

Article

# Extending Structures for Lie 2-Algebras

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**Abstract:** The extending structures problem for strict Lie 2-algebras is studied. To provide the theoretical answer to this problem, this paper introduces the unified product of a given strict Lie 2-algebra  $\mathfrak{g}$  and 2-vector space  $V$ . The unified product includes some interesting products such as semi-direct product, crossed product, and bicrossed product. The paper focuses on crossed and bicrossed products, which give the answer to the extension problem and factorization problem, respectively.

**Keywords:** strict Lie 2-algebra; extending structures problem; crossed product; unified product

## 1. Introduction

Recently, many mathematicians have paid attention to Lie algebra-like structures. In particular, they seek category theoretic analogs of them in [1–3]. A kind of algebra, strict Lie 2-algebra, has appeared in some parts of the articles. Lie 2-algebras play a part in studying algebraic structures on Lie 2-groups, string theory, higher categorical structures, and multisymplectic structures, Courant algebroids, Dirac structures, omni-Lie 2-algebras, and Hom-Lie 2-algebras, and so on [4,5]. For example, Omni-Lie 2-algebras are a kind of special weak Lie 2-algebra. Weak Lie 2-algebras are a categorification of Lie algebras, or an internal category of Lie algebras. This paper is going to study extensions of crossed modules of Lie algebras. Meanwhile, crossed modules of Lie algebras can be identified with strict Lie 2-algebras. The extending structures problem for some algebra objects such as Lie algebras, Hopf algebras, Leibniz algebras, associative algebras, left-symmetric algebras and Lie conformal algebras have been studied in [6–11] respectively.

Lie 2-algebras were first introduced by J. C. Baez and A. S. Crans in 2004. It is a new kind algebra and more sophisticated than the usual Lie algebras. By now, many Lie 2-algebra theories have not been developed. The extension theory of Lie 2-algebras has been characterized by cohomological groups in [12]. However, in [12], one needs the subalgebras to be abelian. This paper will abandon the commutative conditions in [12]. More explicitly, the paper studies the following extending structures problem of strict Lie 2-algebras.

**Problem 1.** Let  $\mathfrak{g} := ((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathcal{L})$  be a strict Lie 2-algebra,  $V := V_1 \xrightarrow{\mu} V_0$  a 2-vector space and  $\mathfrak{c} : \mathfrak{c}_1 \xrightarrow{\epsilon} \mathfrak{c}_0$  a 2-vector space such that  $\mathfrak{g}$  is a 2-vector subspace. Suppose that  $\mathfrak{c}_i = \mathfrak{g}_i \oplus V_i$  as vector spaces for  $i = 0, 1$ . Describe and classify all strict Lie 2-algebra structures on  $\mathfrak{c}$  up to an isomorphism of Lie 2-algebras that stabilizes  $\mathfrak{g}$ .

In fact, this problem generalizes two important algebra problems. One is the extension problem for strict Lie 2-algebras.

**Problem 2.** Given two Lie 2-algebras  $\mathfrak{g} := ((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathcal{L})$ ,  $V := ((V_0, *_0), (V_1, *_1), \mu, *_2)$ . Describe and classify all extensions of  $V$  by  $\mathfrak{g}$  which are strict Lie 2-algebras up to an isomorphism of Lie 2-algebras that stabilizes  $\mathfrak{g}$ .

Here, an extension of  $V$  by  $\mathfrak{g}$  is a Lie 2-algebra  $\mathfrak{c}$  which satisfies short exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathfrak{g}_1 & \xrightarrow{\iota_1} & \mathfrak{c}_1 & \xrightarrow{\pi_1} & V_1 \longrightarrow 0 \\
 & & \downarrow \phi & & \downarrow \varepsilon & & \downarrow \mu \\
 0 & \longrightarrow & \mathfrak{g}_0 & \xrightarrow{\iota_0} & \mathfrak{c}_0 & \xrightarrow{\pi_0} & V_0 \longrightarrow 0
 \end{array}$$

where  $\iota_i, \pi_i$  are linear maps for  $i = 0, 1$ , i.e.,  $Im(\iota_i) = Ker(\pi_i)$ ,  $Ker(\iota_i) = 0$  and  $Im(\pi_i) = V_i$  (Ref. [13]). When  $\mathfrak{g}$  is an abelian Lie 2-algebra, all extensions of  $V$  by  $\mathfrak{g}$ , which are strict Lie 2-algebras up to an isomorphism of Lie 2-algebras that stabilizes  $\mathfrak{g}$ , can be characterized by the second cohomology group  $H^2_{\nabla}(V, \phi)$  defined in [12]. When  $\mathfrak{g}$  is not abelian, all extensions of  $V$  by  $\mathfrak{g}$ , which are strict Lie 2-algebras that stabilizes  $\mathfrak{g}$ , are exactly the non-abelian extensions of  $V$  by  $\mathfrak{g}$  defined in [13]. The other problem is the factorization problem for strict Lie 2-algebras.

**Problem 3.** Let  $\mathfrak{g} := ((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$ ,  $V := ((V_0, *_0), (V_1, *_1), \mu, *_2)$  be two strict Lie 2-algebras and  $\mathfrak{c} : \mathfrak{c}_1 \xrightarrow{\varepsilon} \mathfrak{c}_0$  a 2-vector space such that  $\mathfrak{g}$  is a 2-vector subspace. Suppose that  $\mathfrak{c}_i = \mathfrak{g}_i \oplus V_i$  as vector spaces for  $i = 0, 1$ . Describe and classify all strict Lie 2-algebra structures on  $\mathfrak{c}$  such that  $\mathfrak{g}$  and  $V$  are two sub-Lie 2-algebras of  $\mathfrak{c}$  up to an isomorphism of Lie 2-algebras that stabilizes  $\mathfrak{g}$ .

Therefore, the study of the extending structures problem is of signification and will be useful for investigating the structure theory of Lie 2-algebras. The paper always assumes that  $V \neq 0$ . Two cohomological type objects are constructed by introducing the unified products. Using this unified product, the extension problem and the factorization problem for strict Lie 2-algebras are studied in detail.

An outline of this paper is as follows. In Section 2, the paper provides some preliminaries. The unified product  $\mathfrak{g} \bowtie V$  of a strict Lie 2-algebra  $\mathfrak{g}$  by 2-vector space  $V$  associated with an extending datum  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \sigma, \triangleright_3, \triangleleft_3; j = 0, 1, 2)$  is introduced in Section 3. Then, the paper presents the sufficient and necessary conditions to ensure that  $\mathfrak{g} \bowtie V$  is a Lie 2-algebra. Next, the paper shows that any strict Lie 2-algebra  $\mathfrak{c}$  satisfying the condition in extending structures problem is isomorphic to a unified product of  $\mathfrak{g}$  by  $V$ . Finally, the paper constructs two cohomological type objects, where one is isomorphic to the classification of the extending structures problem. Some special cases of unified products such as crossed product and bicrossed product are introduced in Section 4. Using the crossed product and bicrossed product, the paper describes the extension problem and factorization problem, respectively.

### 2. Preliminaries

In this section, some definitions and results about strict Lie 2-algebras are provided. A Lie 2-algebra is an object of an internal category of Lie algebras. It has been noted in [12,14,15] and elsewhere that the category of strict Lie 2-algebras is equivalent to the category of crossed modules of Lie algebras. A crossed module of Lie algebras is defined as follows.

**Definition 1.** A crossed module of Lie algebras is a quadruple  $((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$  where  $(\mathfrak{g}_i, [\cdot, \cdot]_i)$  for  $i = 0, 1$  are Lie algebras,  $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$  is a Lie algebra homomorphism and  $\mathfrak{L} : \mathfrak{g}_0 \rightarrow gl(\mathfrak{g}_1)$  is a Lie algebra action by derivations, such that for any  $x_i, y_i \in \mathfrak{g}_i$  and  $i = 0, 1$ ,

$$\phi(\mathfrak{L}_{x_0}(x_1)) = [x_0, \phi(x_1)]_0,$$

$$\mathfrak{L}_{\phi(x_1)}(y_1) = [x_1, y_1]_1.$$

These equations are called equivariance and infinitesimal Peiffer, respectively.

The homomorphism of two crossed modules of Lie algebras is given by the following definition.

**Definition 2.** Let  $\mathfrak{g} := ((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$  and  $\mathfrak{g}' := ((\mathfrak{g}'_0, [\cdot, \cdot]'_0), (\mathfrak{g}'_1, [\cdot, \cdot]'_1), \phi', \mathfrak{L}')$  be two strict Lie 2-algebras. A Lie 2-algebra homomorphism  $\Phi : \mathfrak{g} \rightarrow \mathfrak{g}'$  consists of linear maps  $\Phi_i : \mathfrak{g}_i \rightarrow \mathfrak{g}'_i$  for  $i = 0, 1$ , such that the following equalities hold for all  $x_i, y_i \in \mathfrak{g}_i$ ,

- $\Phi_0\phi = \phi'\Phi_1$ ,
- $\Phi_i[x_i, y_i]_i = [\Phi_i(x_i), \Phi_i(y_i)]'_i$ ,
- $\Phi_1(\mathfrak{L}_{x_0}x_1) = \mathfrak{L}'_{\Phi_0(x_0)}\Phi_1(x_1)$ .

If  $\Phi_0, \Phi_1$  are invertible, then  $\Phi$  is an isomorphism.

All crossed modules of Lie algebras and homomorphisms between them form a category. Given a 2-vector space  $V := V_1 \xrightarrow{\mu} V_0$ . One can construct a strict Lie 2-algebra  $gl(\mu) := ((gl(\mu)_1, [\cdot, \cdot]_\mu), (gl(\mu)_0, [\cdot, \cdot]_\Delta), \Delta, \mathfrak{L}^\mu)$ , where  $gl(\mu)_0 = \{(F, f) \in End(V_1) \oplus End(V_0) : \mu F = f\mu\}$ ,  $gl(\mu)_1 = Hom(V_0, V_1)$ ,  $\Delta(A) = (A\mu, \mu A)$ ,  $[(F_0, f_0), (F_1, f_1)] = (F_0F_1 - F_1F_0, f_0f_1 - f_1f_0)$ ,  $[A, B]_\mu = A\mu B - B\mu A$  and  $\mathfrak{L}^\mu_{(F_1, f_1)}A = F_1A - Af_1$  for  $A, B \in gl(\mu)_1$  and  $(F_0, f_0), (F_1, f_1) \in gl(\mu)_0$  (Ref. [12]). If there is a homomorphism  $\rho$  from a Lie 2-algebra  $\mathfrak{g}$  to the Lie 2-algebra  $gl(\mu)$ , then  $V$  is called a representation of the Lie 2-algebra  $\mathfrak{g}$ . Suppose that  $V$  is a representation of a Lie 2-algebra  $\mathfrak{g}$ . Then  $V_1$  is a representation of  $\mathfrak{g}_1$ , and  $V_0$  is a representation of  $\mathfrak{g}_0$ . Moreover, the paper introduces the following concept.

**Definition 3.** Let  $\mathfrak{g} := ((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$  be a strict Lie 2-algebra,  $V := V_1 \xrightarrow{\mu} V_0$  a 2-vector space and  $\mathfrak{c} := \mathfrak{c}_1 \xrightarrow{\varepsilon} \mathfrak{c}_0$  a 2-vector space such that the diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathfrak{g}_1 & \xrightarrow{\iota_1} & \mathfrak{c}_1 & \xrightarrow{\pi_1} & V_1 & \longrightarrow & 0 \\
 \downarrow & & \downarrow \phi & & \downarrow \varepsilon & & \downarrow \mu & & \downarrow \\
 0 & \longrightarrow & \mathfrak{g}_0 & \xrightarrow{\iota_0} & \mathfrak{c}_0 & \xrightarrow{\pi_0} & V_0 & \longrightarrow & 0
 \end{array} \tag{1}$$

commutes, where  $\pi_i : \mathfrak{c}_i \rightarrow V_i$  are the canonical projections of  $\mathfrak{c}_i$  and  $\iota_i : \mathfrak{g}_i \rightarrow \mathfrak{c}_i$  are the inclusion maps for  $i = 0, 1$ . For linear functor  $\varphi = (\varphi_0, \varphi_1) : \mathfrak{c} \rightarrow \mathfrak{c}$ , consider the diagram:

$$\begin{array}{ccccccc}
 & & \mathfrak{g}_1 & \xrightarrow{\iota_1} & \mathfrak{c}_1 & \xrightarrow{\pi_1} & V_1 \\
 & \nearrow & \downarrow \phi & \nearrow \varphi_1 & \downarrow \varepsilon & \nearrow \pi_1 & \downarrow \mu \\
 \mathfrak{g}_1 & \longrightarrow & \mathfrak{c}_1 & \xrightarrow{\pi_1} & V_1 & & \\
 \downarrow & & \downarrow \phi & & \downarrow \varepsilon & & \downarrow \mu \\
 \mathfrak{g}_0 & \longrightarrow & \mathfrak{c}_0 & \xrightarrow{\pi_0} & V_0 & & \\
 \downarrow & & \downarrow \phi & & \downarrow \varepsilon & & \downarrow \mu \\
 \mathfrak{g}_0 & \longrightarrow & \mathfrak{c}_0 & \xrightarrow{\pi_0} & V_0 & & \\
 & \nearrow & \downarrow \phi & \nearrow \varphi_0 & \downarrow \varepsilon & \nearrow \pi_0 & \downarrow \mu \\
 & & \mathfrak{g}_0 & \xrightarrow{\iota_0} & \mathfrak{c}_0 & \xrightarrow{\pi_0} & V_0
 \end{array} \tag{2}$$

The paper calls that  $\varphi$  stabilizes  $((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$  (resp. co-stabilizes  $V_1 \xrightarrow{\mu} V_0$ ) if the left cube (resp. the right cube) of the diagram (2) is commutative.

