariance function $c(s,t)=\int_0^s e^{2\varphi(u)}\,\sigma^2(u)\,du\quad(s<t)$, where $\sigma(t):\mathbb{R}^+\to R^+$ is a continuous function. Therefore,

$$Z(t+\Delta t)-Z(t)=\nu(t+\Delta t)\,e^{\varphi(t+\Delta t)}-\nu(t)\,e^{\varphi(t)}+W\Big[\int_t^{t+\Delta t}e^{2\varphi(u)}\,\sigma^2(u)\,du\Big], \tag{4}$$

where $W(t)$ is the standard Wiener process.

Making use of the assumption of random environment, denoting by $\{X(t),t\geq 0\}$ the stochastic process describing the evolution of the population, from (3), one obtains the following stochastic equation:

$$X(t+\Delta t)-X(t)=X(t)\,\Big[\exp\Big\{e^{-\varphi(t+\Delta t)}\Big[Z(t+\Delta t)-Z(t)\Big]-e^{-\varphi(t+\Delta t)}\Big[e^{\varphi(t+\Delta t)}-e^{\varphi(t)}\Big]\ln X(t)\Big\}-1\Big], \tag{5}$$

from which one derives:

$$X(t+\Delta t)-X(t)=X(t)\sum_{r=1}^{+\infty}\frac{e^{-r\,\varphi(t+\Delta t)}}{r!}\,\Big\{Z(t+\Delta t)-Z(t)-\Big[e^{\varphi(t+\Delta t)}-e^{\varphi(t)}\Big]\ln X(t)\Big\}^r. \tag{6}$$

For $n = 1, 2, \ldots$ let

$$A_n(x,t) = \lim_{\Delta t \to 0} \frac{\mathbb{E}\{[X(t+\Delta t) - X(t)]^n | X(t) = x\}}{\Delta t}$$

be the infinitesimal moments of $X(t)$. Recalling (6), one obtains:

$$\begin{aligned} A_1(x,t) &= x\Big\{\nu'(t) + \beta(t)\,\big[\nu(t) - \ln x\big] + \frac{\sigma^2(t)}{2}\Big\}, \\ A_2(x,t) &= \sigma^2(t)\,x^2, \\ A_n(x,t) &= 0, \qquad n = 3, 4, \ldots \end{aligned} \tag{7}$$

From (7), we note that $X(t)$ is a time-inhomogeneous Gompertz-type diffusion process with space-state $(0, +\infty)$ having infinitesimal drift and variance $A_1(x,t)$ and $A_2(x,t)$, respectively. We remark that the boundaries 0 and $+\infty$ are unattainable states for $X(t)$. The transition pdf $f_X(x,t|x_0,t_0)$ of $X(t)$ is solution of the Fokker–Planck equation with the initial delta condition:

$$\begin{aligned} &\frac{\partial f_X(x,t|x_0,t_0)}{\partial t} = -\frac{\partial}{\partial x}\Big[A_1(x,t)\; f_X(x,t|x_0,t_0)\Big] + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\Big[A_2(x,t)\; f_X(x,t|x_0,t_0)\Big], \\ &\lim_{t \downarrow t_0} f_X(x,t|x_0,t_0) = \delta(t - t_0). \end{aligned} \tag{8}$$

Making use of transformations:

$$y = \ln x, \qquad y_0 = \ln x_0, \qquad f_X(x,t|x_0,t_0) = \frac{1}{x}\, f_U(y,t|y_0,t_0), \tag{9}$$

from (8), one has:

$$\begin{aligned} &\frac{\partial f_U(y,t|y_0,t_0)}{\partial t} = -\frac{\partial}{\partial y}\Big[C_1(y,t)\; f_U(y,t|y_0,t_0)\Big] + \frac{1}{2}\,\frac{\partial^2}{\partial y^2}\Big[C_2(t)\; f_U(y,t|y_0,t_0)\Big], \\ &\lim_{t \downarrow t_0} f_U(y,t|y_0,t_0) = \delta(t - t_0), \end{aligned} \tag{10}$$

with

$$C_1(y,t) = \nu'(t) + \nu(t)\,\beta(t) - \beta(t)\,y, \qquad C_2(t) = \sigma^2(t). \tag{11}$$

Therefore, $f_U(y,t|y_0,t_0)$ is the transition pdf of a time-inhomogeneous Ornstein–Uhlenbeck process $\{U(t), t \geq 0\}$ with state-space $(-\infty, +\infty)$, whose infinitesimal drift and variance are given in (11). Hence, $f_U(y,t|y_0,t_0)$ is a normal pdf with conditional mean and variance:

$$\begin{aligned} M(t|y_0,t_0) &= \mathbb{E}[U(t)|U(t_0) = y_0] = \nu(t) + \big[y_0 - \nu(t_0)\big]\, e^{-[\varphi(t) - \varphi(t_0)]}, \\ V(t|t_0) &= \mathbb{V}ar[U(t)|U(t_0) = y_0] = e^{-2\varphi(t)} \int_{t_0}^{t} e^{2\varphi(u)}\,\sigma^2(u)\,du, \end{aligned} \tag{12}$$

respectively. Recalling (9), the transition pdf of $X(t)$ is lognormal:

$$f_X(x,t|x_0,t_0) = \frac{1}{x}\,\frac{1}{\sqrt{2\pi\, V(t|t_0)}}\,\exp\Big\{-\frac{[\ln x - M(t|\ln x_0,t_0)]^2}{2\,V(t|t_0)}\Big\}, \qquad x, x_0 > 0. \tag{13}$$

Moreover, conditional mean, median and variance are:

$$\begin{aligned}
&\mathbb{E}[X(t)|X(t_0)=x_0] = \exp\Big\{M(t|\ln x_0,t_0)+\frac{V(t|t_0)}{2}\Big\},\\
&\mathbb{M}ed[X(t)|X(t_0)=x_0] = \exp\Big\{M(t|\ln x_0,t_0)\Big\},\\
&\mathbb{V}ar[X(t)|X(t_0)=x_0] = \exp\Big\{2\,M(t|\ln x_0,t_0)+V(t|t_0)\Big\}\,\Big[\exp\{V(t|t_0)\}-1\Big].
\end{aligned} \tag{14}$$

We note that the conditional median of $X(t)$ does not dependent upon $\sigma^2(t)$ and it coincides with the deterministic solution given in (1). Furthermore, when $\sigma^2(t)$ is zero, the conditional mean is identified with the deterministic solution and the conditional variance is zero.

2.3. First-Passage Time Problem for $X(t)$

We consider the FPT problem for the diffusion process $X(t)$ with infinitesimal moments (7). Let

$$\mathcal{T}_X = \begin{cases} \inf_{t\ge t_0}\{t : X(t)\ge S(t)\}, & 0<X(t_0)=x_0<S(t_0),\\[2ex] \inf_{t\ge t_0}\{t : X(t)\le S(t)\}, & X(t_0)=x_0>S(t_0)>0, \end{cases} \tag{15}$$

be the random variable that describes the FPT of $X(t)$ from $X(t_0)=x_0$ to the continuous boundary $S(t)\in C^1(T)$. The FPT pdf $g_X[S(t),t|x_0,t_0] = dP(\mathcal{T}_X\le t)/dt$ is solution of the first-kind Volterra integral equation:

$$\begin{aligned} f_X(x,t|x_0,t_0) = \textstyle\int_{t_0}^t g_X[S(\tau),\tau|x_0,t_0]\, f_X[x,t|S(\tau),\tau]\,d\tau\\ [0<x_0<S(t_0), x\ge S(t)] \text{ or } [x_0>S(t_0), 0<x\le S(t)]. \end{aligned} \tag{16}$$