Let  $([\cdot, \cdot]_{\mathfrak{c}_1}, [\cdot, \cdot]_{\mathfrak{c}_0}, \varepsilon, \mathfrak{L}^\varepsilon)$  and  $([\cdot, \cdot]'_{\mathfrak{c}_1}, [\cdot, \cdot]'_{\mathfrak{c}_0}, \varepsilon', \mathfrak{L}'^\varepsilon)$  be two strict Lie 2-algebra structures on  $\mathfrak{c}_1 \xrightarrow{\varepsilon} \mathfrak{c}_0$  both containing  $((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$  as a sub-Lie 2-algebra. If there exists a Lie 2-algebra homomorphism  $\varphi$  which stabilizes  $((\mathfrak{g}_1, [\cdot, \cdot]_1), (\mathfrak{g}_0, [\cdot, \cdot]_0), \phi, \mathfrak{L})$ , then  $([\cdot, \cdot]_{\mathfrak{c}_1}, [\cdot, \cdot]_{\mathfrak{c}_0}, \varepsilon, \mathfrak{L}^\varepsilon)$  and  $([\cdot, \cdot]'_{\mathfrak{c}_1}, [\cdot, \cdot]'_{\mathfrak{c}_0}, \varepsilon', \mathfrak{L}'^\varepsilon)$  are called equivalent, which is denoted by

$$((\mathfrak{c}_1, [\cdot, \cdot]_{\mathfrak{c}_1}), (\mathfrak{c}_0, [\cdot, \cdot]_{\mathfrak{c}_0}), \varepsilon, \mathfrak{L}^\varepsilon) \equiv ((\mathfrak{c}_1, [\cdot, \cdot]'_{\mathfrak{c}_1}), (\mathfrak{c}_0, [\cdot, \cdot]'_{\mathfrak{c}_0}), \varepsilon', \mathfrak{L}'^\varepsilon).$$

If there exists a Lie 2-algebra isomorphism  $\varphi$  which stabilizes  $((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$  and co-stabilizes  $V_1 \xrightarrow{\mu} V_0$ , i.e., the diagram (2) commutes, then  $([\cdot, \cdot]_{\mathfrak{c}_1}, [\cdot, \cdot]_{\mathfrak{c}_0}, \varepsilon, \mathfrak{L}^\varepsilon)$  and  $([\cdot, \cdot]'_{\mathfrak{c}_1}, [\cdot, \cdot]'_{\mathfrak{c}_0}, \varepsilon', \mathfrak{L}'^\varepsilon)$  are called cohomologous, which is denoted by

$$((\mathfrak{c}_1, [\cdot, \cdot]_{\mathfrak{c}_1}), (\mathfrak{c}_0, [\cdot, \cdot]_{\mathfrak{c}_0}), \varepsilon, \mathfrak{L}^\varepsilon) \approx ((\mathfrak{c}_1, [\cdot, \cdot]'_{\mathfrak{c}_1}), (\mathfrak{c}_0, [\cdot, \cdot]'_{\mathfrak{c}_0}), \varepsilon', \mathfrak{L}'^\varepsilon).$$

It is easy to see that  $\equiv$  and  $\approx$  are equivalence relations on the set of all strict Lie 2-algebra structures on  $\mathfrak{c}_1 \xrightarrow{\varepsilon} \mathfrak{c}_0$  containing  $((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$  as a sub-Lie 2-algebra. The set of all equivalence classes via  $\equiv$  and  $\approx$  are denoted by  $LExd(\mathfrak{c}, \mathfrak{g})$  and  $LExd'(\mathfrak{c}, \mathfrak{g})$  respectively. In addition, it is easy to show that there exists a canonical projection  $LExd'(\mathfrak{c}, \mathfrak{g}) \rightarrow LExd(\mathfrak{c}, \mathfrak{g})$ .

**Proposition 1.** Let  $\mathfrak{g} := ((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$  be a strict Lie 2-algebra,  $V := V_1 \xrightarrow{\mu} V_0$  a 2-vector space and  $\mathfrak{c} := \mathfrak{c}_1 \xrightarrow{\varepsilon} \mathfrak{c}_0$  a 2-vector space such that the diagram (1) commutes. Then  $\mathfrak{c}$  is 2-vector space  $\mathfrak{g}_1 + V_1 \xrightarrow{\phi + \sigma + \mu} \mathfrak{g}_0 + V_0$ , where  $\sigma : V_1 \rightarrow \mathfrak{g}_0$  is a linear map.

**Proof.** By the definition of  $\pi_i$  and  $\iota_i$ ,  $\mathfrak{c}_i = \mathfrak{g}_i + V_i$ . Let  $\varepsilon(x_1) = x_0 + v_0$  for any  $x_1 \in Im(\iota_1) \subseteq \mathfrak{c}_1$ . Since the left square of diagram (1) commutes,  $x_0 + v_0 = \varepsilon(x_1) = \iota_0\phi(x_1) = \phi(x_1)$ . Thus,  $\varepsilon(x_1) = \phi(x_1)$ . Similarly, if  $\varepsilon(v_1) = x_0 + v_0$  for  $v_1 \in V_1$ , then  $v_0 = \mu(v_1)$  as the right square of the diagram (1) commutes. Define  $\sigma : V_1 \rightarrow \mathfrak{g}_0$  by  $\sigma(v) = \pi^*\varepsilon(v)$  for any  $v \in V_1$ , where  $\pi^* : \mathfrak{g}_0 \oplus V_0 \rightarrow \mathfrak{g}_0$  is the canonical projection. Then  $\varepsilon(v_1) = \phi(v_1) + \mu(v_1)$  for any  $v_1 \in V_1$ . Hence  $\varepsilon(x_1 + v_1) = \phi(x_1) + \sigma(v_1) + \mu(v_1)$  for  $x_1 + v_1 \in \mathfrak{c}_1$  and  $\varepsilon = \phi + \sigma + \mu$ .  $\square$

### 3. Unified Products for Lie 2-Algebras

In this section, a unified product of two Lie 2-algebras is introduced. Using this product, the paper provides the theoretical answer to the extending structure problem. 2-vector space  $V_1 \xrightarrow{\mu} V_0$  and Lie 2-algebra  $((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$  are simply denoted by  $V$  and  $\mathfrak{g}$  respectively in the following.

**Definition 4.** Suppose that  $\mathfrak{g}$  is a strict Lie 2-algebra and  $V$  is a 2-vector space. An extending datum of  $\mathfrak{g}$  by  $V$  is a system  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  consisting of one linear map  $\sigma : V_1 \rightarrow \mathfrak{g}_0$  and fourteen bilinear maps

$$\begin{aligned} \triangleright_0 : V_0 \times \mathfrak{g}_0 &\rightarrow \mathfrak{g}_0, & \triangleleft_0 : V_0 \times \mathfrak{g}_0 &\rightarrow V_0, & f_0 : V_0 \times V_0 &\rightarrow \mathfrak{g}_0, & *_0 : V_0 \times V_0 &\rightarrow V_0, \\ \triangleright_1 : V_1 \times \mathfrak{g}_1 &\rightarrow \mathfrak{g}_1, & \triangleleft_1 : V_1 \times \mathfrak{g}_1 &\rightarrow V_1, & f_1 : V_1 \times V_1 &\rightarrow \mathfrak{g}_1, & *_1 : V_1 \times V_1 &\rightarrow V_1, \\ \triangleright_2 : V_0 \times \mathfrak{g}_1 &\rightarrow \mathfrak{g}_1, & \triangleleft_2 : V_0 \times \mathfrak{g}_1 &\rightarrow V_1, & f_2 : V_0 \times V_1 &\rightarrow \mathfrak{g}_1, & *_2 : V_0 \times V_1 &\rightarrow V_1, \\ & & \triangleright_3 : V_1 \times \mathfrak{g}_0 &\rightarrow \mathfrak{g}_1, & \triangleleft_3 : V_1 \times \mathfrak{g}_0 &\rightarrow V_1. \end{aligned}$$

Let  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  be an extending datum. Define a new strict Lie 2-algebra  $\mathfrak{g} \natural_{\Omega(\mathfrak{g}, V)} V$  as follows. As a 2-vector space,  $\mathfrak{g} \natural_{\Omega(\mathfrak{g}, V)} V$  is equal to  $\mathfrak{g}_1 \times V_1 \xrightarrow{\varepsilon} \mathfrak{g}_0 \times V_0$ , where  $\varepsilon(y, w) = (\phi(y) + \sigma(w), \mu(w))$  for  $y \in \mathfrak{g}_1$  and  $w \in V_1$ . The bilinear maps  $\{\cdot, \cdot\}_i : (\mathfrak{g}_i \times V_i) \times (\mathfrak{g}_i \times V_i) \rightarrow \mathfrak{g}_i \times V_i$  and the linear map  $\mathfrak{L}^c : \mathfrak{g}_0 \times V_0 \rightarrow gl(\mathfrak{g}_1 \times V_1)$  is given by

$$\{(x_i, v_i), (y_i, w_i)\}_i = ([x_i, y_i]_i + v_i \triangleright_i y_i - w_i \triangleright_i x_i + f_i(v_i, w_i), v_i *_i w_i + v_i \triangleleft_i y_i - w_i \triangleleft_i x_i) \tag{3}$$

and

$$\mathfrak{L}^c_{(x_0, v_0)}(x_1, v_1) = (\mathfrak{L}_{x_0} x_1 + v_0 \triangleright_2 x_1 - v_1 \triangleright_3 x_0 + f_2(v_0, v_1), v_0 *_2 v_1 + v_0 \triangleleft_2 x_1 - v_1 \triangleleft_3 x_0) \tag{4}$$

respectively, for  $i = 0, 1$  and all  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ . This strict Lie 2-algebra  $\mathfrak{g} \natural_{\Omega(\mathfrak{g}, V)} V$  is called a unified product of  $\mathfrak{g}$  and  $V$ ,  $\Omega(\mathfrak{g}, V)$  is called a Lie 2-extending structure of  $\mathfrak{g}$  by  $V$ . If only one Lie 2-extending structure  $\Omega(\mathfrak{g}, V)$  of  $\mathfrak{g}$  by  $V$  is considered, then  $\mathfrak{g} \natural_{\Omega(\mathfrak{g}, V)} V$  is usually simplified as  $\mathfrak{g} \natural V$  and the Lie 2-extending structure is simply called extending datum. The set of all Lie 2-extending structures of  $\mathfrak{g}$  by  $V$  is denoted by  $\mathcal{L}(\mathfrak{g}, V)$ .

**Theorem 1.** Suppose that  $\mathfrak{g}$  is a strict Lie 2-algebra,  $V$  is a 2-vector space. Then  $\Omega(\mathfrak{g}, V)$  is an extending datum of  $\mathfrak{g}$  by  $V$  such that  $\mathfrak{g} \bowtie V$  is a strict Lie 2-algebra if and only if the following conditions hold for any  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i, u_i \in V_i$  and  $i = 0, 1$ :