Due to (9), one has $g_X[S(t),t|x_0,t_0] = g_U[\ln S(t),t|\ln x_0,t_0]$, where g_U is the FPT density of the Ornstein–Uhlenbeck process with infinitesimal moments (11). Then, making use of the results given in Di Nardo et al. [36], the FPT density $g_X[S(t),t|x_0,t_0]$ is a solution of the following non-singular second-kind Volterra integral equation:

$$\begin{aligned} g_X[S(t),t|x_0,t_0] = -2\varrho\,\Psi_X[S(t),t|x_0,t_0] + 2\varrho\textstyle\int_{t_0}^t g_X[S(\tau),\tau|x_0,t_0]\,\Psi_X[S(t),t|S(\tau),\tau]\,d\tau\\ [0<x_0<S(t_0),\ x\ge S(t)] \text{ or } [x_0>S(t_0),\ 0<x\le S(t)], \end{aligned} \tag{17}$$

with $\varrho=1$ when $x_0<S(t_0)$ and $\varrho=-1$ when $x_0>S(t_0)$, where

$$\begin{aligned} \Psi_X[S(t),t|y,\tau] = \Bigg\{ \frac{S'(t)-\nu'(t)\,S(t)}{2} - S(t)\,\frac{\ln S(t)-\nu(t)}{2}\,\frac{\sigma^2(t)\,e^{2\varphi(t)}-\beta(t)\int_\tau^t e^{2\varphi(u)}\,\sigma^2(u)\,du}{\int_\tau^t e^{2\varphi(u)}\,\sigma^2(u)\,du}\\ +S(t)\,\frac{\ln y-\nu(\tau)}{2}\,\frac{\sigma^2(t)\,e^{\varphi(t)+\varphi(\tau)}}{\int_\tau^t e^{2\varphi(u)}\,\sigma^2(u)\,du}\Bigg\}\, f_X[S(t),t|y,\tau]. \end{aligned} \tag{18}$$

An effective numerical procedure to obtain $g_X[S(t),t|x_0,t_0]$ via (17) is given in Di Nardo et al. [36] and Buonocore et al. [37]. Furthermore, if the boundary $S(t)$ is chosen as

$$S_1(t) = \exp\Big\{\nu(t) + d_1 e^{-\varphi(t)}\int_0^t e^{2\varphi(u)}\,\sigma^2(u)\,du + d_2\,e^{-\varphi(t)}\Big\}, \qquad t>0,\ d_1,d_2\in\mathbb{R}, \tag{19}$$

then

$$\begin{aligned} g_X[S_1(t),t|x_0,t_0] = |\ln S_1(t_0)-\ln x_0|\,\frac{e^{\varphi(t)+\varphi(t_0)}\,\sigma^2(t)}{\int_{t_0}^t e^{2\varphi(u)}\,\sigma^2(u)\,du}\,S_1(t)\,f_X[S_1(t),t|x_0,t_0]\\ [0<x_0<S_1(t_0)] \text{ or } [x_0>S_1(t_0)], \end{aligned} \tag{20}$$

with $f_X(x,t|x_0,t_0)$ given in (13). Moreover, one has:

$$\int_{t_0}^{+\infty} g_X[S_1(t),t|x_0,t_0]\,dt = \begin{cases} 1, & d_1\big[\ln S_1(t_0)-\ln x_0\big]\le 0,\\[2ex] \exp\Big\{-2d_1 e^{\varphi(t_0)}\big[\ln S_1(t_0)-\ln x_0\big]\Big\}, & d_1\big[\ln S_1(t_0)-\ln x_0\big]>0. \end{cases} \tag{21}$$

Note that, if $d_1 = d_2 = 0$ in (19), then $S_1(t) = e^{\nu(t)}$ coincides with the carrying capacity and, from (20), it follows that $g_X[e^{\nu(t)},t|x_0,t_0]$ does not depend on $\nu(t)$, but it depends only on $\nu(t_0)$.

2.4. Asymptotic Behavior for $X(t)$

We analyze the asymptotic behavior of the process $X(t)$ when the functions $\beta(t)$ and $\sigma^2(t)$ admit finite limits as the time increases by distinguishing two different cases: *(a)* $\nu(t)$ admits a finite limit and *(b)* $\nu(t)$ is a periodic function of period Q.

Case *(a)* We assume that

$$\lim_{t\to+\infty}\beta(t)=\beta,\qquad \lim_{t\to+\infty}\sigma^2(t)=\sigma^2,\qquad \lim_{t\to+\infty}\nu(t)=\nu\qquad (\beta>0,\sigma>0,\nu\in\mathbb{R}). \tag{22}$$

From (2) and (12), one has

$$\lim_{t\to+\infty}\varphi(t)=+\infty,\qquad \lim_{t\to+\infty}M(t|\ln x_0,t_0)=\nu,\qquad \lim_{t\to+\infty}V(t|t_0)=\frac{\sigma^2}{2\,\beta}, \tag{23}$$

so that, from (13), one obtains the steady-state density:

$$W_X(x)=\lim_{t\to+\infty}f_X(x,t|x_0,t_0)=\frac{1}{x}\sqrt{\frac{\beta}{\pi\,\sigma^2}}\,\exp\Big\{-\frac{\beta\,(\ln x-\nu)^2}{\sigma^2}\Big\},\qquad x>0. \tag{24}$$

Making use of (23) in (14), the asymptotic mean, median and variance follow:

$$\begin{aligned} &\lim_{t\to+\infty}\mathbb{E}[X(t)|X(t_0)=x_0]=\exp\Big\{\nu+\frac{\sigma^2}{4\beta}\Big\},\\ &\lim_{t\to+\infty}\mathbb{M}ed[X(t)|X(t_0)=x_0]=e^{\nu},\\ &\lim_{t\to+\infty}\mathbb{V}ar[X(t)|X(t_0)=x_0]=\exp\Big\{2\,\nu+\frac{\sigma^2}{2\beta}\Big\}\Big[\exp\Big\{\frac{\sigma^2}{2\beta}\Big\}-1\Big]. \end{aligned} \tag{25}$$

Under the assumptions (22), we analyze the asymptotic behavior of the FPT density $g_X(S,t|x_0,t_0)$ through a constant boundary $S(t)=S$, with $S>0$. As the threshold S is progressively moved away from the starting point of the process $X(t)$, an exponential approximation is shown to hold for the FPT pdf $g_X(S,t|x_0,t_0)$ (cf. [38,39]). Indeed, if $0<x_0<S$ and $S>e^{\nu}$ or if $x_0>S$ and $0<S<e^{\nu}$, the following behavior holds:

$$g_X(S,t|x_0,t_0)\simeq\widetilde{g}_X(S,t)=R_X(S)\,e^{-R_X(S)\,(t-t_0)}, \tag{26}$$

where

$$R_X(S)=-2\varrho\lim_{t\to+\infty}\Psi_X[S(t),t|y,\tau]=\varrho\,\beta\,S\,(\ln S-\nu)\,W_X(S), \tag{27}$$

with $\Psi_X[S(t),t|y,\tau]$ given in (18) and $\varrho=1$ when $0<x_0<S$ and $\varrho=-1$ when $x_0>S>0$.

Case *(b)* We suppose that

$$\lim_{t\to+\infty}\beta(t)=\beta,\qquad \lim_{t\to+\infty}\sigma^2(t)=\sigma^2,\qquad \nu(t+k\,Q)=\nu(t)\qquad (\beta>0,\sigma>0,k\in\mathbb{N}_0). \tag{28}$$

From (2) and (12), one has

$$\lim_{k\to+\infty}\varphi(t+kQ)=+\infty,\qquad \lim_{k\to+\infty}M(t+kQ|\ln x_0,t_0)=\nu(t),\qquad \lim_{k\to+\infty}V(t+kQ|t_0)=\frac{\sigma^2}{2\,\beta},\tag{29}$$

so that, from (13), one obtains the asymptotic density:

$$W_X(x,t)=\lim_{k\to+\infty}f_X(x,t+kQ|x_0,t_0)=\frac{1}{x}\sqrt{\frac{\beta}{\pi\,\sigma^2}}\,\exp\Big\{-\frac{\beta\,[\ln x-\nu(t)]^2}{\sigma^2}\Big\},\qquad x>0.\tag{30}$$

Making use of (29) in (14) the asymptotic mean, median and variance follow:

$$\begin{aligned}&\lim_{k\to+\infty}\mathbb{E}[X(t+kQ)|X(t_0)=x_0]=\exp\Big\{\nu(t)+\frac{\sigma^2}{4\beta}\Big\},\\&\lim_{k\to+\infty}\mathbb{M}ed[X(t+kQ)|X(t_0)=x_0]=e^{\nu(t)},\\&\lim_{k\to+\infty}\mathbb{V}ar[X(t+kQ)|X(t_0)=x_0]=\exp\Big\{2\,\nu(t)+\frac{\sigma^2}{2\beta}\Big\}\,\Big[\exp\Big\{\frac{\sigma^2}{2\beta}\Big\}-1\Big].\end{aligned}\tag{31}$$