- (L1)  $f_i(v_i, v_i) = 0, \quad v_i *_i v_i = 0$ ;
- (L2)  $(V_i, \triangleleft_i)$  is a right  $\mathfrak{g}_i$ -module;
- (L3)  $v_i \triangleright_i [x_i, y_i]_i = [v_i \triangleright_i x_i, y_i]_i + [x_i, v_i \triangleright_i y_i]_i + (v_i \triangleleft_i x_i) \triangleright_i y_i - (v_i \triangleleft_i y_i) \triangleright_i x_i$ ;
- (L4)  $(v_i *_i w_i) \triangleleft_i x_i = v_i *_i (w_i \triangleleft_i x_i) + (v_i \triangleleft_i x_i) *_i w_i + v_i \triangleleft_i (w_i \triangleright_i x_i) - w_i \triangleleft_i (w_i \triangleright_i x_i)$ ;
- (L5)  $(v_i *_i w_i) \triangleright_i x_i = v_i \triangleright_i (w_i \triangleright_i x_i) - w_i \triangleright_i (v_i \triangleright_i x_i) + [x_i, f_i(v_i, w_i)]_i + f_i(v_i, w_i \triangleleft_i x_i) + f_i(v_i \triangleleft_i x_i, w_i)$ ;
- (L6)  $f_i(v_i, w_i *_i u_i) + f_i(w_i, u_i *_i v_i) + f_i(u_i, v_i *_i w_i) + v_i \triangleright_i f_i(w_i, u_i) + w_i \triangleright_i f_i(u_i, v_i) + u_i \triangleright_i f_i(v_i, w_i) = 0$ ;
- (L7)  $v_i *_i (w_i *_i u_i) + w_i *_i (u_i *_i v_i) + u_i *_i (v_i *_i w_i) + v_i \triangleleft_i f_i(w_i, u_i) + w_i \triangleleft_i f_i(u_i, v_i) + u_i \triangleleft_i f_i(v_i, w_i) = 0$ ;
- (L8)  $\mu(v_1) \triangleright_0 \phi(x_1) = \phi(v_1 \triangleright_1 x_1) + \sigma(v_1 \triangleleft_1 x_1) + [\phi(x_1), \sigma(v_1)]_0$ ;
- (L9)  $(\mu(v_1) \triangleleft_0 \phi(x_1) = \mu(v_1 \triangleleft_1 x_1)$ ;
- (L10)  $\mu(w_1) \triangleright_0 \sigma(v_1) - \mu(v_1) \triangleright_0 \sigma(w_1) = [\sigma(v_1), \sigma(w_1)]_0 - \phi(f_1(v_1, w_1)) - \sigma(v_1 *_1 w_1) + f_0(\mu(v_1), \mu(w_1))$ ;
- (L11)  $\mu(w_1) \triangleleft_0 \sigma(v_1) - \mu(v_1) \triangleleft_0 \sigma(w_1) = \mu(v_1) *_0 \mu(w_1) - \mu(v_1 *_1 w_1)$ ;
- (L12)  $\mathfrak{L}_{x_0}(v_1 \triangleright_1 x_1) = (v_1 \triangleleft_1 x_1) \triangleright_3 x_0 + v_1 \triangleright_1 (\mathfrak{L}_{x_0} x_1) + [x_1, v_1 \triangleright_3 x_0]_1 - (v_1 \triangleleft_3 x_0) \triangleright_1 x_1$ ;
- (L13)  $(v_1 \triangleleft_3 x_0) \triangleleft_1 x_1 = (v_1 \triangleleft_1 x_1) \triangleleft_3 x_0 + v_1 \triangleleft_1 (\mathfrak{L}_{x_0} x_1)$ ;
- (L14)  $\mathfrak{L}_{x_0} f_1(v_1, w_1) = (v_1 *_1 w_1) \triangleright_3 x_0 + w_1 \triangleright_1 (v_1 \triangleright_3 x_0) - f_1(v_1 \triangleleft_3 x_0, w_1) - v_1 \triangleright_1 (w_1 \triangleright_3 x_0) - f_1(v_1, w_1 \triangleleft_3 x_0)$ ;
- (L15)  $(v_1 *_1 w_1) \triangleleft_3 x_0 + w_1 \triangleleft_1 (v_1 \triangleright_3 x_0) = (v_1 \triangleleft_3 x_0) *_1 w_1 + v_1 \triangleleft_1 (w_1 \triangleright_3 x_0) + v_1 *_1 (w_1 \triangleleft_3 x_0)$ ;
- (L16)  $v_0 \triangleright_2 [x_1, y_1]_1 = [v_0 \triangleright_2 x_1, y_1]_1 + (v_0 \triangleleft_2 x_1) \triangleright_1 y_1 + [x_1, v_0 \triangleright_2 y_1]_1 - (v_0 \triangleleft_2 y_1) \triangleright_1 x_1$ ;
- (L17)  $v_0 \triangleleft_2 [x_1, y_1]_1 = (v_0 \triangleleft_2 x_1) \triangleleft_1 y_1 - (v_0 \triangleleft_2 y_1) \triangleleft_1 x_1$ ;
- (L18)  $v_0 \triangleright_2 (v_1 \triangleright_1 x_1) + f_2(v_0, v_1 \triangleleft_1 x_1) - v_1 \triangleright_1 (v_0 \triangleright_2 x_1) + f_1(v_0 \triangleleft_2 x_1, v_1) + [x_1, f_2(v_0, v_1)]_1 - (v_0 *_2 v_1) \triangleright_1 x_1 = 0$ ;
- (L19)  $v_0 \triangleleft_2 (v_1 \triangleright_1 x_1) + v_0 *_2 (v_1 \triangleleft_1 x_1) - v_1 \triangleleft_1 (v_0 \triangleright_2 x_1) + (v_0 \triangleleft_2 x_1) *_1 v_1 - (v_0 *_2 v_1) \triangleleft_1 x_1 = 0$ ;
- (L20)  $v_0 \triangleright_2 f_1(v_1, w_1) + f_2(v_0, v_1 *_1 w_1) + w_1 \triangleright_1 f_2(v_0, v_1) = f_1(v_0 *_2 v_1, w_1) + v_1 \triangleright_1 f_2(v_0, w_1) + f_1(v_1, v_0 *_2 w_1)$ ;
- (L21)  $v_0 \triangleleft_2 f_1(v_1, w_1) + v_0 *_2 (v_1 *_1 w_1) + w_1 \triangleleft_1 f_2(v_0, v_1) = (v_0 *_2 v_1) *_1 w_1 + v_1 \triangleleft_1 f_2(v_0, w_1) + v_1 *_1 (v_0 *_2 w_1)$ ;
- (L22)  $\mathfrak{L}_{x_0}(v_1 \triangleright_3 y_0) = v_1 \triangleright_3 [x_0, y_0]_0 + (v_1 \triangleleft_3 y_0) \triangleright_3 x_0 + \mathfrak{L}_{y_0}(v_1 \triangleright_3 x_0) - (v_1 \triangleleft_3 x_0) \triangleright_3 y_0$ ;
- (L23)  $v_1 \triangleleft_3 [x_0, y_0]_0 = (v_1 \triangleleft_3 x_0) \triangleleft_3 y_0 - (v_1 \triangleleft_3 y_0) \triangleleft_3 x_0$ ;
- (L24)  $\mathfrak{L}_{v_0 \triangleright_0 x_0} x_1 = (v_0 \triangleleft_2 x_1) \triangleright_3 x_0 - (v_0 \triangleleft_0 x_0) \triangleright_2 x_1 - \mathfrak{L}_{x_0}(v_0 \triangleright_2 x_1) + v_0 \triangleright_2 \mathfrak{L}_{x_0} x_1$ ;
- (L25)  $(v_0 \triangleleft_0 x_0) \triangleleft_2 x_1 = (v_0 \triangleleft_2 x_1) \triangleleft_3 x_0 + v_0 \triangleleft_2 \mathfrak{L}_{x_0} x_1$ ;
- (L26)  $v_1 \triangleright_3 (v_0 \triangleright_0 x_0) - v_0 \triangleright_2 (v_1 \triangleright_3 x_0) = f_2(v_0 \triangleleft_0 x_0, v_1) + \mathfrak{L}_{x_0} f_2(v_0, v_1) - (v_0 *_2 v_1) \triangleright_3 x_0 + f_2(v_0, v_1 \triangleleft_3 x_0)$ ;
- (L27)  $v_1 \triangleleft_3 (v_0 \triangleright_0 x_0) - v_0 \triangleleft_2 (v_1 \triangleright_3 x_0) = (v_0 \triangleleft_0 x_0) *_2 v_1 - (v_0 *_2 v_1) \triangleleft_3 x_0 + v_0 *_2 (v_1 \triangleleft_3 x_0)$ ;
- (L28)  $\mathfrak{L}_{f_0(v_0, w_0)} x_1 + (v_0 *_0 w_0) \triangleright_2 x_1 - v_0 \triangleright_2 (w_0 \triangleright_2 x_1) - f_2(v_0, w_0 \triangleleft_2 x_1) + w_0 \triangleright_2 (v_0 \triangleright_2 x_1) + f_2(w_0, v_0 \triangleleft_2 x_1) = 0$ ;
- (L29)  $(v_0 *_0 w_0) \triangleleft_2 x_1 - v_0 \triangleleft_2 (w_0 \triangleright_2 x_1) - v_0 *_2 (w_0 \triangleleft_2 x_1) + w_0 \triangleleft_2 (v_0 \triangleright_2 x_1) + w_0 *_2 (v_0 \triangleleft_2 x_1) = 0$ ;
- (L30)  $-v_1 \triangleright_3 f_0(v_0, w_0) + f_2(v_0 *_0 w_0, v_1) - v_0 \triangleright_2 f_2(w_0, v_1) - f_2(v_0, w_0 *_2 v_1) + w_0 \triangleright_2 f_2(v_0, v_1) + f_2(w_0, v_0 *_2 v_1) = 0$ ;
- (L31)  $-v_1 \triangleleft_3 f_0(v_0, w_0) + (v_0 *_0 w_0) *_2 v_1 - v_0 \triangleleft_2 f_2(w_0, v_1) - v_0 *_2 (w_0 *_2 v_1) + w_0 \triangleleft_2 f_2(v_0, v_1) + w_0 *_2 (v_0 *_2 v_1) = 0$ ;
- (L32)  $\mu(v_1) \triangleright_0 x_0 = \phi(v_1 \triangleright_3 x_0) + \sigma(v_1 \triangleleft_3 x_0) + [x_0, \sigma(v_1)]_0$ ;
- (L33)  $\mu(v_1) \triangleleft_0 x_0 = \mu(v_1 \triangleleft_3 x_0)$ ;
- (L34)  $v_0 \triangleright_0 \phi(x_1) = \phi(v_0 \triangleright_2 x_1) + \sigma(v_0 \triangleleft_2 x_1)$ ;
- (L35)  $v_0 \triangleleft_0 \phi(x_1) = \mu(v_0 \triangleleft_2 x_1)$ ;
- (L36)  $\phi(f_2(v_0, v_1)) + \sigma(v_0 *_2 v_1) - v_0 \triangleright_0 \sigma(v_1) - f_0(v_0, \mu(v_1)) = 0$ ;
- (L37)  $\mu(v_0 *_2 v_1) - v_0 \triangleleft_0 \sigma(v_1) - v_0 *_0 \mu(v_1) = 0$ ;
- (L38)  $v_1 \triangleright_3 \phi(x_1) = v_1 \triangleright_1 x_1$ ;
- (L39)  $v_1 \triangleleft_3 \phi(x_1) = v_1 \triangleleft_1 x_1$ ;
- (L40)  $\mu(v_1) \triangleright_2 x_1 = v_1 \triangleright_1 x_1 - \mathfrak{L}_{\sigma(v_1)} x_1$ ;
- (L41)  $\mu(v_1) \triangleleft_2 x_1 = v_1 \triangleleft_1 x_1$ ;
- (L42)  $w_1 \triangleright_3 \sigma(v_1) = f_2(\mu(v_1), w_1) - f_1(v_1, w_1)$ ;

(L43)  $w_1 \triangleleft_3 \sigma(v_1) = \mu(v_1) *_2 w_1 - v_1 *_1 w_1.$

**Proof.** By ([6] Theorem 2.2),  $\mathfrak{g}_i \times V_i$  is a Lie algebra if and only if the conditions (L1)–(L7) hold. Thus,  $\mathfrak{g} \bowtie V$  is a strict Lie 2-algebra if and only if  $\varepsilon : \mathfrak{g}_1 \times V_1 \rightarrow \mathfrak{g}_0 \times V_0$  is a Lie algebra homomorphism,  $\mathfrak{L}^c : \mathfrak{g}_0 \times V_0 \rightarrow gl(\mathfrak{g}_1 \times V_1)$  is a Lie algebra action by derivations and satisfying equivariance and infinitesimal Peiffer, i.e.,

$$\varepsilon\{(x_1, v_1), (y_1, w_1)\}_1 = \{\varepsilon(x_1, v_1), \varepsilon(y_1, w_1)\}_0, \tag{5}$$

$$\mathfrak{L}^c_{(x_0, v_0)}\{(x_1, v_1), (y_1, w_1)\}_1 = \{\mathfrak{L}^c_{(x_0, v_0)}(x_1, v_1), (y_1, w_1)\}_1 + \{(x_1, v_1), \mathfrak{L}^c_{(x_0, v_0)}(y_1, w_1)\}_1, \tag{6}$$

$$\mathfrak{L}^c_{\{(x_0, v_0), (y_0, w_0)\}_0}(x_1, v_1) = \mathfrak{L}^c_{(x_0, v_0)}\mathfrak{L}^c_{(y_0, w_0)}(x_1, v_1) - \mathfrak{L}^c_{(y_0, w_0)}\mathfrak{L}^c_{(x_0, v_0)}(x_1, v_1), \tag{7}$$

$$\varepsilon(\mathfrak{L}^c_{(x_0, v_0)}(x_1, v_1)) = \{(x_0, v_0), \varepsilon(x_1, v_1)\}_0, \tag{8}$$

$$\mathfrak{L}^c_{\varepsilon(x_1, v_1)}(y_1, w_1) = \{(x_1, v_1), (y_1, w_1)\}_1 \tag{9}$$

for  $i = 0, 1$  and all  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ . Since  $(x_i, v_i) = (x_i, 0) + (0, v_i)$  in  $\mathfrak{g} \bowtie V$ , Equations (5)–(9) hold if and only if they hold for the set  $\{(x_i, 0) | x_i \in \mathfrak{g}_i\} \cup \{(0, v_i) | v_i \in V_i\}$ . First, Equation (5) holds for  $(x_1, 0), (y_1, 0)$  as

$$\begin{aligned} \varepsilon\{(x_1, 0), (y_1, 0)\}_1 - \{\varepsilon(x_1, 0), \varepsilon(y_1, 0)\}_0 &= \varepsilon([x_1, y_1]_1, 0) - \{(\phi(x_1), 0), (\phi(y_1), 0)\}_0 \\ &= (\phi[x_1, y_1]_1 - [\phi(x_1), \phi(y_1)]_0, 0) = (0, 0). \end{aligned}$$

Since

$$\begin{aligned} &\varepsilon\{(x_1, 0), (0, v_1)\}_1 - \{\varepsilon(x_1, 0), \varepsilon(0, v_1)\}_0 \\ &= \varepsilon(-v_1 \triangleright_1 x_1, -v_1 \triangleleft_1 x_1) - \{(\phi(x_1), 0), (\sigma(v_1), \mu(v_1))\}_0 \\ &= (-\phi(v_1 \triangleright_1 x_1) - \sigma(v_1 \triangleleft_1 x_1) - [\phi(x_1), \sigma(v_1)]_0 + \mu(v_1) \triangleright_0 \phi(x_1), -\mu(v_1 \triangleleft_1 x_1) + \mu(v_1) \triangleleft_0 \phi(x_1)), \end{aligned}$$

Equation (5) holds for  $(x_1, 0), (0, v_1)$  if and only if (L8) and (L9) hold. Equation (5) holds for  $(0, v_1), (0, w_1)$  if and only if (L10) and (L11) hold, since

$$\begin{aligned} &\varepsilon\{(0, v_1), (0, w_1)\}_1 - \{\varepsilon(0, v_1), \varepsilon(0, w_1)\}_0 \\ &= \varepsilon(f_1(v_1, w_1), v_1 *_1 w_1) - \{(\sigma(v_1), \mu(v_1)), (\sigma(w_1), \mu(w_1))\}_0 \\ &= (\phi(f_1(v_1, w_1)) + \sigma(v_1 *_1 w_1) - [\sigma(v_1), \sigma(w_1)]_0 - \mu(v_1) \triangleright_0 \sigma(w_1) + \mu(w_1) \triangleright_0 \sigma(v_1) \\ &\quad - f_0(\mu(v_1), \mu(w_1)), \mu(v_1 *_1 w_1) - \mu(v_1) \triangleleft_0 \sigma(w_1) + \mu(w_1) \triangleleft_0 \sigma(v_1) - \mu(v_1) *_0 \mu(w_1)). \end{aligned}$$

Then, Equation (6) holds for  $(x_0, 0), (x_1, 0), (y_1, 0)$  as

$$\begin{aligned} &\mathfrak{L}^c_{(x_0, 0)}\{(x_1, 0), (y_1, 0)\}_1 - \{\mathfrak{L}^c_{(x_0, 0)}(x_1, 0), (y_1, 0)\}_1 - \{(x_1, 0), \mathfrak{L}^c_{(x_0, 0)}(y_1, 0)\}_1 \\ &= \mathfrak{L}^c_{(x_0, 0)}([x_1, y_1]_1, 0) - \{(\mathfrak{L}_{x_0} x_1, 0), (y_1, 0)\}_1 - \{(x_1, 0), (\mathfrak{L}_{x_0} y_1, 0)\}_1 \\ &= (\mathfrak{L}_{x_0}[x_1, y_1]_1 - [\mathfrak{L}_{x_0} x_1, y_1]_1 - [x_1, \mathfrak{L}_{x_0} y_1]_1, 0) = (0, 0). \end{aligned}$$

Since, for  $(x_0, 0), (x_1, 0), (0, v_1),$

$$\begin{aligned} &\mathfrak{L}^c_{(x_0, 0)}\{(x_1, 0), (0, v_1)\}_1 - \{\mathfrak{L}^c_{(x_0, 0)}(x_1, 0), (0, v_1)\}_1 - \{(x_1, 0), \mathfrak{L}^c_{(x_0, 0)}(0, v_1)\}_1 \\ &= \mathfrak{L}^c_{(x_0, 0)}(-v_1 \triangleright_1 x_1, -v_1 \triangleleft_1 x_1) - \{(\mathfrak{L}_{x_0} x_1, 0), (0, v_1)\}_1 + \{(x_1, 0), (v_1 \triangleright_3 x_0, v_1 \triangleleft_3 x_0)\}_1 \\ &= (-\mathfrak{L}_{x_0}(v_1 \triangleright_1 x_1) + (v_1 \triangleleft_1 x_1) \triangleright_3 x_0 + v_1 \triangleright_1 (\mathfrak{L}_{x_0} x_1) + [x_1, v_1 \triangleright_3 x_0]_1 - (v_1 \triangleleft_3 x_0) \triangleright_1 x_1, \\ &\quad (v_1 \triangleleft_1 x_1) \triangleleft_3 x_0 + v_1 \triangleleft_1 (\mathfrak{L}_{x_0} x_1) - (v_1 \triangleleft_3 x_0) \triangleleft_1 x_1), \end{aligned}$$

Equation (6) holds for  $(x_0, 0), (x_1, 0), (0, v_1)$  if and only if (L12) and (L13) hold. Equation (6) holds for  $(x_0, 0), (0, v_1), (0, w_1)$  if and only if (L14) and (L15) hold, since

$$\begin{aligned} & \mathfrak{L}^c_{(x_0,0)}\{(0, v_1), (0, w_1)\}_1 - \{\mathfrak{L}^c_{(x_0,0)}(0, v_1), (0, w_1)\}_1 - \{(0, v_1), \mathfrak{L}^c_{(x_0,0)}(0, w_1)\}_1 \\ = & \mathfrak{L}^c_{(x_0,0)}(f_1(v_1, w_1), v_1 *_1 w_1) + \{(v_1 \triangleright_3 x_0, v_1 \triangleleft_3 x_0), (0, w_1)\}_1 + \{(0, v_1), (w_1 \triangleright_3 x_0, w_1 \triangleleft_3 x_0)\}_1 \\ = & (\mathfrak{L}_{x_0} f_1(v_1, w_1) - (v_1 *_1 w_1) \triangleright_3 x_0 - w_1 \triangleright_1 (v_1 \triangleright_3 x_0) + f_1(v_1 \triangleleft_3 x_0, w_1) + v_1 \triangleright_1 (w_1 \triangleright_3 x_0) \\ & + f_1(v_1, w_1 \triangleleft_3 x_0), -(v_1 *_1 w_1) \triangleleft_3 x_0 - w_1 \triangleleft_1 (v_1 \triangleright_3 x_0) + (v_1 \triangleleft_3 x_0) *_1 w_1 + v_1 \triangleleft_1 (w_1 \triangleright_3 x_0) \\ & + v_1 *_1 (w_1 \triangleleft_3 x_0)). \end{aligned}$$

Equation (6) holds for  $(0, v_0), (x_1, 0), (y_1, 0)$  if and only if (L16) and (L17) hold, as

$$\begin{aligned} & \mathfrak{L}^c_{(0,v_0)}\{(x_1, 0), (y_1, 0)\}_1 - \{\mathfrak{L}^c_{(0,v_0)}(x_1, 0), (y_1, 0)\}_1 - \{(x_1, 0), \mathfrak{L}^c_{(0,v_0)}(y_1, 0)\}_1 \\ = & \mathfrak{L}^c_{(0,v_0)}([x_1, y_1]_1, 0) - \{(v_0 \triangleright_2 x_1, v_0 \triangleleft_2 x_1), (y_1, 0)\}_1 - \{(x_1, 0), (v_0 \triangleright_2 y_1, v_0 \triangleleft_2 y_1)\}_1 \\ = & (v_0 \triangleright_2 [x_1, y_1]_1 - [v_0 \triangleright_2 x_1, y_1]_1 - (v_0 \triangleleft_2 x_1) \triangleright_1 y_1 - [x_1, v_0 \triangleright_2 y_1]_1 + (v_0 \triangleleft_2 y_1) \triangleright_1 x_1, \\ & v_0 \triangleleft_2 [x_1, y_1]_1 - (v_0 \triangleleft_2 x_1) \triangleleft_1 y_1 + (v_0 \triangleleft_2 y_1) \triangleleft_1 x_1). \end{aligned}$$

Equation (6) holds for  $(0, v_0), (x_1, 0), (0, v_1)$  if and only if (L18) and (L19) hold. Indeed,

$$\begin{aligned} & \mathfrak{L}^c_{(0,v_0)}\{(x_1, 0), (0, v_1)\}_1 - \{\mathfrak{L}^c_{(0,v_0)}(x_1, 0), (0, v_1)\}_1 - \{(x_1, 0), \mathfrak{L}^c_{(0,v_0)}(0, v_1)\}_1 \\ = & \mathfrak{L}^c_{(0,v_0)}(-v_1 \triangleright_1 x_1, -v_1 \triangleleft_1 x_1) - \{(v_0 \triangleright_2 x_1, v_0 \triangleleft_2 x_1), (0, v_1)\}_1 - \{(x_1, 0), (f_2(v_0, v_1), v_0 *_2 v_1)\}_1 \\ = & (-v_0 \triangleright_2 (v_1 \triangleright_1 x_1) - f_2(v_0, v_1 \triangleleft_1 x_1) + v_1 \triangleright_1 (v_0 \triangleright_2 x_1) - f_1(v_0 \triangleleft_2 x_1, v_1) - [x_1, f_2(v_0, v_1)]_1 \\ & + (v_0 *_2 v_1) \triangleright_1 x_1, -v_0 \triangleleft_2 (v_1 \triangleright_1 x_1) - v_0 *_2 (v_1 \triangleleft_1 x_1) + v_1 \triangleleft_1 (v_0 \triangleright_2 x_1) - (v_0 \triangleleft_2 x_1) *_1 v_1 \\ & + (v_0 *_2 v_1) \triangleleft_1 x_1) \end{aligned}$$

for  $(0, v_0), (x_1, 0), (0, v_1)$ . Notice that

$$\begin{aligned} & \mathfrak{L}^c_{(0,v_0)}\{(0, v_1), (0, w_1)\}_1 - \{\mathfrak{L}^c_{(0,v_0)}(0, v_1), (0, w_1)\}_1 - \{(0, v_1), \mathfrak{L}^c_{(0,v_0)}(0, w_1)\}_1 \\ = & \mathfrak{L}^c_{(0,v_0)}(f_1(v_1, w_1), v_1 *_1 w_1) - \{(f_2(v_0, v_1), v_0 *_2 v_1), (0, w_1)\}_1 - \{(0, v_1), (f_2(v_0, w_1), v_0 *_2 w_1)\}_1 \\ = & (v_0 \triangleright_2 f_1(v_1, w_1) + f_2(v_0, v_1 *_1 w_1) + w_1 \triangleright_1 f_2(v_0, v_1) - f_1(v_0 *_2 v_1, w_1) - v_1 \triangleright_1 f_2(v_0, w_1) \\ & - f_1(v_1, v_0 *_2 w_1), v_0 \triangleleft_2 f_1(v_1, w_1) + v_0 *_2 (v_1 *_1 w_1) + w_1 \triangleleft_1 f_2(v_0, v_1) - (v_0 *_2 v_1) *_1 w_1 \\ & - v_1 \triangleleft_1 f_2(v_0, w_1) - v_1 *_1 (v_0 *_2 w_1)) \end{aligned}$$

for  $(0, v_0), (0, v_1), (0, w_1)$ . Thus, Equation (6) holds for  $(0, v_0), (0, v_1), (0, w_1)$  if and only if (L20) and (L21) hold. Now Equation (7) holds for  $(x_0, 0), (y_0, 0), (x_1, 0)$  as

$$\begin{aligned} & \mathfrak{L}^c_{\{(x_0,0),(y_0,0)\}_0}(x_1, 0) - \mathfrak{L}^c_{(x_0,0)}\mathfrak{L}^c_{(y_0,0)}(x_1, 0) + \mathfrak{L}^c_{(y_0,0)}\mathfrak{L}^c_{(x_0,0)}(x_1, 0) \\ = & \mathfrak{L}^c_{([x_0,y_0]_0,0)}(x_1, 0) - \mathfrak{L}^c_{(x_0,0)}(\mathfrak{L}_{y_0} x_1, 0) + \mathfrak{L}^c_{(y_0,0)}(\mathfrak{L}_{x_0} x_1, 0) \\ = & (\mathfrak{L}_{[x_0,y_0]_0} x_1 - \mathfrak{L}_{x_0} \mathfrak{L}_{y_0} x_1 + \mathfrak{L}_{y_0} \mathfrak{L}_{x_0} x_1, 0) = (0, 0). \end{aligned}$$

Since

$$\begin{aligned} & \mathfrak{L}^c_{\{(x_0,0),(y_0,0)\}_0}(0, v_1) - \mathfrak{L}^c_{(x_0,0)}\mathfrak{L}^c_{(y_0,0)}(0, v_1) + \mathfrak{L}^c_{(y_0,0)}\mathfrak{L}^c_{(x_0,0)}(0, v_1) \\ = & \mathfrak{L}^c_{([x_0,y_0]_0,0)}(0, v_1) + \mathfrak{L}^c_{(x_0,0)}(v_1 \triangleright_3 y_0, v_1 \triangleleft_3 y_0) - \mathfrak{L}^c_{(y_0,0)}(v_1 \triangleright_3 x_0, v_1 \triangleleft_3 x_0) \\ = & (-v_1 \triangleright_3 [x_0, y_0]_0 + \mathfrak{L}_{x_0}(v_1 \triangleright_3 y_0) - (v_1 \triangleleft_3 y_0) \triangleright_3 x_0 - \mathfrak{L}_{y_0}(v_1 \triangleright_3 x_0) + (v_1 \triangleleft_3 x_0) \triangleright_3 y_0, \\ & -v_1 \triangleleft_3 [x_0, y_0]_0 - (v_1 \triangleleft_3 y_0) \triangleleft_3 x_0 + (v_1 \triangleleft_3 x_0) \triangleleft_3 y_0) \end{aligned}$$

for  $(x_0, 0), (y_0, 0), (0, v_1)$ , Equation (7) holds for  $(x_0, 0), (y_0, 0), (0, v_1)$  if and only if (L22) and (L23) hold. As

$$\begin{aligned} & \mathfrak{L}^c_{\{(x_0,0),(0,v_0)\}_0}(x_1, 0) - \mathfrak{L}^c_{(x_0,0)}\mathfrak{L}^c_{(0,v_0)}(x_1, 0) + \mathfrak{L}^c_{(0,v_0)}\mathfrak{L}^c_{(x_0,0)}(x_1, 0) \\ &= \mathfrak{L}^c_{(-v_0 \triangleright_0 x_0, -v_0 \triangleleft_0 x_0)}(x_1, 0) - \mathfrak{L}^c_{(x_0,0)}(v_0 \triangleright_2 x_1, v_0 \triangleleft_2 x_1) + \mathfrak{L}^c_{(0,v_0)}(\mathfrak{L}_{x_0} x_1, 0) \\ &= (-\mathfrak{L}_{v_0 \triangleright_0 x_0} x_1 - (v_0 \triangleleft_0 x_0) \triangleright_2 x_1 - \mathfrak{L}_{x_0}(v_0 \triangleright_2 x_1) + (v_0 \triangleleft_2 x_1) \triangleright_3 x_0 + v_0 \triangleright_2 \mathfrak{L}_{x_0} x_1, \\ & \quad - (v_0 \triangleleft_0 x_0) \triangleleft_2 x_1 + (v_0 \triangleleft_2 x_1) \triangleleft_3 x_0 + v_0 \triangleleft_2 \mathfrak{L}_{x_0} x_1), \end{aligned}$$

Equation (7) holds for  $(x_0, 0), (0, v_0), (x_1, 0)$  if and only if (L24) and (L25) hold. Indeed,

$$\begin{aligned} & \mathfrak{L}^c_{\{(x_0,0),(0,v_0)\}_0}(0, v_1) - \mathfrak{L}^c_{(x_0,0)}\mathfrak{L}^c_{(0,v_0)}(0, v_1) + \mathfrak{L}^c_{(0,v_0)}\mathfrak{L}^c_{(x_0,0)}(0, v_1) \\ &= \mathfrak{L}^c_{(-v_0 \triangleright_0 x_0, -v_0 \triangleleft_0 x_0)}(0, v_1) - \mathfrak{L}^c_{(x_0,0)}(f_2(v_0, v_1), v_0 *_2 v_1) + \mathfrak{L}^c_{(0,v_0)}(-v_1 \triangleright_3 x_0, -v_1 \triangleleft_3 x_0) \\ &= (v_1 \triangleright_3 (v_0 \triangleright_0 x_0) - f_2(v_0 \triangleleft_0 x_0, v_1) - \mathfrak{L}_{x_0} f_2(v_0, v_1) + (v_0 *_2 v_1) \triangleright_3 x_0 - v_0 \triangleright_2 (v_1 \triangleright_3 x_0) \\ & \quad - f_2(v_0, v_1 \triangleleft_3 x_0), v_1 \triangleleft_3 (v_0 \triangleright_0 x_0) - (v_0 \triangleleft_0 x_0) *_2 v_1 + (v_0 * _2 v_1) \triangleleft_3 x_0 - v_0 \triangleleft_2 (v_1 \triangleright_3 x_0) \\ & \quad - v_0 *_2 (v_1 \triangleleft_3 x_0)) \end{aligned}$$

for  $(x_0, 0), (0, v_0), (0, v_1)$ . Then Equation (7) holds for  $(x_0, 0), (0, v_0), (0, v_1)$  if and only if (L26) and (L27) hold. Because

$$\begin{aligned} & \mathfrak{L}^c_{\{(0,v_0),(0,w_0)\}_0}(x_1, 0) - \mathfrak{L}^c_{(0,v_0)}\mathfrak{L}^c_{(0,w_0)}(x_1, 0) + \mathfrak{L}^c_{(0,w_0)}\mathfrak{L}^c_{(0,v_0)}(x_1, 0) \\ &= \mathfrak{L}^c_{(f_0(v_0,w_0), v_0 *_0 w_0)}(x_1, 0) - \mathfrak{L}^c_{(0,v_0)}(w_0 \triangleright_2 x_1, w_0 \triangleleft_2 x_1) + \mathfrak{L}^c_{(0,w_0)}(v_0 \triangleright_2 x_1, v_0 \triangleleft_2 x_1) \\ &= (\mathfrak{L}_{f_0(v_0,w_0)} x_1 + (v_0 *_0 w_0) \triangleright_2 x_1 - v_0 \triangleright_2 (w_0 \triangleright_2 x_1) - f_2(v_0, w_0 \triangleleft_2 x_1) + w_0 \triangleright_2 (v_0 \triangleright_2 x_1) \\ & \quad + f_2(w_0, v_0 \triangleleft_2 x_1), (v_0 *_0 w_0) \triangleleft_2 x_1 - v_0 \triangleleft_2 (w_0 \triangleright_2 x_1) - v_0 *_2 (w_0 \triangleleft_2 x_1) + w_0 \triangleleft_2 (v_0 \triangleright_2 x_1) \\ & \quad + w_0 *_2 (v_0 \triangleleft_2 x_1)) \end{aligned}$$

for  $(0, v_0), (0, w_0), (x_1, 0)$ , Equation (7) holds for  $(0, v_0), (0, w_0), (x_1, 0)$  if and only if (L28) and (L29) hold. Since

$$\begin{aligned} & \mathfrak{L}^c_{\{(0,v_0),(0,w_0)\}_0}(0, v_1) - \mathfrak{L}^c_{(0,v_0)}\mathfrak{L}^c_{(0,w_0)}(0, v_1) + \mathfrak{L}^c_{(0,w_0)}\mathfrak{L}^c_{(0,v_0)}(0, v_1) \\ &= \mathfrak{L}^c_{(f_0(v_0,w_0), v_0 *_0 w_0)}(0, v_1) - \mathfrak{L}^c_{(0,v_0)}(f_2(w_0, v_1), w_0 *_2 v_1) + \mathfrak{L}^c_{(0,w_0)}(f_2(v_0, v_1), v_0 *_2 v_1) \\ &= (-v_1 \triangleright_3 f_0(v_0, w_0) + f_2(v_0 *_0 w_0, v_1) - v_0 \triangleright_2 f_2(w_0, v_1) - f_2(v_0, w_0 *_2 v_1) + w_0 \triangleright_2 f_2(v_0, v_1) \\ & \quad + f_2(w_0, v_0 *_2 v_1), -v_1 \triangleleft_3 f_0(v_0, w_0) + (v_0 *_0 w_0) *_2 v_1 - v_0 \triangleleft_2 f_2(w_0, v_1) - v_0 *_2 (w_0 *_2 v_1) \\ & \quad + w_0 \triangleleft_2 f_2(v_0, v_1) + w_0 *_2 (v_0 *_2 v_1)), \end{aligned}$$