Under the assumptions (28), we analyze the asymptotic behavior of the FPT density $g_X(S,t|x_0,t_0)$ through a constant boundary $S(t)=S$, with $S>0$. From (18), one has:

$$R_X(S,t)=-2\varrho\lim_{k\to+\infty}\Psi_X[S(t+kQ),t|y,\tau]=\varrho\,S\,\Big\{\nu'(t)+\beta[\ln S-\nu(t)]\Big\}\,W_X(S,t),\tag{32}$$

with $\varrho=1$ when $0<x_0<S$ and $\varrho=-1$ when $x_0>S>0$. Furthermore, denoting by $\xi(t)=\exp\{\nu(t)-\nu'(t)/\beta\}$, we have that $R_X(S,t)>0$ if $0<x_0<S$ and $S>\xi(t)$ or if $x_0>S$ and $0<S<\xi(t)$ for all $t\geq t_0$. As the threshold S moves away from x_0, a time-inhomogeneous exponential approximation holds for the FPT pdf $g_X(S,t|x_0,t_0)$ (cf. [38,39]). Indeed, if $0<x_0<S$ and $S>\xi(t)$ or if $x_0>S$ and $0<S<\xi(t)$, for all $t\geq t_0$, one has:

$$g_X(S,t|x_0,t_0)\simeq\widetilde{g}_X(S,t)=R_X(S,t)\,\exp\Big\{-\int_{t_0}^{t}R_X(S,u)\,du\Big\},\tag{33}$$

with $R_X(S,t)$ given in (32).

3. A Special Gompertz-Type Growth with Periodic Carrying Capacity

In the deterministic and stochastic Gompertz-type model, we assume that the growth rate $\beta(t)=\beta$ $(\beta>0)$ and the carrying capacity $e^{\nu(t)}$ is a periodic function of period Q, such that

$$\nu(t)=\ln\Big[a+b\,\sin\Big(\frac{2\pi\,t}{Q}\Big)\Big]\qquad(a>b\geq 0).\tag{34}$$

In Figure 1, we plot the deterministic curves $x(t)$, given in (1), for $x_0=1$ and $x_0=5$ (solid blue curves); the dashed curve represents the periodic carrying capacity $e^{\nu(t)}=e+0.3\,\sin(2\pi\,t)$.

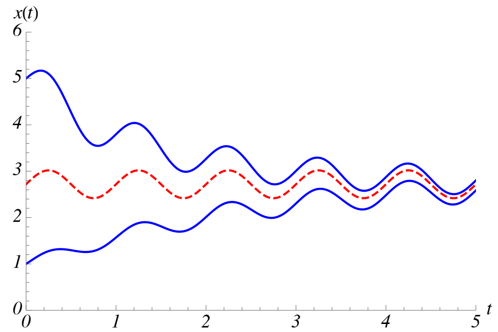

Figure 1. The deterministic curves $x(t)$, given in (1), with $t_0 = 0$, $\beta = 0.6$, $\nu(t) = \ln[e + 0.3\sin(2\pi\, t)]$ and for $x_0 = 1$ and $x_0 = 5$ (solid blue curves); the dashed curve represents the carrying capacity $e^{\nu(t)}$.

Moreover, in the stochastic Gompertz model $X(t)$ with infinitesimal moments (7), we consider two cases: *(i)* $\sigma^2(t) = \sigma^2$ and *(ii)* $\sigma^2(t) = \sigma^2\,(1 - e^{-2\beta\, t})^2$, with $\sigma > 0$.

Case *(i)* Let $\sigma^2(t) = \sigma^2$, with σ real positive constant. In this case, from (12) one has:

$$M(t|y_0, t_0) = \nu(t) + [y_0 - \nu(t_0)]\, e^{-\beta(t-t_0)}, \qquad V(t|t_0) = \frac{\sigma^2}{2\,\beta}\Big(1 - e^{-2\beta(t-t_0)}\Big). \tag{35}$$

For the same choices as Figure 1, with $x_0 = 5$ and $\sigma^2 = 2$, on the left of Figure 2, we plot the transition pdf, given in (13), as function of t when $x = 2$ and $x = 4$ (solid blue curves); the dashed curves represent the corresponding asymptotic densities, obtained from (30). On the right of Figure 2, for the same choices as Figure 1 with $x_0 = 5$ and $\sigma^2 = 2$, the conditional mean and median (14) are compared with the corresponding asymptotic behaviors, given in (31). Note that the median coincides with the deterministic curve plotted in Figure 1.

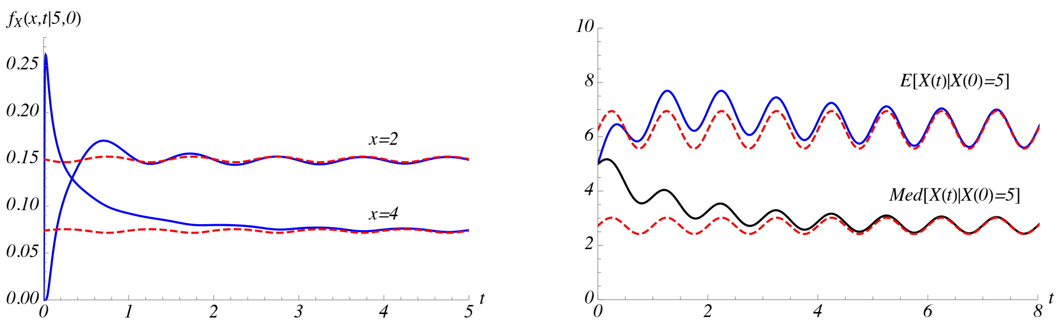

Figure 2. For the same choices as Figure 1, with $x_0 = 5$ and $\sigma^2 = 2$, on the left the transition densities are plotted for $x = 2$ and $x = 4$, whereas, on the right, the conditional mean and median (solid blue curves) are shown. The dashed curves indicate the corresponding asymptotic behaviors.

On the left of Figure 3, the FPT pdf $g_X[S_1(t), t|x_0, t_0]$ through the carrying capacity $S_1(t) = e^{\nu(t)}$, given in (20), is plotted as function of t for the same choices as Figure 1, with $x_0 = 5$ and $\sigma^2 = 2$. Note that, in this case, the shape of the FPT pdf is not affected by the periodicity of the boundary. Moreover, on the right of Figure 3, we plot the asymptotic behavior $\widetilde{g}_X(S, t)$, given in (33), of the FPT density $g_X(S, t|x_0, t_0)$ through the constant boundary $S = 20$ when $x_0 < S$.

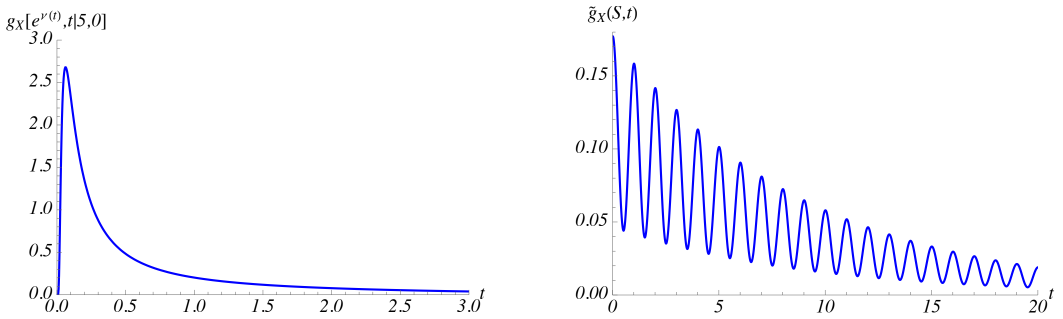

Figure 3. The FPT density through the carrying capacity $S_1(t) = e^{\nu(t)} = e + 0.3\,\sin(2\pi\, t)$ is plotted on the left for the same choices as Figure 1, with $x_0 = 5$ and $\sigma^2 = 2$. On the right, the asymptotic behavior of the FPT pdf through $S = 20$ is shown.