Equation (7) holds for  $(0, v_0), (0, w_0), (0, v_1)$  if and only if (L30) and (L31) hold. Next Equation (8) holds for  $(x_0, 0), (x_1, 0)$  as

$$\begin{aligned} & \varepsilon(\mathfrak{L}^c_{(x_0,0)}(x_1, 0)) - \{(x_0, 0), \varepsilon(x_1, 0)\}_0 \\ &= \varepsilon(\mathfrak{L}_{x_0} x_1, 0) - \{(x_0, 0), (\phi(x_1), 0)\}_0 \\ &= (\phi(\mathfrak{L}_{x_0} x_1) - [x_0, \phi(x_1)]_0, 0) = (0, 0). \end{aligned}$$

As

$$\begin{aligned} & \varepsilon(\mathfrak{L}^c_{(x_0,0)}(0, v_1)) - \{(x_0, 0), \varepsilon(0, v_1)\}_0 \\ &= \varepsilon(-v_1 \triangleright_3 x_0, -v_1 \triangleleft_3 x_0) - \{(x_0, 0), (\sigma(v_1), \mu(v_1))\}_0 \\ &= (-\phi(v_1 \triangleright_3 x_0) - \sigma(v_1 \triangleleft_3 x_0) - [x_0, \sigma(v_1)]_0 + \mu(v_1) \triangleright_0 x_0, -\mu(v_1 \triangleleft_3 x_0) + \mu(v_1) \triangleleft_0 x_0), \end{aligned}$$



Equation (8) holds for  $(x_0, 0), (0, v_1)$  if and only if (L32) and (L33) hold. Indeed,

$$\begin{aligned} & \varepsilon(\mathfrak{L}_{(0,v_0)}^c(x_1, 0)) - \{(0, v_0), \varepsilon(x_1, 0)\}_0 \\ &= \varepsilon(v_0 \triangleright_2 x_1, v_0 \triangleleft_2 x_1) - \{(0, v_0), (\phi(x_1), 0)\}_0 \\ &= (\phi(v_0 \triangleright_2 x_1) + \sigma(v_0 \triangleleft_2 x_1) - v_0 \triangleright_0 \phi(x_1), \mu(v_0 \triangleleft_2 x_1) - v_0 \triangleleft_0 \phi(x_1)) \end{aligned}$$

for  $(0, v_0), (x_1, 0)$ . Thus, Equation (8) holds for  $(0, v_0), (x_1, 0)$  if and only if (L34) and (L35) hold. Equation (8) holds for  $(0, v_0), (0, v_1)$  if and only if (L36) and (L37) hold, since

$$\begin{aligned} & \varepsilon(\mathfrak{L}_{(0,v_0)}^c(0, v_1)) - \{(0, v_0), \varepsilon(0, v_1)\}_0 \\ &= \varepsilon(f_2(v_0, v_1), v_0 *_2 v_1) - \{(0, v_0), (\sigma(v_1), \mu(v_1))\}_0 \\ &= (\phi(f_2(v_0, v_1)) + \sigma(v_0 *_2 v_1) - v_0 \triangleright_0 \sigma(v_1) - f_0(v_0, \mu(v_1)), \\ & \quad \mu(v_0 *_2 v_1) - v_0 \triangleleft_0 \sigma(v_1) - v_0 *_0 \mu(v_1)). \end{aligned}$$

Finally, Equation (9) holds for  $(x_1, 0), (y_1, 0)$  as

$$\begin{aligned} & \mathfrak{L}_{\varepsilon(x_1,0)}^c(y_1, 0) - \{(x_1, 0), (y_1, 0)\}_1 \\ &= \mathfrak{L}_{(\phi(x_1),0)}^c(y_1, 0) - ([x_1, y_1]_1, 0) \\ &= (\mathfrak{L}_{\phi(x_1)} y_1 - [x_1, y_1]_1, 0) = (0, 0). \end{aligned}$$

Indeed,

$$\begin{aligned} & \mathfrak{L}_{\varepsilon(x_1,0)}^c(0, v_1) - \{(x_1, 0), (0, v_1)\}_1 \\ &= \mathfrak{L}_{(\phi(x_1),0)}^c(0, v_1) + (v_1 \triangleright_1 x_1, v_1 \triangleleft_1 x_1) \\ &= (-v_1 \triangleright_3 \phi(x_1) + v_1 \triangleright_1 x_1, -v_1 \triangleleft_3 \phi(x_1) + v_1 \triangleleft_1 x_1) \end{aligned}$$

for  $(x_1, 0), (0, v_1)$ . Thus, Equation (9) holds for  $(x_1, 0), (0, v_1)$  if and only if (L38) and (L39) hold. Since, for  $(0, v_1), (x_1, 0)$ ,

$$\begin{aligned} & \mathfrak{L}_{\varepsilon(0,v_1)}^c(x_1, 0) - \{(0, v_1), (x_1, 0)\}_1 \\ &= \mathfrak{L}_{(\sigma(v_1),\mu(v_1))}^c(x_1, 0) - (v_1 \triangleright_1 x_1, v_1 \triangleleft_1 x_1) \\ &= (\mathfrak{L}_{\sigma(v_1)} x_1 + \mu(v_1) \triangleright_2 x_1 - v_1 \triangleright_1 x_1, \mu(v_1) \triangleleft_2 x_1 - v_1 \triangleleft_1 x_1), \end{aligned}$$

Equation (9) holds for  $(0, v_1), (x_1, 0)$  if and only if (L40) and (L41) hold. Equation (9) holds for  $(0, v_1), (0, w_1)$  if and only if (L42) and (L43) hold, since

$$\begin{aligned} & \mathfrak{L}_{\varepsilon(0,v_1)}^c(0, w_1) - \{(0, v_1), (0, w_1)\}_1 \\ &= \mathfrak{L}_{(\sigma(v_1),\mu(v_1))}^c(0, w_1) - (f_1(v_1, w_1), v_1 *_1 w_1) \\ &= (-w_1 \triangleright_3 \sigma(v_1) + f_2(\mu(v_1), w_1) - f_1(v_1, w_1), -w_1 \triangleleft_3 \sigma(v_1) + \mu(v_1) *_2 w_1 - v_1 *_1 w_1). \end{aligned}$$

By now the proof is completed.  $\square$

**Example 1.** Let  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  be an extending datum of Lie 2-algebra  $\mathfrak{g}$  by 2-vector space  $V$  such that  $\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma$  for  $j = 0, 1, 2$  are trivial maps, i.e.,  $v_i \triangleright_i x_i = 0, v_i \triangleleft_i x_i = 0, f_i(v_i, w_i) = 0, v_i *_i w_i = 0, \sigma(v_1) = 0, v_0 \triangleright_2 x_1 = 0, v_0 \triangleleft_2 x_1 = 0, f_2(v_0, v_1) = 0, v_0 *_2 v_1 = 0, v_1 \triangleright_3 x_0 = 0, v_1 \triangleleft_3 x_0 = 0$  for  $i = 0, 1$  and  $x_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ . Then  $\mathfrak{g}_1^c V$  is a Lie 2-algebra with  $\varepsilon = \phi \times \mu$ , the brackets and derivation given by  $\{(x_i, v_i), (y_i, w_i)\}_i = ([x_i, y_i]_i, 0)$  and  $\mathfrak{L}_{(x_0,v_0)}^c(x_1, v_1) = (\mathfrak{L}_{x_0} x_1, 0)$  respectively, for  $i = 0, 1$  and all  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ . The paper calls this Lie 2-extending structure the trivial extending structure of  $\mathfrak{g}$  by  $V$ .

In the sequel, the paper uses the following convention: if one of the maps  $\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma$  for  $j = 0, 1, 2$  of an extending datum  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  is trivial then the paper will omit it from  $(\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$ .

**Example 2.** Let  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  be an extending datum of Lie 2-algebra  $\mathfrak{g}$  by 2-vector space  $V$  such that  $\triangleright_j, f_j, *_j, \sigma, \triangleright_3$  for  $j = 0, 1, 2$  are trivial maps. Then  $\Omega(\mathfrak{g}, V) = (\triangleleft_k; k = 0, 1, 2, 3)$  is a Lie 2-extending structure of  $\mathfrak{g}$  by  $V$  if and only if  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mu)$  is a representation of Lie 2-algebra  $\mathfrak{g}$  on 2-vector space  $V$ , where  $\rho_1(x_1)v_0 = -v_0 \triangleleft_2 x_1$ ,  $\rho_0^0(x_0)v_0 = -v_0 \triangleleft_0 x_0$  and  $\rho_0^1(x_0)v_1 = -v_1 \triangleleft_3 x_0$  for  $x_i \in \mathfrak{g}_i, v_i \in V_i$ . In this case, the associative unified product  $\mathfrak{g} \ltimes_{\Omega(\mathfrak{g}, V)} V$  is the semi-direct product  $((\mathfrak{g}_1 \oplus_{\rho_0^1 \circ \phi} V_1, \{\cdot, \cdot\}_1), (\mathfrak{g}_0 \oplus_{\rho_0^0} V_0, \{\cdot, \cdot\}_0), \phi \times \mu, \mathcal{L}^c)$ , where

$$\begin{aligned} \{(x_0, v_0), (y_0, w_0)\}_0 &= ([x_0, y_0]_0, \rho_0^0(x_0)w_0 - \rho_0^0(y_0)v_0), \\ \{(x_1, v_1), (y_1, w_1)\}_1 &= ([x_1, y_1]_1, (\rho_0^1\phi)(x_1)w_1 - (\rho_0^1\phi)(y_1)v_1), \\ \mathcal{L}_{(x_0, v_0)}^c(x_1, v_1) &= (\mathcal{L}_{x_0}x_1, \rho_0^1(x_0)v_1 - \rho_1(x_1)v_0) \end{aligned}$$

for  $i = 0, 1$  and  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ .

Let  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2) \in \mathcal{L}(\mathfrak{g}, V)$  be a Lie 2-extending structure and  $\mathfrak{g} \ltimes V$  be the associated unified product. Then the canonical inclusion

$$\iota = (\iota_0, \iota_1) : \mathfrak{g} \rightarrow \mathfrak{g} \ltimes V, \quad \iota_i(x_i) = (x_i, 0)$$

is an injective Lie 2-algebra homomorphism. Hence,  $\mathfrak{g}$  can be viewed as a sub-Lie 2-algebra of  $\mathfrak{g} \ltimes V$  by the identification  $\mathfrak{g} \cong \iota(\mathfrak{g}) = \mathfrak{g} \times \{0\}$ . Conversely, the paper will prove that any strict Lie 2-algebra structure on 2-vector space  $\mathfrak{c}$  containing  $\mathfrak{g}$  as a sub-Lie 2-algebra is isomorphic to a unified product.

**Theorem 2.** Let  $\mathfrak{g}$  be a strict Lie 2-algebra and  $\mathfrak{c}$  a 2-vector space containing  $\mathfrak{g}$  as a 2-vector subspace. Suppose that  $(\{\cdot, \cdot\}_i, \varepsilon, \mathcal{L}^c; i = 0, 1)$  is a strict Lie 2-algebra structure on  $\mathfrak{c}$  such that  $\mathfrak{g}$  is a sub-Lie 2-algebra in  $(\mathfrak{c}, \{\cdot, \cdot\}_i, \varepsilon, \mathcal{L}^c; i = 0, 1)$  of  $\mathfrak{g}$  by a 2-vector space  $V$ . Then there is an isomorphism of Lie 2-algebras  $(\mathfrak{c}, \{\cdot, \cdot\}_i, \varepsilon, \mathcal{L}^c; i = 0, 1) \cong \mathfrak{g} \ltimes V$  which stabilizes  $\mathfrak{g}$  and co-stabilizes  $V$ .

**Proof.** Let  $p_i : \mathfrak{c}_i \rightarrow \mathfrak{g}_i$  be linear maps such that  $p_i(x_i) = x_i$  for  $i = 0, 1$  and  $x_i \in \mathfrak{g}_i$ . Then  $V_i := \text{Ker}(p_i)$  is a subspace of  $\mathfrak{c}_i$  and a complement of  $\mathfrak{g}_i$  in  $\mathfrak{c}_i$ . Define the extending datum of  $\mathfrak{g}$  by  $V$  by the following formulas:

$$\begin{aligned} \triangleright_i &= \triangleright_{p_i} : V_i \times \mathfrak{g}_i \rightarrow \mathfrak{g}_i, & v_i \triangleright_i x_i &:= p_i(\{v_i, x_i\}), \\ \triangleleft_i &= \triangleleft_{p_i} : V_i \times \mathfrak{g}_i \rightarrow V_i, & v_i \triangleleft_i x_i &:= \{v_i, x_i\} - p_i(\{v_i, x_i\}), \\ f_i &= f_{p_i} : V_i \times V_i \rightarrow \mathfrak{g}_i, & f_i(v_i, w_i) &:= p_i(\{v_i, w_i\}), \\ *_i &= *_i : V_i \times V_i \rightarrow V_i, & v_i *_i w_i &:= \{v_i, w_i\} - p_i(\{v_i, w_i\}), \\ \triangleright_2 &: V_0 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_1, & v_0 \triangleright_2 x_1 &:= p_1(\mathcal{L}_{v_0}^c x_1), \\ \triangleleft_2 &: V_0 \times \mathfrak{g}_1 \rightarrow V_1, & v_0 \triangleleft_2 x_1 &:= (\mathcal{L}_{v_0}^c x_1) - p_1(\mathcal{L}_{v_0}^c x_1), \\ f_2 &: V_0 \times V_1 \rightarrow \mathfrak{g}_1, & f_2(v_0, v_1) &:= p_1(\mathcal{L}_{v_0}^c v_1), \\ *_2 &: V_0 \times V_1 \rightarrow V_1, & v_0 *_2 v_1 &:= \mathcal{L}_{v_0}^c v_1 - p_1(\mathcal{L}_{v_0}^c v_1), \\ \triangleright_3 &: V_1 \times \mathfrak{g}_0 \rightarrow \mathfrak{g}_1, & v_1 \triangleright_3 x_0 &:= -p_1(\mathcal{L}_{x_0}^c v_1), \\ \triangleleft_3 &: V_1 \times \mathfrak{g}_0 \rightarrow V_1, & v_1 \triangleleft_3 x_0 &:= -\mathcal{L}_{x_0}^c v_1 + p_1(\mathcal{L}_{x_0}^c v_1) \\ \sigma &: V_1 \rightarrow \mathfrak{g}_0, & \sigma(v_1) &:= p_0(\varepsilon(v_1)) \end{aligned}$$

for any  $i = 0, 1, x_i \in \mathfrak{g}_i$  and  $v_i, w_i \in V_i$ . First, the above maps are all well defined. This paper shall prove that  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  is a Lie 2-extending structure of  $\mathfrak{g}$  by  $V$  and

$$\Psi = (\Psi_0, \Psi_1) : \mathfrak{g} \bowtie V \rightarrow \mathfrak{c}, \quad \Psi_i(x_i, v_i) := x_i + v_i$$