Case *(ii)* Let

$$\sigma^2(t) = \sigma^2\,\left(1 - e^{-2\beta\, t}\right)^2, \tag{36}$$

where σ is a real positive constant. In this case, $\sigma^2(t)$ is an increasing monotonic function which tends to σ^2 as $t \to +\infty$. From (12), one has:

$$M(t|y_0, t_0) = \nu(t) + [y_0 - \nu(t_0)]\, e^{-\beta(t-t_0)}, \quad V(t|t_0) = \frac{\sigma^2\, e^{-2\beta t}}{\beta}\,\Big[\sinh(2\beta\, t) - \sinh(2\beta\, t_0) - 2\beta(t - t_0)\Big]. \tag{37}$$

For the same choices as Figure 1, with $x_0 = 5$ and $\sigma^2(t)$ given in (36) with $\sigma^2 = 2$, on the left of Figure 4, we plot the transition densities, given in (13), as function of t when $x = 2$ and $x = 4$ (solid blue curves); the dashed curves represent the corresponding asymptotic densities, obtained from (30). On the right of Figure 4, for the same choices as Figure 1 with $x_0 = 5$ and $\sigma^2(t) = 2\left(1 - e^{-1.2\,t}\right)^2$, the conditional mean and median (14) are compared with the corresponding asymptotic behaviors, given in (31). Note that, also in this case, the median coincides with the deterministic curve plotted in Figure 1. By comparing Figures 2 and 4, we remark that the asymptotic behaviors are the same, the medians are the same, whereas some differences are highlighted on the averages. Furthermore, the transient behaviors of the transition pdf of Figure 4 are delayed with respect to those shown in Figure 2.

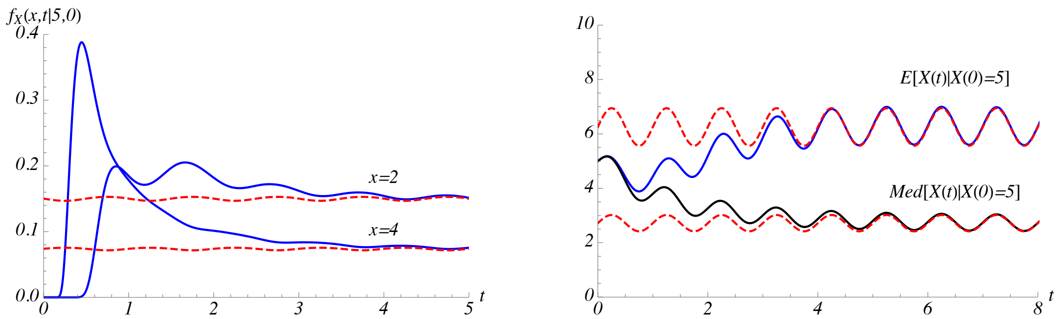

Figure 4. As in Figure 2, with $\sigma^2(t) = 2\left(1 - e^{-1.2\,t}\right)^2$.

Finally, in Figure 5, the FPT density $g_X[S_1(t), t|x_0, t_0]$, given in (20), through the carrying capacity $S_1(t) = e^{\nu(t)} = e + 0.3\,\sin(2\pi\, t)$, is plotted as function of t for the same choices as Figure 1 with $x_0 = 5$ and $\sigma^2(t) = 2\left(1 - e^{-1.2\,t}\right)^2$. Note that the FPT density of Figure 5 is delayed with respect one showed on the left of Figure 3.

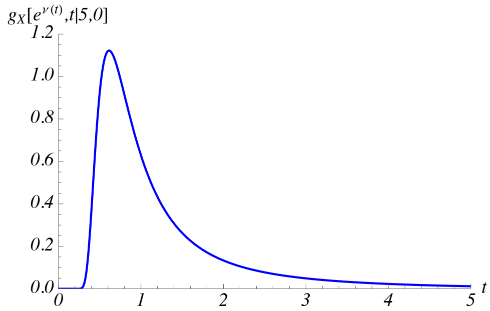

Figure 5. The FPT density (20) through the carrying capacity $S_1(t) = e^{\nu(t)}$ is plotted for the same choices as Figure 1 with $x_0 = 5$ and $\sigma^2(t) = 2\big(1 - e^{-1.2\,t}\big)^2$.

4. Time-Inhomogeneous Restricted Gompertz-Type Growth

In this section, to take into account immigration effects, occurring when the population is not isolated, we analyze a Gompertz-type diffusion process restricted by a reflecting boundary in the zero state. We remark that in the classical Gompertz process the state zero is unattainable, so that, to build the restricted process, we first shift the original Gompertz process and then we introduce in this last one a reflecting condition in the zero state. For the obtained restricted diffusion process, we determine the transition pdf and its moments; furthermore, we analyze the FPT problem and we study the asymptotic behavior.

4.1. Deterministic Evolution

Starting from (1), we perform the transformation $y(t) = x(t) - e^{\nu(t)}$, so that $y(t)$ evolves according to the following law:

$$y(t) = e^{\nu(t)} \left[\exp\Big\{ \big[\ln\big(x_0 + e^{\nu(t_0)}\big) - \nu(t_0)\big]\, e^{-[\varphi(t)-\varphi(t_0)]} \Big\} - 1 \right], \qquad t \ge t_0 \ge 0, \tag{38}$$

with $y(t_0) = x_0 \ge 0$ and $\varphi(t)$ given in (2). Since $x_0 \ge 0$, one has $y(t) \ge 0$ for all $t \ge t_0$ and $\lim_{t\to+\infty} y(t) = 0$, so that the population size decreases to zero and the extinction takes place when t tends to infinity. Therefore, differently from the Gompertz-type model, analyzed in Section 2, in this case, $e^{\nu(t)}$ does not represent the carrying capacity, rather it can be interpreted as a regulation function that influences the population dynamics. From (38), we have:

$$\frac{dy(t)}{dt} = \big[y(t) + e^{\nu(t)}\big] \Big\{ \nu'(t) + \beta(t)\, \big[\nu(t) - \ln\big(y(t) + e^{\nu(t)}\big)\big] \Big\} - \nu'(t)\, e^{\nu(t)}, \qquad y(t_0) = x_0.$$

By using (38), the population density at time $t + \Delta t$ can be expressed as follows:

$$\begin{aligned} y(t+\Delta t) + e^{\nu(t+\Delta t)} = \Big[y(t) + e^{\nu(t)}\Big] \exp\Big\{ & e^{-\varphi(t+\Delta t)} \Big[\nu(t+\Delta t)\, e^{\varphi(t+\Delta t)} - \nu(t)\, e^{\varphi(t)}\Big] \\ & - e^{-\varphi(t+\Delta t)} \Big[e^{\varphi(t+\Delta t)} - e^{\varphi(t)}\Big] \ln\Big[y(t) + e^{\nu(t)}\Big] \Big\}. \end{aligned} \tag{39}$$

Starting from (39), a stochastic restricted time-inhomogeneous diffusion process can be constructed.

4.2. Stochastic Evolution

Under the same assumption of random environment of Section 2.2, starting from (39), we consider the stochastic process $\{\widetilde{Y}(t), t \geq 0\}$ that satisfies the following stochastic equation:

$$\begin{aligned}\big[\widetilde{Y}(t+\Delta t) + e^{\nu(t+\Delta t)}\big] - \big[\widetilde{Y}(t) + e^{\nu(t)}\big] = \Big[\widetilde{Y}(t) + e^{\nu(t)}\Big] \Big[\exp\Big\{e^{-\varphi(t+\Delta t)}\Big[Z(t+\Delta t) - Z(t)\Big] \\ -e^{-\varphi(t+\Delta t)}\Big[e^{\varphi(t+\Delta t)} - e^{\varphi(t)}\Big] \ln\big[\widetilde{Y}(t) + e^{\nu(t)}\big]\Big\} - 1\Big],\end{aligned} \tag{40}$$

with $Z(t+\Delta t) - Z(t)$ given in (4). Proceeding as in Section 2.2, we obtain the infinitesimal moments $B_n(x,t)$ $(n = 1, 2, \ldots)$ of $\widetilde{Y}(t)$:

$$\begin{aligned}&B_1(x,t) = \big[x + e^{\nu(t)}\big] \Big\{\nu'(t) + \beta(t)\,\nu(t) - \beta(t)\,\ln\big[x + e^{\nu(t)}\big] + \frac{\sigma^2(t)}{2}\Big\} - \nu'(t)\, e^{\nu(t)}, \\ &B_2(x,t) = \sigma^2(t)\,\big[x + e^{\nu(t)}\big]^2 \\ &B_n(x,t) = 0, \qquad n = 3, 4, \ldots\end{aligned} \tag{41}$$

Hence, $\widetilde{Y}(t)$ is a time-inhomogeneous diffusion process with space-state $(-e^{\nu(t)}, +\infty)$ having infinitesimal drift and variance $B_1(x,t)$ and $B_2(x,t)$, respectively. The transition pdf $f_{\widetilde{Y}}(x,t|x_0,t_0)$ is

$$f_{\widetilde{Y}}(x,t|x_0,t_0) = f_X\big[x + e^{\nu(t)}, t\big|x_0 + e^{\nu(t_0)}, t_0\big], \tag{42}$$

with $f_X(x,t|x_0,t_0)$ given in (13).