is an isomorphism of Lie 2-algebras that stabilizes  $\mathfrak{g}$  and co-stabilizes  $V$ . It is easy to verify that for  $i = 0, 1$  and  $z_i \in \mathfrak{c}_i, \Psi_i^{-1} = (\Psi_0^{-1}, \Psi_1^{-1}) : \mathfrak{c} \rightarrow \mathfrak{g} \times V, \Psi_i^{-1}(z_i) := (p_i(z_i), z_i - p_i(z_i))$  is an inverse of  $\Psi : \mathfrak{g} \times V \rightarrow \mathfrak{c}$  as 2-vector space. Therefore, there is a unique strict Lie 2-algebra structure on  $\mathfrak{g} \times V$  such that  $\Psi$  is an isomorphism of strict Lie 2-algebras and this unique Lie 2-algebra structure is given by

$$\{(x_i, v_i), (y_i, w_i)\}_i := \Psi_i^{-1}(\{\Psi_i(x_i, v_i), \Psi_i(y_i, w_i)\}_i),$$

$$\mathfrak{L}_{(x_0, v_0)}^{\mathfrak{c}}(x_1, v_1) := \Psi_1^{-1}(\mathfrak{L}_{\Psi_0(x_0, v_0)}^{\mathfrak{c}} \Psi_1(x_1, v_1))$$

for all  $x_i, y_i \in \mathfrak{g}_i$  and  $v_i, w_i \in V_i$ . Then the proof is sufficient to prove that this Lie 2-algebra structure coincides with the one defined by (3) and (4) associated with the system  $(\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$ . Indeed, for any  $x_i, y_i \in \mathfrak{g}_i$  and  $v_i, w_i \in V_i$ ,

$$\begin{aligned} & \{(x_i, v_i), (y_i, w_i)\}_i \\ &= \Psi_i^{-1}(\{\Psi_i(x_i, v_i), \Psi_i(y_i, w_i)\}_i) \\ &= \Psi_i^{-1}(\{x_i, y_i\}_i + \{x_i, w_i\}_i + \{v_i, y_i\}_i + \{v_i, w_i\}_i) \\ &= (p_i(\{x_i, y_i\}_i) + p_i(\{x_i, w_i\}_i) + p_i(\{v_i, y_i\}_i) + p_i(\{v_i, w_i\}_i), \{x_i, y_i\}_i - p_i(\{x_i, y_i\}_i) \\ & \quad + \{x_i, w_i\}_i - p_i(\{x_i, w_i\}_i) + \{v_i, y_i\}_i - p_i(\{v_i, y_i\}_i) + \{v_i, w_i\}_i - p_i(\{v_i, w_i\}_i)) \\ &= ([x_i, y_i]_i - w_i \triangleright_i x_i + v_i \triangleright_i y_i + f_i(v_i, w_i), v_i \triangleleft_i y_i + v_i *_i w_i - w_i \triangleleft_i x_i)_i, \end{aligned}$$

$$\begin{aligned} & \mathfrak{L}_{(x_0, v_0)}^{\mathfrak{c}}(x_1, v_1) \\ &= \Psi_1^{-1}(\mathfrak{L}_{x_0}^{\mathfrak{c}} x_1 + \mathfrak{L}_{v_0}^{\mathfrak{c}} v_1 + \mathfrak{L}_{v_0}^{\mathfrak{c}} x_1 + \mathfrak{L}_{x_0}^{\mathfrak{c}} v_1) \\ &= (p_1(\mathfrak{L}_{x_0}^{\mathfrak{c}} x_1) + p_1(\mathfrak{L}_{x_0}^{\mathfrak{c}} v_1) + p_1(\mathfrak{L}_{v_0}^{\mathfrak{c}} x_1) + p_1(\mathfrak{L}_{v_0}^{\mathfrak{c}} v_1), \mathfrak{L}_{x_0}^{\mathfrak{c}} x_1 - p_1(\mathfrak{L}_{x_0}^{\mathfrak{c}} x_1) \\ & \quad + \mathfrak{L}_{x_0}^{\mathfrak{c}} v_1 - p_1(\mathfrak{L}_{x_0}^{\mathfrak{c}} v_1) + \mathfrak{L}_{v_0}^{\mathfrak{c}} x_1 - p_1(\mathfrak{L}_{v_0}^{\mathfrak{c}} x_1) + \mathfrak{L}_{v_0}^{\mathfrak{c}} v_1 - p_1(\mathfrak{L}_{v_0}^{\mathfrak{c}} v_1)) \\ &= (\mathfrak{L}_{x_0}^{\mathfrak{c}} x_1 - v_1 \triangleright_3 x_0 + v_0 \triangleright_2 x_1 + f_2(v_0, v_1), -v_1 \triangleleft_3 x_0 + v_0 \triangleleft_2 x_1 + v_0 *_2 v_1). \end{aligned}$$

Moreover, the following diagram is commutative

$$\begin{array}{ccccc} & & \mathfrak{g}_1 & \xrightarrow{i_1} & \mathfrak{c}_1 & \longrightarrow & V_1 & . \\ & & \uparrow & \downarrow & \uparrow & & \uparrow & \\ \mathfrak{g}_1 & \xrightarrow{i_{\mathfrak{g}_1}} & \mathfrak{g}_1 \bowtie V_1 & \longrightarrow & V_1 & & & \\ & & \downarrow & & \downarrow & & \downarrow & \\ & & \mathfrak{g}_0 & \xrightarrow{i_0} & \mathfrak{c}_0 & \longrightarrow & V_0 & \\ & & \uparrow & \downarrow & \uparrow & & \uparrow & \\ \mathfrak{g}_0 & \xrightarrow{i_{\mathfrak{g}_0}} & \mathfrak{g}_0 \bowtie V_0 & \longrightarrow & V_0 & & & \end{array} \tag{10}$$

The proof is completed now.  $\square$

By Theorem 2, the classification of all strict Lie 2-algebra structure on  $\mathfrak{c}$  that containing  $\mathfrak{g}$  as a sub-Lie 2-algebra reduces to the classification of all unified products  $\mathfrak{g} \bowtie V$  associated with all Lie 2-extending structures  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$ , for a given 2-vector space  $V$  such that  $V_i$  is a complement of  $\mathfrak{g}_i$  in  $\mathfrak{c}_i$ .

**Lemma 1.** Suppose that  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  and  $\Omega'(\mathfrak{g}, V) = (\triangleleft'_j, \triangleright'_j, f'_j, *_j, \sigma', \triangleleft'_3, \triangleright'_3; j = 0, 1, 2)$  are two Lie 2-extending structures of  $\mathfrak{g}$  by  $V$  and  $\mathfrak{g} \bowtie V, \mathfrak{g} \bowtie' V$  are the associated unified

products. Then there exists a bijection between the set of all morphisms of Lie 2-algebras  $\Phi : \mathfrak{g} \bowtie V \rightarrow \mathfrak{g} \bowtie' V$  which stabilizes  $\mathfrak{g}$  and the set of  $(r_i, \tau_i; i = 0, 1)$ , where  $r_i : V_i \rightarrow \mathfrak{g}_i$  and  $\tau_i : V_i \rightarrow V_i$  are linear maps satisfying the following compatibility conditions for  $i = 0, 1$  and any  $x_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ :

- (M1)  $\tau_i(v_i) \triangleleft'_i x_i = \tau(v_i \triangleleft_i x_i)$ ;
- (M2)  $r_i(v_i \triangleleft_i x_i) = \{r_i(v_i), x_i\} - v_i \triangleright_i x_i + \tau(v_i) \triangleright'_i x_i$ ;
- (M3)  $\tau(v_i *_i w_i) = \tau(v_i) *_i' \tau(w_i) + \tau(v_i) \triangleleft'_i r_i(w_i) - \tau(w_i) \triangleleft'_i r_i(v_i)$ ;
- (M4)  $r_i(v_i *_i w_i) = \{r_i(v_i), r_i(w_i)\} + \tau(v_i) \triangleright'_i r_i(w_i) - \tau(w_i) \triangleright'_i r_i(v_i) + f'_i(\tau(v_i), \tau(w_i)) - f_i(v_i, w_i)$ ,
- (M5)  $\mu \tau_1(v_1) = \tau_0 \mu(v_1)$ ,
- (M6)  $\phi(r_1(v_1)) + \sigma'(\tau_1(v_1)) = \sigma(v_1) + r_0(\mu(v_1))$ ,
- (M7)  $v_1 \triangleright_3 x_0 + r_1(v_1 \triangleleft_3 x_0) = \tau_1(v_1) \triangleright'_3 x_0 - \mathfrak{L}_{x_0} r_1(v_1)$ ,
- (M8)  $\tau_1(v_1 \triangleleft_3 x_0) = \tau_1(v_1) \triangleleft'_3 x_0$ ,
- (M9)  $v_0 \triangleright_2 x_1 + r_1(v_0 \triangleleft_2 x_1) = \mathfrak{L}_{r_0(v_0)} x_1 + \tau_0(v_0) \triangleright'_2 x_1$ ,
- (M10)  $\tau_1(v_0 \triangleleft_2 x_1) = \tau_0(v_0) \triangleleft'_2 x_1$ ,
- (M11)  $f_2(v_0, v_1) + r_1(v_0 *_2 v_1) = \mathfrak{L}_{r_0(v_0)} r_1(v_1) + \tau_0(v_0) \triangleright'_2 r_1(v_1) - \tau_1(v_1) \triangleright'_3 r_0(v_0) + f'_2(\tau_0(v_0), \tau_1(v_1))$ ,
- (M12)  $\tau_1(v_0 *_2 v_1) = \tau_0(v_0) \triangleleft'_2 r_1(v_1) - \tau_1(v_1) \triangleleft'_3 r_0(v_0) + \tau_0(v_0) *_2' \tau_1(v_1)$ .

Under the above bijection the morphism of Lie 2-algebras  $\Phi = \Phi_{(r_i, \tau_i; i=0,1)} = (\Phi_0, \Phi_1) : \mathfrak{g} \bowtie V \rightarrow \mathfrak{g} \bowtie' V$  corresponding to  $(r_i, \tau_i; i = 0, 1)$  is given by:

$$\Phi_i(x_i, v_i) = (x_i + r_i(v_i), \tau_i(v_i)),$$

for any  $x_i \in \mathfrak{g}_i, v_i \in V_i$  and  $i = 0, 1$ . Moreover,  $\Phi = \Phi_{(r_i, \tau_i; i=0,1)}$  is an isomorphism if and only if  $\tau_i : V_i \rightarrow V_i$  is an isomorphism and  $\Phi = \Phi_{(r_i, \tau_i; i=0,1)}$  co-stabilizes  $V$  if and only if  $\tau_i = Id_{V_i}$ .

**Proof.** Suppose that  $\Phi = (\Phi_0, \Phi_1) : \mathfrak{g} \bowtie V \rightarrow \mathfrak{g} \bowtie' V$  is a linear functor such that

$$\begin{array}{ccccc}
 & & \mathfrak{g}_1 & \xrightarrow{i_{\mathfrak{g}_1}} & \mathfrak{g}_1 \bowtie' V_1 \\
 & \nearrow i_{\mathfrak{g}_1} & & \searrow \Phi_1 & \downarrow \\
 \mathfrak{g}_1 & \xrightarrow{i_{\mathfrak{g}_1}} & \mathfrak{g}_1 \bowtie V_1 & & \\
 \downarrow & & \downarrow & & \downarrow i_{\mathfrak{g}_0} \\
 & \nearrow i_{\mathfrak{g}_0} & \mathfrak{g}_0 & \xrightarrow{i_{\mathfrak{g}_0}} & \mathfrak{g}_0 \bowtie' V_0 \\
 \mathfrak{g}_0 & \xrightarrow{i_{\mathfrak{g}_0}} & \mathfrak{g}_0 \bowtie V_0 & & \\
 & & & & \downarrow \Phi_0
 \end{array}
 \tag{11}$$

commutes. Then it is uniquely determined by linear maps  $r_i : V_i \rightarrow \mathfrak{g}_i$  and  $\tau_i : V_i \rightarrow V_i$  such that  $\Phi_i(x_i, v_i) = (x_i + r_i(v_i), \tau_i(v_i))$  for  $i = 0, 1$  and all  $x_i \in \mathfrak{g}_i, v_i \in V_i$ . In fact, let  $\Phi_i(0, v_i) = (r_i(v_i), \tau_i(v_i)) \in \mathfrak{g}_i \times V_i$  for all  $v_i \in V_i$ . Then  $\Phi_i(x_i, v_i) = (x_i + r_i(v_i), \tau_i(v_i)) \in \mathfrak{g}_i \times V_i$ . Next, the paper proves that  $\Phi = \Phi_{(r_i, \tau_i; i=0,1)}$  is a morphism of strict Lie 2-algebras if and only if the compatibility conditions (M1)–(M12) hold. It is sufficient to prove the equations

$$\Phi_i(\{(x_i, v_i), (y_i, w_i)\}_i) = \{\Phi(x_i, v_i), \Phi(y_i, w_i)\}_i, \tag{12}$$

$$(\phi + \sigma' + \mu)\Phi_1(x_1, v_1) = \Phi_0(\phi(x_1) + \sigma(v_1), \mu(v_1)), \tag{13}$$

$$\Phi_1(\mathfrak{L}'_{(x_0, v_0)}(x_1, v_1)) = \mathfrak{L}'_{\Phi_0(x_0, v_0)}\Phi_1(x_1, v_1) \tag{14}$$

hold for the set  $\{(x_i, 0) | x_i \in \mathfrak{g}_i\} \cup \{(0, v_i) | v_i \in V_i\}$  and  $i = 0, 1$ . By ([6], Lemma 2.5), Equation (12) holds if and only if conditions (M1)–(M4) hold. First, consider Equation (13). It is easy to see that Equation (13) holds for  $(x_1, 0) \in \mathfrak{g}_1 \times V_1$ . Equation (13) holds for  $(0, v_1) \in \mathfrak{g}_1 \times V_1$  if and only if (M5) and (M6) hold, since

$$\begin{aligned}
 & (\phi + \sigma' + \mu)\Phi_1(0, v_1) - \Phi_0(\sigma(v_1), \mu(v_1)) \\
 = & (\phi(r_1(v_1)) + \sigma'(\tau_1(v_1)) - \sigma(v_1) - r_0(\mu(v_1)), \mu\tau_1(v_1) - \tau_0\mu(v_1)).
 \end{aligned}$$

Now consider Equation (14). It is easy to see that Equation(14) holds for  $(x_i, 0) \in \mathfrak{g}_i \times V_i$ . Since, for  $(x_0, 0) \in \mathfrak{g}_0 \times V_0$  and  $(0, v_1) \in \mathfrak{g}_1 \times V_1$ ,

$$\begin{aligned} & \Phi_1(\mathfrak{L}_{(x_0,0)}^c(0, v_1)) - \mathfrak{L}_{\Phi_0(x_0,0)}^c \Phi_1(0, v_1) \\ &= \Phi_1(-v_1 \triangleright_3 x_0, -v_1 \triangleleft_3 x_0) - \mathfrak{L}_{(x_0,0)}^c(r_1(v_1), \tau_1(v_1)) \\ &= (-v_1 \triangleright_3 x_0 - r_1(v_1 \triangleleft_3 x_0) - \mathfrak{L}_{x_0} r_1(v_1) + \tau_1(v_1) \triangleright'_3 x_0, -\tau_1(v_1 \triangleleft_3 x_0) + \tau_1(v_1) \triangleleft'_3 x_0), \end{aligned}$$

Equation (14) holds for  $(x_0, 0) \in \mathfrak{g}_0 \times V_0$  and  $(0, v_1) \in \mathfrak{g}_1 \times V_1$  if and only if (M7) and (M8) hold. Similarly, it is easy to check that Equation (14) holds for  $(0, v_0) \in \mathfrak{g}_0 \times V_0$  and  $(x_1, 0) \in \mathfrak{g}_1 \times V_1$  if and only if (M9) and (M10) hold; Equation (14) holds for  $(0, v_0) \in \mathfrak{g}_0 \times V_0$  and  $(0, v_1) \in \mathfrak{g}_1 \times V_1$  if and only if (M11) and (M12) hold.

Assume that  $\tau_i : V_i \rightarrow V_i$  is bijective. Then  $\Phi_{(r_i, \tau_i; i=0,1)}$  is an isomorphism of Lie 2-algebras with the inverse given by  $\Phi_{(r_i, \tau_i; i=0,1)}^{-1} = (\Phi_0^{-1}, \Phi_1^{-1})$ , where  $\Phi_i^{-1}(x_i, v_i) = (x_i - r_i(\tau^{-1}(v_i)), \tau^{-1}(v_i))$  for  $x_i \in \mathfrak{g}_i$  and  $v_i \in V_i$ . Conversely, assume that  $\Phi_{(r_i, \tau_i; i=0,1)} = (\Phi_0, \Phi_1)$  is isomorphic. Then  $\Phi_i$  is an isomorphism of Lie algebras for  $i = 0, 1$ . By the proof of ([6], Lemma 2.5),  $\tau_i$  is a bijection for  $i = 0, 1$ . The last assertion is trivial, and the proof is completed now.  $\square$

**Definition 5.** Let  $\mathfrak{g}$  be a strict Lie 2-algebra and  $V$  a 2-vector space. If there exists linear maps  $r_i : V_i \rightarrow \mathfrak{g}_i$  and  $\tau_i : V_i \rightarrow V_i$  for  $i = 0, 1$  such that Lie 2-algebra extending structure  $\Omega'(\mathfrak{g}, V) = (\triangleleft'_j, \triangleright'_j, f'_j, *_j, \triangleleft'_3, \triangleright'_3, \sigma'; j = 0, 1, 2)$  can be yield from another Lie 2-algebra extending structure  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  using  $(r_i, \tau_i; i = 0, 1)$  via:

$$\begin{aligned} v_i \triangleleft'_i x_i &= \tau_i(\tau_i^{-1}(v_i) \triangleleft_i x_i) \\ v_i \triangleright'_i x_i &= r_i(\tau_i^{-1}(v_i) \triangleleft_i x_i) + \tau_i^{-1}(v_i) \triangleright_i x_i + [x_i, r_i(\tau_i^{-1}(v_i))]_i \\ f'_i(v_i, w_i) &= f_i(\tau_i^{-1}(v_i), \tau_i^{-1}(w_i)) + r_i(\tau_i^{-1}(v_i)) *_i \tau_i^{-1}(w_i) + [r_i(\tau_i^{-1}(v_i)), r_i(\tau_i^{-1}(w_i))]_i \\ &\quad - r_i(\tau_i^{-1}(v_i) \triangleleft_i r_i(\tau_i^{-1}(w_i))) - \tau_i^{-1}(v_i) \triangleright_i r_i(\tau_i^{-1}(w_i)) + r(\tau_i^{-1}(w_i) \triangleleft_i \tau_i^{-1}(v_i)) \\ &\quad + \tau_i^{-1}(w_i) \triangleright_i r_i(\tau_i^{-1}(v_i)) \\ v_i *_i' w_i &= \tau_i(\tau_i^{-1}(v_i) *_i \tau_i^{-1}(w_i)) - \tau_i(\tau_i^{-1}(v_i) \triangleleft_i r_i(\tau_i^{-1}(w_i))) + \tau_i(\tau_i^{-1}(w_i) \triangleleft_i r_i(\tau_i^{-1}(v_i))) \\ \sigma'(v_1) &= \sigma(\tau_1^{-1}(v_1)) + r_0(\mu(\tau_1^{-1}(v_1))) - \phi(r_1(\tau_1^{-1}(v_1))) \\ &= \sigma(\tau_1^{-1}(v_1)) + r_0(\tau_0^{-1}(\mu(v_1))) - \phi(r_1(\tau_1^{-1}(v_1))) \\ v_1 \triangleright'_3 x_0 &= r_1(\tau_1^{-1}(v_1) \triangleleft_3 x_0) + \tau_1^{-1}(v_1) \triangleright_3 x_0 + \mathfrak{L}_{x_0} r_1(\tau_1^{-1}(v_1)) \\ v_1 \triangleleft'_3 x_0 &= \tau_1(\tau_1^{-1}(v_1) \triangleleft_3 x_0) \\ v_0 \triangleright'_2 x_1 &= r_1(\tau_0^{-1}(v_0) \triangleleft_2 x_1) - \mathfrak{L}_{r_0(\tau_0^{-1}(v_0))} x_1 + \tau_0^{-1}(v_0) \triangleright_2 x_1 \\ v_0 \triangleleft'_2 x_1 &= \tau_1(\tau_0^{-1}(v_0) \triangleleft_2 x_1) \\ f'_2(v_0, v_1) &= r_1(\tau_0^{-1}(v_0) *_2 \tau_1^{-1}(v_1) + \tau_1^{-1}(v_1) \triangleleft_3 r_0(\tau_0^{-1}(v_0)) - \tau_0^{-1}(v_0) \triangleleft_2 r_1(\tau_1^{-1}(v_1))) \\ &\quad + f_2(\tau_0^{-1}(v_0), \tau_1^{-1}(v_1)) + \tau_1^{-1}(v_1) \triangleright_3 r_0(\tau_0^{-1}(v_0)) - \tau_0^{-1}(v_0) \triangleright_2 r_1(\tau_1^{-1}(v_1)) \\ &\quad + \mathfrak{L}_{r_0(\tau_0^{-1}(v_0))} r_0(\tau_1^{-1}(v_1)) \\ v_0 *_2' v_1 &= \tau_1(\tau_0^{-1}(v_0) *_2 \tau_1^{-1}(v_1) + \tau_1^{-1}(v_1) \triangleleft_3 r_0(\tau_0^{-1}(v_0)) - \tau_0^{-1}(v_0) \triangleleft_2 r_1(\tau_1^{-1}(v_1))) \end{aligned}$$

for  $i = 0, 1$  and any  $x_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ , then  $\Omega(\mathfrak{g}, V)$  and  $\Omega'(\mathfrak{g}, V)$  are said to be equivalent, which is denoted by  $\Omega(\mathfrak{g}, V) \equiv \Omega'(\mathfrak{g}, V)$ . In particular, if  $\tau_i = Id_{V_i}$  for  $i = 0, 1$ , then  $\Omega(\mathfrak{g}, V)$  and  $\Omega'(\mathfrak{g}, V)$  are called cohomologous, which is denoted by  $\Omega(\mathfrak{g}, V) \approx \Omega'(\mathfrak{g}, V)$ .

The paper concludes this section by the following theorem, which provides an answer to the extending structures problem of strict Lie 2-algebras.

**Theorem 3.** Suppose that  $\mathfrak{g}$  is a strict Lie 2-algebra,  $V$  is a 2-vector space and  $\mathfrak{c}$  is a 2-vector space which contains  $\mathfrak{g}$  as a 2-subspace and  $V_i$  is a complement of  $\mathfrak{g}_i$  in  $\mathfrak{c}_i$  for  $i = 0, 1$ . Then

1. the relation  $\equiv$  is an equivalence relation on the set  $\mathcal{L}(\mathfrak{g}, V)$  of all Lie 2-extending structures of  $\mathfrak{g}$  by  $V$ , and the map  $\mathcal{H}_{\mathfrak{g}}^2(V, \mathfrak{g}) := \mathcal{L}(\mathfrak{g}, V) / \equiv \rightarrow \text{LExt}d(\mathfrak{c}, \mathfrak{g})$ , given by

$$\overline{(\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)} \mapsto (\mathfrak{g} \sharp V, \{\cdot, \cdot\}, \phi + \sigma + \mu, \mathfrak{L}^{\mathfrak{g} \sharp V}),$$

is bijective, where  $\overline{(\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)}$  is the equivalence class of  $(\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  under the equivalent relation  $\equiv$ .

2. the relation  $\approx$  is an equivalent relation on the set  $\mathcal{L}(\mathfrak{g}, V)$  of all Lie 2-extending structures of  $\mathfrak{g}$  by  $V$ , and the mapping  $\mathcal{H}^2(V, \mathfrak{g}) := \mathcal{L}(\mathfrak{g}, V) / \approx \rightarrow \text{LExt}d'(\mathfrak{c}, \mathfrak{g})$  given by

$$\overline{\overline{(\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)}} \mapsto (\mathfrak{g} \sharp V, \{\cdot, \cdot\}, \phi + \sigma + \mu, \mathfrak{L}^{\mathfrak{g} \sharp V})$$

is a bijection, where  $\overline{\overline{(\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)}}$  is the equivalence class of  $(\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  under the equivalent relation  $\approx$ .

**Proof.** It follows from Theorem 1, Theorem 2 and Lemma 1.  $\square$

#### 4. Special Cases of Unified Products

In this section, two special cases of unified products are studied. One corresponds to the extension problem and the other corresponds to the factorization problem.

##### 4.1. Crossed Products and the Extension Problem

Let  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  be the extending datum of  $\mathfrak{g}$  by  $V$  such that  $\triangleleft_k$  are trivial maps for  $k = 0, 1, 2, 3$ . Then  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2) = (\triangleright_j, f_j, *_j, \triangleright_3, \sigma; j = 0, 1, 2)$  is a Lie 2-extending structure of  $\mathfrak{g}$  by  $V$  if and only if  $(V, \mu, *_j; j = 0, 1, 2)$  is a strict Lie 2-algebra and the following compatibilities hold for  $i = 0, 1$  and any  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ :

- $f_0(v_0, v_0) = 0;$
- $v_0 \triangleright_0 [x_0, y_0]_0 = [v_0 \triangleright_0 x_0, y_0]_0 + [x_0, v_0 \triangleright_0 y_0]_0;$
- $(v_0 *_0 w_0) \triangleright_0 x_0 = v_0 \triangleright_0 (w_0 \triangleright_0 x_0) - w_0 \triangleright_0 (v_0 \triangleright_0 x_0) + [x_0, f_0(v_0, w_0)]_0;$
- $f_0(v_0, w_0 *_0 u_0) + f_0(w_0, u_0 *_0 v_0) + f_0(u_0, v_0 *_0 w_0) + v_0 \triangleright_0 f_0(w_0, u_0) + w_0 \triangleright_0 f_0(u_0, v_0) + u_0 \triangleright_0 f_0(v_0, w_0) = 0;$
- $v_0 \triangleright_2 [x_1, y_1]_1 = [v_0 \triangleright_2 x_1, y_1]_1 + [x_1, v_0 \triangleright_2 y_1]_1;$
- $\mathfrak{L}_{x_0}(v_1 \triangleright_3 y_0) = v_1 \triangleright_3 [x_0, y_0]_0 + \mathfrak{L}_{y_0}(v_1 \triangleright_3 x_0);$
- $\mathfrak{L}_{v_0 \triangleright_0 x_0} x_1 = -\mathfrak{L}_{x_0}(v_0 \triangleright_2 x_1) + v_0 \triangleright_1 \mathfrak{L}_{x_0} x_1;$
- $v_1 \triangleright_3 (v_0 \triangleright_0 x_0) - v_0 \triangleright_2 (v_1 \triangleright_3 x_0) = \mathfrak{L}_{x_0} f_2(v_0, v_1) - (v_0 *_2 v_1) \triangleright_3 x_0;$
- $\mathfrak{L}_{f_0(v_0, w_0)} x_1 + (v_0 *_0 w_0) \triangleright_2 x_1 - v_0 \triangleright_2 (w_0 \triangleright_2 x_1) + w_0 \triangleright_2 (v_0 \triangleright_2 x_1) = 0;$
- $-v_1 \triangleright_3 f_0(v_0, w_0) + f_2(v_0 *_0 w_0, v_1) - v_0 \triangleright_2 f_2(w_0, v_1) - f_2(v_0, w_0 *_2 v_1) + w_0 \triangleright_2 f_2(v_0, v_1) + f_2(w_0, v_0 *_2 v_1) = 0;$
- $\mu(v_1) \triangleright_0 x_0 = \phi(v_1 \triangleright_3 x_0) + [x_0, \sigma(v_1)]_0;$
- $v_0 \triangleright_0 \phi(x_1) = \phi(v_0 \triangleright_2 x_1);$
- $\phi(f_2(v_0, v_1)) + \sigma(v_0 *_2 v_1) = v_0 \triangleright_0 \sigma(v_1) + f_0(v_0, \mu(v_1));$
- $v_1 \triangleright_3 \phi(x_1) = v_1 \triangleright_1 x_1;$
- $\mu(v_1) \triangleright_2 x_1 = v_1 \triangleright_1 x_1 - \mathfrak{L}_{\sigma(v_1)} x_1;$
- $w_1 \triangleright_3 \sigma(v_1) = f_2(\mu(v_1), w_1) - f_1(v_1, w_1).$

In this case, the associated unified product  $\mathfrak{g} \sharp_{\Omega(\mathfrak{g}, V)} V = \mathfrak{g} \sharp_{\triangleleft}^{\#} V$  is called the crossed product of the Lie 2-algebras  $\mathfrak{g}$  and  $V$ . A system  $(\mathfrak{g}, V, \triangleright_j, f_j, \triangleright_3, \sigma; j = 0, 1, 2)$  consisting of two strict Lie 2-algebras

$\mathfrak{g}, V$ , seven bilinear maps  $\triangleright_i : V_i \times \mathfrak{g}_i \rightarrow \mathfrak{g}_i, f_i : V_i \times V_i \rightarrow \mathfrak{g}_i, \triangleright_2 : V_0 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_1, \triangleright_3 : V_1 \times \mathfrak{g}_0 \rightarrow \mathfrak{g}_1, f_2 : V_0 \times V_1 \rightarrow \mathfrak{g}_1$  for  $i = 0, 1$  and one linear map  $\sigma : V_1 \rightarrow \mathfrak{g}_0$  satisfying the above compatibility conditions will be called a crossed system of Lie 2-algebras. The crossed product associated with the crossed system  $(\mathfrak{g}, V, \triangleright_j, f_j, \triangleright_3, \sigma; j = 0, 1, 2)$  is the strict Lie 2-algebra  $\mathfrak{g} \#_{\triangleleft}^f V = \mathfrak{g} \times V$  with the brackets and derivation given by:

$$\{(x_i, v_i), (y_i, w_i)\}_i := ([x_i, y_i]_i + v_i \triangleright_i y_i - w_i \triangleright_i x_i + f_i(v_i, w_i), v_i *_i w_i),$$

$$\mathfrak{L}_{(x_0, v_0)}(x_1, v_1) := (\mathfrak{L}_{x_0} x_1 + v_0 \triangleright_2 x_1 - v_1 \triangleright_3 x_0 + f_2(v_0, v_1), v_0 *_2 v_1)$$

for  $i = 0, 1$  and  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ . Then  $\mathfrak{g} \cong \mathfrak{g} \times \{0\}$  is an ideal of the Lie 2-algebra  $\mathfrak{g} \#_{\triangleleft}^f V$  since  $\{(x_i, 0), (y_i, w_i)\}_i := ([x_i, y_i]_i - w_i \triangleright_i x_i, 0), \mathfrak{L}_{(x_0, 0)}(x_1, v_1) := (\mathfrak{L}_{x_0} x_1 - v_1 \triangleright_3 x_0, 0)$ , and  $\mathfrak{L}_{(x_0, v_0)}(x_1, 0) := (\mathfrak{L}_{x_0} x_1 + v_0 \triangleright_2 x_1, 0)$ . Conversely, crossed products describe all Lie 2-algebra structures on the 2-vector space  $\mathfrak{c}$  such that a given strict Lie 2-algebra  $\mathfrak{g}$  is an ideal of  $\mathfrak{c}$ .