In order to describe the population dynamics, we consider the stochastic process $\{Y(t), t \geq 0\}$, obtained by restrict the space-state of $\widetilde{Y}(t)$ to the interval $[0, +\infty)$, with 0 reflecting boundary. This ensures that the population does not become extinct. The transition pdf $r_Y(x,t|x_0,t_0)$ of the restricted diffusion process $Y(t)$ is a solution of the Fokker–Planck equation and satisfies the reflecting condition in zero state and the initial delta condition:

$$\begin{aligned}&\frac{\partial r_Y(x,t|x_0,t_0)}{\partial t} = -\frac{\partial}{\partial x}\Big[B_1(x,t)\; r_Y(x,t|x_0,t_0)\Big] + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\Big[B_2(x,t)\; r_Y(x,t|x_0,t_0)\Big], \\ &\lim_{x\downarrow 0}\Big\{B_1(x,t)\; r_Y(x,t|x_0,t_0) - \frac{1}{2}\,\frac{\partial}{\partial x}\Big[B_2(x,t)\; r_Y(x,t|x_0,t_0)\Big]\Big\} = 0, \\ &\lim_{t\downarrow t_0} r_Y(x,t|x_0,t_0) = \delta(t - t_0),\end{aligned} \tag{43}$$

with $B_i(x,t)$ $(i = 1, 2)$ given in (41). The second equation in (43) expresses a zero-flux condition at $x = 0$ for the restricted process $Y(t)$. Making use of transformations:

$$y = \ln\big[x + e^{\nu(t)}\big], \qquad y_0 = \ln\big[x_0 + e^{\nu(t_0)}\big], \qquad r_Y(x,t|x_0,t_0) = \frac{1}{x + e^{\nu(t)}}\, r_U(y,t|y_0,t_0), \tag{44}$$

from (43), one has:

$$\begin{aligned}&\frac{\partial r_U(y,t|y_0,t_0)}{\partial t} = -\frac{\partial}{\partial y}\Big[C_1(y,t)\; r_U(y,t|y_0,t_0)\Big] + \frac{1}{2}\,\frac{\partial^2}{\partial y^2}\Big[C_2(t)\; r_U(y,t|y_0,t_0)\Big], \\ &\lim_{y\downarrow \nu(t)}\Big\{C_1(y,t)\; r_U(y,t|y_0,t_0) - \frac{C_2(t)}{2}\,\frac{\partial}{\partial x} r_U(y,t|y_0,t_0)\Big\} - \nu'(t)\, r_U(y,t|y_0,t_0) = 0, \\ &\lim_{t\downarrow t_0} r_U(y,t|y_0,t_0) = \delta(t - t_0),\end{aligned} \tag{45}$$

with $C_1(y,t)$ and $C_2(t)$ given in (11). Therefore, $r_U(y,t|y_0,t_0)$ is the transition pdf of a restricted time-inhomogeneous Ornstein–Uhlenbeck process restricted to the state-space $[\nu(t), +\infty)$ by a reflecting condition in $\nu(t)$. The second equation in (45) expresses a zero-flux condition at $y = \nu(t)$ for the restricted Ornstein–Uhlenbeck. As proved in Buonocore et al. [34], one has

$$r_U(y,t|y_0,t_0) = f_U(y,t|y_0,t_0) + f_U[2\nu(t) - y, t|y_0,t_0], \qquad y \geq \nu(t),\ y_0 \geq \nu(t_0),$$

where $f_U(y,t|y_0,t_0)$ is a normal pdf with conditional mean and variance given in (12). Recalling (44), the transition pdf of $Y(t)$ is

$$\begin{aligned} & r_Y(x,t|x_0,t_0) = \frac{1}{x+e^{\nu(t)}}\,\frac{1}{\sqrt{2\pi\,V(t|t_0)}}\,\exp\Big\{-\frac{[\ln(x+e^{\nu(t)}) - M(t|\ln(x_0+e^{\nu(t_0)}),t_0)]^2}{2\,V(t|t_0)}\Big\} \\ & + \frac{1}{x+e^{\nu(t)}}\,\frac{1}{\sqrt{2\pi\,V(t|t_0)}}\,\exp\Big\{-\frac{[2\,\nu(t) - \ln(x+e^{\nu(t)}) - M(t|\ln(x_0+e^{\nu(t_0)}),t_0)]^2}{2\,V(t|t_0)}\Big\}, \\ & x, x_0 > 0. \end{aligned} \tag{46}$$

Moreover, setting $y_0 = \ln(x_0 + e^{\nu(t_0)})$, the conditional mean and second order moment are:

$$\begin{aligned} & \mathbb{E}[Y(t)|Y(t_0) = x_0] = \frac{1}{2}\,\exp\Big\{M(t|y_0,t_0) + \frac{V(t|t_0)}{2}\Big\}\,\Big\{1 - \mathrm{Erf}\Big[\frac{\nu(t) - M(t|y_0,t_0) - V(t|t_0)}{\sqrt{2\,V(t|t_0)}}\Big]\Big\} \\ & + \frac{e^{2\nu(t)}}{2}\,\exp\Big\{-M(t|y_0,t_0) + \frac{V(t|t_0)}{2}\Big\}\,\Big\{1 + \mathrm{Erf}\Big[\frac{\nu(t) - M(t|y_0,t_0) + V(t|t_0)}{\sqrt{2\,V(t|t_0)}}\Big]\Big\} - e^{\nu(t)}, \end{aligned} \tag{47}$$

$$\begin{aligned} & \mathbb{E}[Y^2(t)|Y(t_0) = x_0] = \frac{1}{2}\,\exp\Big\{2\,M(t|y_0,t_0) + 2\,V(t|t_0)\Big\}\,\Big\{1 - \mathrm{Erf}\Big[\frac{\nu(t) - M(t|y_0,t_0) - 2\,V(t|t_0)}{\sqrt{2\,V(t|t_0)}}\Big]\Big\} \\ & + \frac{e^{4\nu(t)}}{2}\,\exp\Big\{-2\,M(t|y_0,t_0) + 2\,V(t|t_0)\Big\}\,\Big\{1 + \mathrm{Erf}\Big[\frac{\nu(t) - M(t|y_0,t_0) + 2\,V(t|t_0)}{\sqrt{2\,V(t|t_0)}}\Big]\Big\} \\ & - e^{2\,\nu(t)} - 2\,e^{\nu(t)}\,\mathbb{E}[Y(t)|Y(t_0) = x_0], \end{aligned} \tag{48}$$

where $\mathrm{Erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-z^2}\,dz$ denotes the error function. Furthermore, when $\sigma^2(t)$ is zero, the conditional mean is identified with the deterministic solution (38) and the conditional variance is zero.

4.3. First-Passage Time Problem for $Y(t)$

We consider the FPT problem for the restricted diffusion process $Y(t)$ with infinitesimal moments (41). Let

$$\mathcal{T}_Y = \begin{cases} \inf_{t \geq t_0}\{t : Y(t) \geq S(t)\}, & 0 \leq Y(t_0) = x_0 < S(t_0), \\ \inf_{t \geq t_0}\{t : Y(t) \leq S(t)\}, & Y(t_0) = x_0 > S(t_0) \geq 0, \end{cases} \tag{49}$$

be the random variable that describes the FPT of $Y(t)$ from $Y(t_0) = x_0$ to the continuous boundary $S(t) \in C^1(T)$ and let $g_Y[S(t),t|x_0,t_0] = dP(\mathcal{T}_Y \leq t)/dt$.

If $x_0 > S(t_0) \geq 0$, the reflecting boundary 0 does not affect the FPT through $S(t)$, so that

$$g_Y[S(t),t|x_0,t_0] = g_X\big[S(t) + e^{\nu(t)}, t|x_0 + e^{\nu(t_0)}, t_0\big], \qquad x_0 > S(t_0) \geq 0, \tag{50}$$

where g_X is the FPT pdf of the diffusion process $X(t)$ analyzed in Section 2.4. In this case, the FPT density through zero state is of interest and the related pdf can be obtained from (20). Indeed, setting $d_1 = d_2 = 0$ in (19), from (20) and (50), one has:

$$\begin{aligned} g_Y(0,t|x_0,t_0) &= g_X\big[e^{\nu(t)}, t|x_0 + e^{\nu(t_0)}, t_0\big] \\ &= \Big\{\ln[x_0 + e^{\nu(t_0)}] - \nu(t_0)\Big\}\,\frac{e^{\varphi(t)+\varphi(t_0)}\,\sigma^2(t)}{\int_{t_0}^t e^{2\varphi(u)}\,\sigma^2(u)\,du}\,e^{\nu(t)}\,f_X[e^{\nu(t)}, t|x_0 + e^{\nu(t_0)}, t_0], \quad x_0 > 0, \end{aligned} \tag{51}$$

with f_X given in (13). Moreover, recalling (21), from (51), it follows that

$$\int_{t_0}^{+\infty} g_Y(0,t|x_0,t_0)\,dt = \int_{t_0}^{+\infty} g_X\left[e^{\nu(t)},t|x_0+e^{\nu(t_0)},t_0\right]dt = 1,$$

so that the first passage through zero state is a certain event. Instead, when $0 \le x_0 < S(t_0)$, the reflecting boundary 0 affects the FPT through $S(t)$. Indeed, the FPT pdf is a solution of the first-kind Volterra integral equation:

$$r_Y(x,t|x_0,t_0) = \int_{t_0}^{t} g_Y[S(\tau),\tau|x_0,t_0]\,r_Y[x,t|S(\tau),\tau]\,d\tau \qquad [0 \le x_0 < S(t_0), x \ge S(t)]. \tag{52}$$

In this case, making use of the results given in Buonocore et al. [34], the FPT density $g_Y[S(t),t|x_0,t_0]$ is a solution of the following non-singular second-kind Volterra integral equation:

$$g_Y[S(t),t|x_0,t_0] = -2\Psi_Y[S(t),t|x_0,t_0] + 2\int_{t_0}^{t} g_Y[S(\tau),\tau|x_0,t_0]\,\Psi_Y[S(t),t|S(\tau),\tau]\,d\tau, \quad 0 \le x_0 < S(t_0), \tag{53}$$

where

$$\begin{aligned}\Psi_Y[S(t),t|y,\tau] \;=\; \Bigg\{ & \frac{S'(t)-\nu'(t)\,S(t)}{2} \\ & -[S(t)+e^{\nu(t)}]\,\frac{\ln[S(t)+e^{\nu(t)}]-\nu(t)}{2}\,\frac{\sigma^2(t)\,e^{2\varphi(t)}-\beta(t)\int_\tau^t e^{2\varphi(u)}\,\sigma^2(u)\,du}{\int_\tau^t e^{2\varphi(u)}\,\sigma^2(u)\,du} \\ & +[S(t)+e^{\nu(t)}]\,\frac{\ln[y+e^{\nu(\tau)}]-\nu(\tau)}{2}\,\frac{\sigma^2(t)\,e^{\varphi(t)+\varphi(\tau)}}{\int_\tau^t e^{2\varphi(u)}\,\sigma^2(u)\,du}\Bigg\}\, r_Y[S(t),t|y,\tau],\end{aligned} \tag{54}$$

with $r_Y[x,t|x_0,t_0]$ given in (46). Equation (53) can be used to obtain a numerical evaluation of g_Y.

4.4. Asymptotic Behavior for $Y(t)$

We analyze the asymptotic behavior of the process $Y(t)$ in the same cases of Section 2.4.

Case *(a)* Under the assumptions (22), the relations (23) again hold. Hence, from (46), one obtains the steady-state density:

$$W_Y(x) = \lim_{t\to+\infty} r_Y(x,t|x_0,t_0) = \frac{2}{x+e^\nu}\sqrt{\frac{\beta}{\pi\,\sigma^2}}\,\exp\Big\{-\frac{\beta\,[\ln(x+e^\nu)-\nu]^2}{\sigma^2}\Big\}, \qquad x > 0. \tag{55}$$

Making use of (23) in (47) and (48), the asymptotic mean and second order moment follow:

$$\begin{aligned}&\lim_{t\to+\infty}\mathbb{E}[Y(t)|Y(t_0)=x_0] = e^\nu\left\{\exp\Big\{\frac{\sigma^2}{4\beta}\Big\}\left[1+\mathrm{Erf}\Big(\frac{\sigma}{2\sqrt{\beta}}\Big)\right]-1\right\},\\ &\lim_{t\to+\infty}\mathbb{E}[Y^2(t)|Y(t_0)=x_0] = e^{2\nu}\left\{1-2\exp\Big\{\frac{\sigma^2}{4\beta}\Big\}\left[1+\mathrm{Erf}\Big(\frac{\sigma}{2\sqrt{\beta}}\Big)\right]+\exp\Big\{\frac{\sigma^2}{\beta}\Big\}\left[1+\mathrm{Erf}\Big(\frac{\sigma}{\sqrt{\beta}}\Big)\right]\right\}.\end{aligned} \tag{56}$$

Under the assumptions (22), we analyze the asymptotic behavior of the FPT density $g_Y(S,t|x_0,t_0)$ through a constant boundary $S(t)=S$, with $0 \le x_0 < S$. From (54), one has:

$$R_Y(S) = -2\lim_{t\to+\infty}\Psi_Y[S(t),t|y,\tau] = \beta(S+e^\nu)\,\left[\ln(S+e^\nu)-\nu\right]W_Y(S) \tag{57}$$

with $W_Y(x)$ given in (55). Hence, if $0 \le x_0 < S$, the FPT pdf $g_Y(S,t|x_0,t_0)$ admits the following asymptotic behavior:

$$g_Y(S,t|x_0,t_0) \simeq \widetilde{g}_Y(S,t) = R_Y(S)\,e^{-R_Y(S)\,(t-t_0)}, \tag{58}$$

with $R_Y(S)$ given in (57).

Case *(b)* Under the assumptions (28), relations (29) are satisfied, so that, from (46), one obtains the following asymptotic density:

$$W_Y(x,t) = \lim_{k\to+\infty} r_X(x,t+kQ|x_0,t_0) = \frac{2}{x+e^{\nu(t)}}\sqrt{\frac{\beta}{\pi\sigma^2}}\exp\Big\{-\frac{\beta\,[\ln(x+e^{\nu(t)})-\nu(t)]^2}{\sigma^2}\Big\}, \qquad x>0. \tag{59}$$

Making use of (23) in (47) and (48), the asymptotic mean and second order moment follow:

$$\begin{aligned}
&\lim_{t\to+\infty}\mathbb{E}[Y(t)|Y(t_0)=x_0] = e^{\nu(t)}\Big\{\exp\Big\{\frac{\sigma^2}{4\beta}\Big\}\Big[1+\mathrm{Erf}\Big(\frac{\sigma}{2\sqrt{\beta}}\Big)\Big]-1\Big\},\\
&\lim_{t\to+\infty}\mathbb{E}[Y^2(t)|Y(t_0)=x_0] = e^{2\nu(t)}\Big\{1-2\exp\Big\{\frac{\sigma^2}{4\beta}\Big\}\Big[1+\mathrm{Erf}\Big(\frac{\sigma}{2\sqrt{\beta}}\Big)\Big]+exp\Big\{\frac{\sigma^2}{\beta}\Big\}\Big[1+\mathrm{Erf}\Big(\frac{\sigma}{\sqrt{\beta}}\Big)\Big]\Big\}.
\end{aligned} \tag{60}$$

Under the assumptions (28), we analyze the asymptotic behavior of the FPT density $g_Y(S,t|x_0,t_0)$ through a constant boundary $S(t)=S$, with $0\le x_0<S$. From (54), one has:

$$R_Y(S,t) = -2\lim_{k\to+\infty}\Psi_Y[S(t+kQ),t|y,\tau] = \Big\{S\,\nu'(t)+\beta\big[S+e^{\nu(t)}\big]\big[\ln(S+e^{\nu(t)})-\nu(t)\big]\Big\}\,W_Y(S,t), \tag{61}$$

with $W_Y(x,t)$ given in (59). Therefore, if $0\le x_0<S$ and if the boundary S satisfies the condition

$$S\,\nu'(t)+\beta\big[S+e^{\nu(t)}\big]\big[\ln(S+e^{\nu(t)})-\nu(t)\big]>0$$

for all $t\ge t_0$, then the FPT pdf $g_Y(S,t|x_0,t_0)$ admits the following asymptotic behavior:

$$g_Y(S,t|x_0,t_0)\simeq\widetilde{g}_Y(S,t)=R_Y(S,t)\,\exp\Big\{-\int_{t_0}^{t}R_Y(S,u)\,du\Big\}, \tag{62}$$

with $R_Y(S,t)$ given in (61).

5. A Special Restricted Gompertz-Type Growth with Periodic Regulation Function

In the deterministic and stochastic restricted Gompertz model, we assume that the growth rate $\beta(t)=\beta$ $(\beta>0)$ and the function $\nu(t)$ is expressed as in (34). In Figure 6, we plot the deterministic curves $y(t)$, given in (38), for $x_0=1$ and $x_0=5$.

As in Section 3, for the restricted process $Y(t)$ with infinitesimal moments (41), we consider two cases: *(i)* $\sigma^2(t)=\sigma^2$ and *(ii)* $\sigma^2(t)=\sigma^2\,(1-e^{-2\beta t})^2$, with $\sigma>0$.

Case *(i)* Let $\sigma^2(t)=\sigma^2$, with σ a real positive constant. Equation (35) holds. For the same choices of Figure 6 with $x_0=5$, in Figure 7, we plot the transition pdf, given in (46), as a function of t for $x=2$ and $x=4$ (solid blue curves) for two different choices of σ^2. The dashed curves represent the corresponding asymptotic densities, obtained from (59).

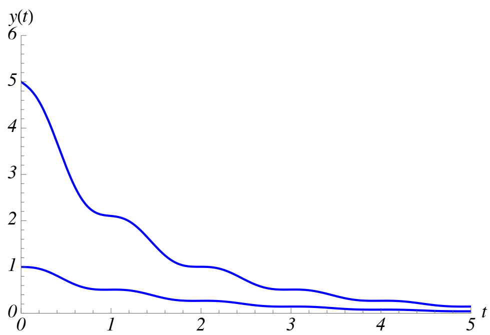

Figure 6. The deterministic curves $y(t)$, given in (38), with $t_0=0$, $\beta=0.6$, $\nu(t)=\ln[e+0.3\sin(2\pi t)]$ and for $x_0=1$ and $x_0=5$.

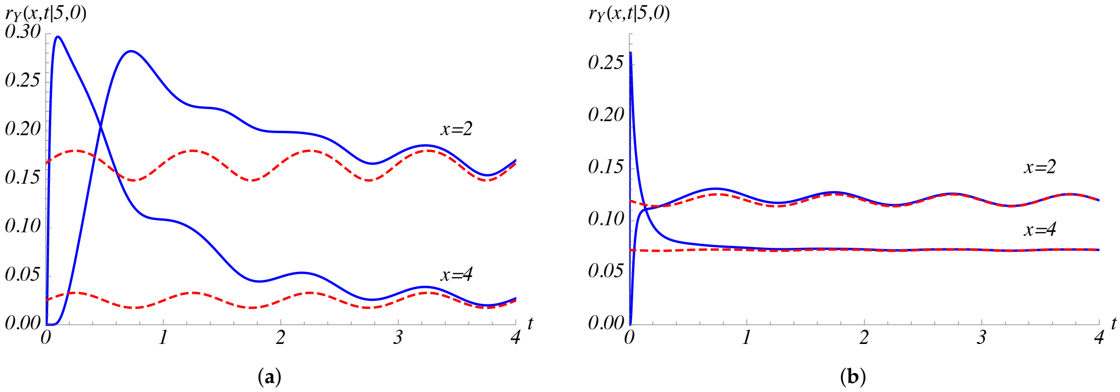

Figure 7. The transition densities, given in (46), are plotted as a function of t for the same choices as Figure 6 with $x_0 = 5$ for $x = 2$ and $x = 4$ (solid blue curves). The dashed curves indicate the corresponding asymptotic densities. (**a**) $\sigma^2 = 0.2$; (**b**) $\sigma^2 = 2$.

Moreover, for the same choices as Figure 6, with $x_0 = 5$, in Figure 8, the conditional mean given in (47) is compared with the corresponding asymptotic behaviors, given in (60) for $\sigma^2 = 0.2$ and $\sigma^2 = 2$. We note that the value of σ^2 influences the behavior of conditional mean function; indeed, small values of σ^2 induce a decrease in population size. Furthermore, due to the effect of the reflecting boundary in zero state, the mean size of the population always remains positive when t increases, so avoiding the extinction.

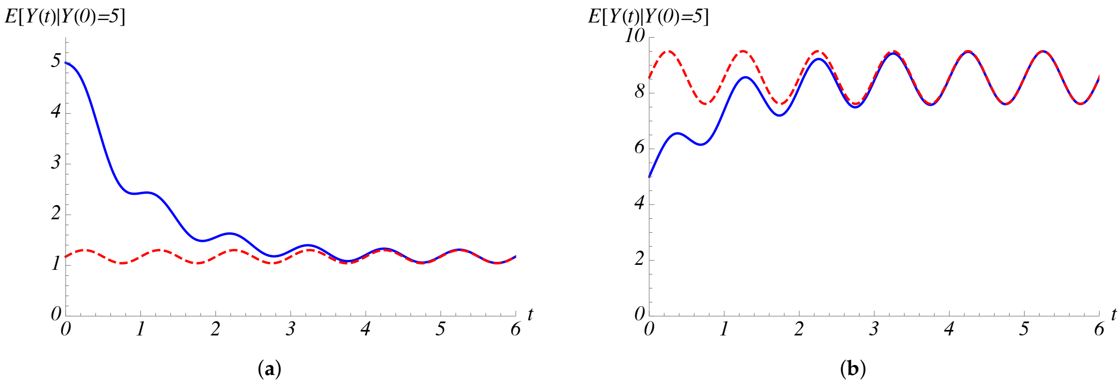

Figure 8. The conditional mean, given in (47), is compared with the corresponding asymptotic behavior (60) for the same choices as Figure 6 with $x_0 = 5$. (**a**) $\sigma^2 = 0.2$; (**b**) $\sigma^2 = 2$.

On the left of Figure 9, the FPT pdf $g_Y(0, t|x_0, t_0)$, given in (51), is plotted as function of t for the same choices as Figure 6 with $x_0 = 5$ and $\sigma^2 = 2$; note that the shapes of the FPT pdf are not affected by the periodicity of $\nu(t)$. Furthermore, on the right of Figure 9, the asymptotic behavior $\widetilde{g}_Y(S, t)$ of the FPT density $g_Y(S, t|x_0, t_0)$, given in (62), is shown with $S = 20$ and $x_0 < S$.

Case *(ii)* We choose $\sigma^2(t)$ as in (36). Relation (37) holds again. For the same choices as Figure 6, with $x_0 = 5$, in Figure 10, the transition pdf, given in (46), is plotted as a function of t, with $\sigma^2(t)$ given in (36) for $x = 2$ and $x = 4$ (solid blue curves) and two different choices of σ^2. The dashed curves represent the corresponding asymptotic densities, obtained from (59). We note that the transient behaviors in Figure 10 are delayed with respect to those shown in Figure 7. Moreover, for the same choices as Figure 6, with $x_0 = 5$, in Figure 11, the conditional mean (47) is compared with the corresponding asymptotic behavior, given in (60). By comparing Figures 8 and 11, we note that the asymptotic behaviors of the averages are the same; moreover, the behaviors of the conditional mean function are strongly influenced by $\sigma^2(t)$.

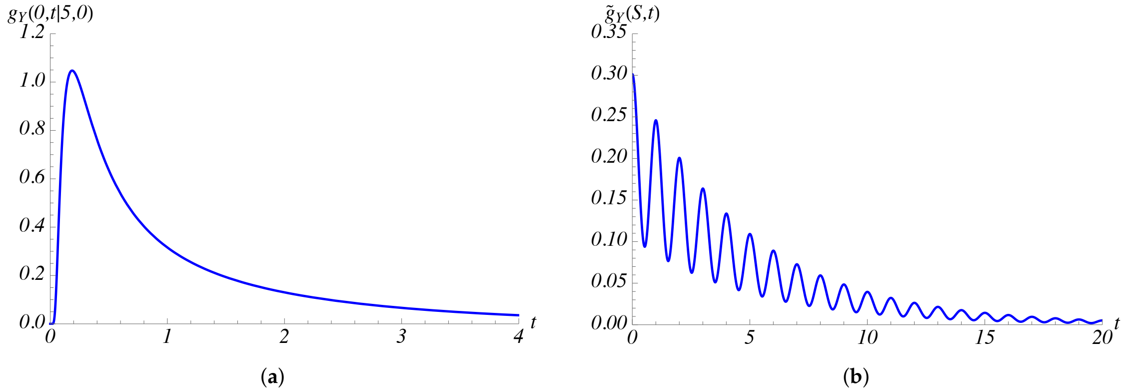

Figure 9. The FPT density (51) is plotted on the left for the same choices as Figure 6 with $x_0 = 5$. The asymptotic behavior $\widetilde{g}_Y(S,t)$ of the FPT density $g_Y(S,t|x_0,t_0)$ is shown on the right, given in (62), with $S = 20$ and $x_0 < S$. (**a**) $\sigma^2 = 2$; (**b**) $\sigma^2 = 2$.

In Figure 12, the FPT density $g_Y(0,t|x_0,t_0)$, given in (51), is plotted as a function of t for the same choices as Figure 1 with $x_0 = 5$ and $\sigma^2(t) = 2\big(1 - e^{-1.2\,t}\big)^2$. Note that the FPT density of Figure 12 is delayed with respect to that shown on the left of Figure 9.

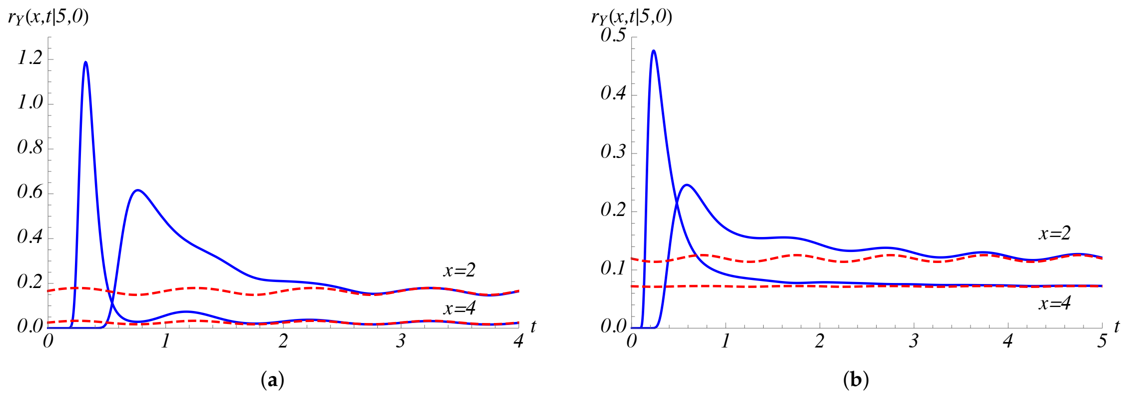

Figure 10. The transition densities, given in (46), are plotted as function of t for the same choices as Figure 6 with $x_0 = 5$, for $x = 2$ and $x = 4$ (solid blue curves). The dashed curves indicate the corresponding asymptotic densities. (**a**) $\sigma^2(t) = 0.2\big(1 - e^{-1.2\,t}\big)^2$; (**b**) $\sigma^2(t) = 2\big(1 - e^{-1.2\,t}\big)^2$.

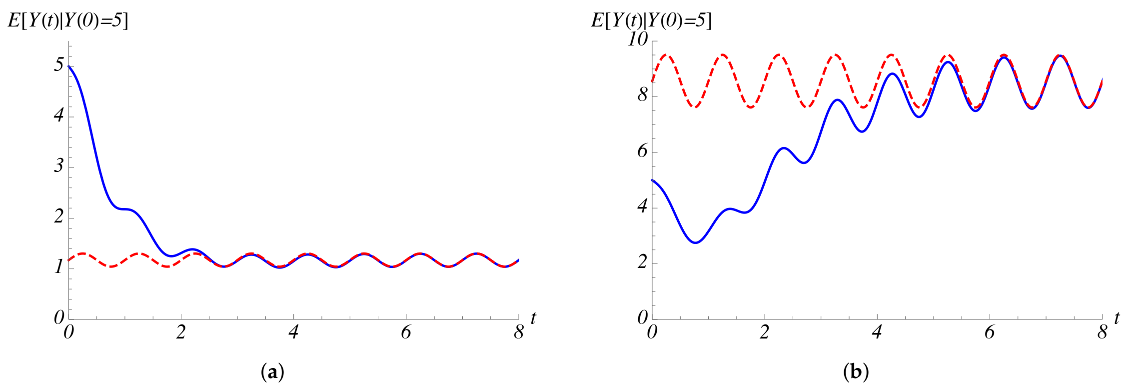

Figure 11. The conditional mean, given in (47), is compared with the corresponding asymptotic behavior (60) for the same choices as Figure 6 with $x_0 = 5$. (**a**) $\sigma^2(t) = 0.2\big(1 - e^{-1.2\,t}\big)^2$; (**b**) $\sigma^2(t) = 2\big(1 - e^{-1.2\,t}\big)^2$.

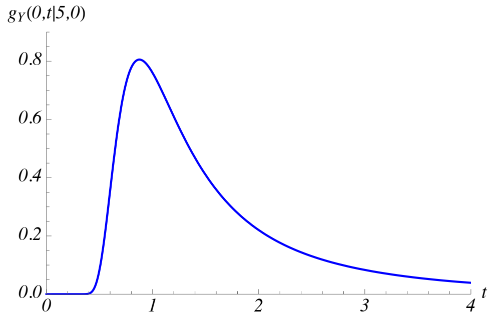

Figure 12. The FPT density (51) through the zero state is plotted for the same choices as Figure 6 with $x_0 = 5$ and $\sigma^2(t) = 2(1 - e^{-1.2\,t})^2$.

6. Conclusions

In this paper, we consider two different time-inhomogeneous diffusion processes useful to model the evolution of a population in a random environment. They arise as approximations of the solution of deterministic Gompertz-type growth models. The first considered stochastic model is a Gompertz-type diffusion process $X(t)$ with growth rate $\beta(t)$, carrying capacity $e^{\nu(t)}$ and noise intensity $\sigma^2(t)$, whose conditional median coincides with the deterministic solution. The second stochastic model is a shifted Gompertz diffusion process $Y(t)$, restricted to the interval $[0, +\infty)$, where zero is a reflecting boundary; the growth rate $\beta(t)$, the regulation function $\nu(t)$ and the noise intensity $\sigma^2(t)$ are time-dependent. For both processes, particular attention is dedicated to analyzing the first-passage time problem and the asymptotic behavior of the transition densities and of the FPT densities through a constant boundary S in two cases: *(a)* for asymptotically constant functions $\beta(t), \nu(t), \sigma^2(t)$ and for *(b)* for asymptotically constant functions $\beta(t), \sigma^2(t)$ with $\nu(t)$ periodic function. In particular, for the considered stochastic models with $\beta(t) = \beta$ and $\nu(t) = \ln[a + b\sin(2\pi t/Q)]$, some comparisons are carried out for *(i)* $\sigma^2(t) = \sigma^2$ and for *(ii)* $\sigma^2(t) = \sigma^2\,(1 - e^{-2\beta\,t})^2$ in order to highlight both the role of $\sigma^2(t)$ and the effect of the reflecting condition on the population growth.

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