**Corollary 1.** *Let  $\mathfrak{g}$  be a strict Lie 2-algebra,  $\mathfrak{c}$  a 2-vector space containing  $\mathfrak{g}$  as a 2-vector subspace. Then any strict Lie 2-algebra structure on  $\mathfrak{c}$  that contains  $\mathfrak{g}$  as an ideal is isomorphic to a crossed product of Lie 2-algebras  $\mathfrak{g} \#_{\triangleleft}^f V$ .*

**Proof.** Let  $(\{\cdot, \cdot\}_i, \varepsilon, \mathfrak{L}^{\mathfrak{c}}; i = 0, 1)$  be a strict Lie 2-algebra structure on  $\mathfrak{c}$  such that  $\mathfrak{g}$  is an ideal of  $\mathfrak{c}$ . In particular,  $\mathfrak{g}$  is a 2-vector subalgebra of  $\mathfrak{c}$ . By Theorem 2, the paper obtains the Lie 2-extending structure  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$ , where the action  $\triangleleft_j$  for  $j = 0, 1, 2$  are trivial. Indeed, for any  $v_i \in V_i$  and  $x_i \in \mathfrak{g}_i, \{x_i, v_i\}_i \in \mathfrak{g}_i, \mathfrak{L}_{v_0} v_1 \in \mathfrak{g}_1$  and  $\mathfrak{L}_{v_0} x_1 \in \mathfrak{g}_1$  and hence  $p_i(\{x_i, v_i\}_i) = \{x_i, v_i\}_i, p_1(\mathfrak{L}_{v_0} v_1) = \mathfrak{L}_{v_0} v_1$  and  $p_1(\mathfrak{L}_{v_0} x_1) = \mathfrak{L}_{v_0} x_1$ . Thus, the unified product  $\mathfrak{g} \#_{\Omega(\mathfrak{g}, V)}^f V = \mathfrak{g} \#_{\triangleleft}^f V$  is the crossed product of the Lie 2-algebras  $\mathfrak{g}$  and 2-vector space  $V := Ker(p_1) \xrightarrow{\varepsilon - p_1 \varepsilon} Ker(p_0)$ .  $\square$

**Remark 1.** *By Corollary 1, all crossed products of Lie 2-algebras  $\mathfrak{g} \#_{\triangleleft}^f V$  give the theoretical answer to Problem 2. In fact, a crossed product of Lie 2-algebras  $\mathfrak{g} \#_{\triangleleft}^f V$  also corresponds to the non-abelian extension structure defined in [13]. Define  $\mu_0(v_0)(x_0) = v_0 \triangleright_0 x_0, \mu_0(v_0)(x_1) = v_0 \triangleright_2 x_1, \mu_1(v_1)(x_0) = v_1 \triangleright_3 x_0, \omega(v_0, w_0) = -f_0(v_0, w_0), \nu(v_0, v_1) = -f_2(v_0, v_1), \varphi(v_1) = \sigma(v_1)$  for  $v_i, w_i \in V_i, x_i \in \mathfrak{g}_i$  and  $i = 0, 1$ , where  $\mu_i, \omega, \nu, \varphi$  are linear maps defined in [13]. The result follows.*

In particular, if  $\mathfrak{g}$  is an abelian Lie 2-algebra, i.e.,  $\mathfrak{g}$  is a 2-vector space. Then the set of all extending structures of the abelian Lie 2-algebra  $\mathfrak{g}$  by the 2-vector space  $V$  is parameterized by the set of all  $(\triangleright_j, f_j, *_j, \triangleright_3, \sigma; j = 0, 1, 2)$ , such that  $(V, \mu, *_j; j = 0, 1, 2)$  is a strict Lie 2-algebra,  $(\mathfrak{g}, \triangleright_j, j = 0, 2, 3)$  is a left  $V$ -module and  $f_i : V_i \times V_i \rightarrow V_i, f_2 : V_0 \times V_1 \rightarrow V_1$  are bilinear maps such that  $f_i(v_i, v_i) = 0$  and

- $f_0(v_0, w_0 *_0 u_0) + f_0(w_0, u_0 *_0 v_0) + f_0(u_0, v_0 *_0 w_0) + v_0 \triangleright_0 f_0(w_0, u_0) + w_0 \triangleright_0 f_0(u_0, v_0) + u_0 \triangleright_0 f_0(v_0, w_0) = 0;$
- $v_0 \triangleright_2 f_1(v_1, w_1) + f_2(v_0, v_1 *_1 w_1) + w_1 \triangleright_1 f_2(v_0, v_1) = f_1(v_0 *_2 v_1, w_1) + v_1 \triangleright_1 f_2(v_0, w_1) + f_1(v_1, v_0 *_2 w_1);$
- $v_1 \triangleright_3 (v_0 \triangleright_0 x_0) - v_0 \triangleright_2 (v_1 \triangleright_3 x_0) = \mathfrak{L}_{x_0} f_2(v_0, v_1) - (v_0 *_2 v_1) \triangleright_3 x_0;$
- $v_1 \triangleright_3 f_0(v_0, w_0) = f_2(v_0 *_0 w_0, v_1) - v_0 \triangleright_2 f_2(w_0, v_1) - f_2(v_0, w_0 *_2 v_1) + w_0 \triangleright_2 f_2(v_0, v_1) + f_2(w_0, v_0 *_2 v_1);$
- $\phi(f_2(v_0, v_1)) + \sigma(v_0 *_2 v_1) - v_0 \triangleright_0 \sigma(v_1) - f_0(v_0, \mu(v_1)) = 0;$
- $\mu(v_1) \triangleright_2 x_1 = v_1 \triangleright_1 x_1;$
- $w_1 \triangleright_3 \sigma(v_1) = f_2(\mu(v_1), w_1) - f_1(v_1, w_1)$

for  $i = 0, 1$  and  $u_i, v_i, w_i \in V_i, x_i \in \mathfrak{g}_i$ . For such  $(\triangleright_j, f_j, *_j, \triangleright_3, \sigma; j = 0, 1, 2)$ , the brackets and the derivation of the extending structure on  $\mathfrak{c} \cong \mathfrak{g} \times V$  are given by

$$\{(x_i, v_i), (y_i, w_i)\}_i := (v_i \triangleright_i y_i - w_i \triangleright_i x_i + f_i(v_i, w_i), v_i *_i w_i),$$

$$\mathfrak{L}_{(x_0, v_0)}(x_1, v_1) := (v_0 \triangleright_2 x_1 - v_1 \triangleright_3 x_0 + f_2(v_0, v_1), v_0 *_2 v_1)$$

respectively, where  $i = 0, 1$  and  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ .

By ([12], Theorem 5.6), the second cohomology group  $H^2_{\nabla}(V, \phi)$  classifies 2-extensions. To study the relation between  $H^2_{\nabla}(V, \phi)$  and Lie 2-algebra  $\mathfrak{g} \natural V$ , the paper needs a result of [12].

**Lemma 2** ([12], Proposition 5.3). *Let  $\rho$  be a 2-representation of strict Lie 2-algebra  $\mathfrak{g}$  on  $V$ . Given a triple  $(\omega, \alpha, \varphi) \in ((\mathfrak{g}_0^* \wedge \mathfrak{g}_0^*) \otimes V_0) \oplus (\mathfrak{g}_0^* \otimes \mathfrak{g}_1^* \otimes V_1) \oplus (\mathfrak{g}_1^* \otimes V_0)$ . Then*

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{t_1} & \mathfrak{g}_1 \oplus_{\rho_0^1 \phi} V_1 & \xrightarrow{\pi_1} & \mathfrak{g}_1 \longrightarrow 0, \\ & & \downarrow \mu & & \downarrow \varepsilon & & \downarrow \phi \\ 0 & \longrightarrow & V_0 & \xrightarrow{t_0} & \mathfrak{g}_0 \oplus_{\rho_0^0} V_0 & \xrightarrow{\pi_0} & \mathfrak{g}_0 \longrightarrow 0 \end{array}$$

with

$$\begin{aligned} \omega_1(x_1, y_1) &:= \rho_1(y_1)\varphi(x_1) + \alpha(\phi(x_1); y_1), \\ \varepsilon(x_1, v_1) &= (\phi(x_1), \mu(v_1) + \varphi(x_1)), \\ \mathfrak{L}_{(x_0, v_0)}^\varepsilon(x_1, v_1) &= (\mathfrak{L}_{x_0} x_1, \rho_0^1(x_0)v_1 - \rho_1(x_1)v_0 - \alpha(x_0; x_1)) \end{aligned}$$

is a 2-extensions if and only if the following equations are satisfied

- (i)  $\rho_0^0(x_0)\omega(y_0, z_0) - \omega([x_0, y_0]_0, z_0) + \circlearrowleft = 0;$
- (ii)  $\rho_1(x_1)\varphi(y_1) + \alpha(\phi(y_1); x_1) + \circlearrowleft = 0;$
- (iii)  $\rho_0^1(\phi(x_1))(\rho_1(y_1)\varphi(z_1) + \alpha(\phi(z_1); y_1)) - \rho_1(z_1)\varphi([x_1, y_1]) - \alpha(\mu([x_1, y_1]); z_1) + \circlearrowleft = 0;$
- (iv)  $\omega(x_0, \phi(x_1)) = \mu\alpha(x_0; x_1) + \rho_0^0(x_0)\varphi(x_1) - \varphi(\mathfrak{L}_{x_0} x_1);$
- (v)  $\alpha([x_0, y_0]; x_1) - \rho_1(x_1)\omega(x_0, x_1) = \rho_0^1(x_0)\alpha(y_0; x_1) - \alpha(y_0; \mathfrak{L}_{x_0} x_1) + \circlearrowleft = 0;$
- (vi) For the contraction  $a_{x_0} := \alpha(x_0; -) \in \mathfrak{g}_1^* \otimes V_1$  seen as a 1-cocycle with values in  $\rho_0^1\phi$ ;

for  $i = 0, 1$  and  $x_i, y_i, z_i \in \mathfrak{g}_i, v_i \in V_i$ . Here,  $\circlearrowleft$  stands for cyclic permutations.

**Proposition 2.** *If  $\mathfrak{g}$  is an abelian Lie 2-algebra, then Lie 2-algebra  $\mathfrak{g} \natural V$  is the Lie 2-algebra defined by a second cohomology group  $(\omega, \alpha, \varphi) \in H^2_{\nabla}(V, \phi)$  as in Lemma 2.*

**Proof.** Given a Lie 2-algebra  $\mathfrak{g} \natural V$ . Let  $\omega = -f_0, \alpha = -f_2, \varphi = \sigma$ . Then the paper obtains a Lie 2-algebra determined by the  $(\omega, \alpha, \varphi) \in H^2_{\nabla}(V, \phi)$ . Conversely, suppose that  $\mathfrak{c}$  is a Lie 2-algebra determined by a the  $(\omega, \alpha, \varphi) \in H^2_{\nabla}(V, \phi)$ . Let  $f_0 = -\omega, f_2 = -\alpha, \sigma = \varphi$ . Then the paper obtains a Lie 2-algebra  $\mathfrak{g} \natural V$ . Thus, the conclusion follows.  $\square$

Moreover, this case Lie 2-algebra is also correspondence with the abelian extension of  $V$  by  $\mathfrak{g}$  which is defined in [14].

#### 4.2. Bicrossed Products and the Factorization Problem

Let  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2)$  be the extending datum of  $\mathfrak{g}$  by  $V$  such that  $\sigma$  and  $f_j$  are trivial maps for  $j = 0, 1, 2$ . Then  $\Omega(\mathfrak{g}, V) = (\triangleleft_j, \triangleright_j, f_j, *_j, \triangleleft_3, \triangleright_3, \sigma; j = 0, 1, 2) = (\triangleleft_j, \triangleright_j, *_j, \triangleleft_3, \triangleright_3; j = 0, 1, 2)$  is a Lie 2-extending structure of  $\mathfrak{g}$  by  $V$  if and only if  $(V, \mu, *_j; j = 0, 1, 2)$  is a strict Lie 2-algebra,  $\mathfrak{g}$  is a left  $V$ -module under  $((\triangleright_0, \triangleright_2), \triangleright_3) : V \rightarrow gl(\phi), V$  is a right  $\mathfrak{g}$ -module under  $((\triangleleft_0, \triangleleft_3), \triangleleft_2) : \mathfrak{g} \rightarrow gl(\mu)$  and the following compatibilities hold for  $i = 0, 1$  and any  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ :

- $v_i \triangleright_i [x_i, y_i]_i = [v_i \triangleright_i x_i, y_i]_i + [x_i, v_i \triangleright_i y_i]_i + (v_i \triangleleft_i x_i) \triangleright_i y_i - (v_i \triangleleft_i y_i) \triangleright_i x_i;$
- $(v_i *_i w_i) \triangleleft_i x_i = v_i *_i (w_i \triangleleft_i x_i) + (v_i \triangleleft_i x_i) *_i w_i + v_i \triangleleft_i (w_i \triangleright_i x_i) - w_i \triangleleft_i (w_i \triangleright_i x_i);$
- $\mathfrak{L}_{x_0}(v_1 \triangleright_1 x_1) = (v_1 \triangleleft_1 x_1) \triangleright_3 x_0 + v_1 \triangleright_1 (\mathfrak{L}_{x_0} x_1) + [x_1, v_1 \triangleright_3 x_0]_1 - (v_1 \triangleleft_3 x_0) \triangleright_1 x_1;$



- $(v_1 *_1 w_1) \triangleleft_3 x_0 + w_1 \triangleleft_1 (v_1 \triangleright_3 x_0) = (v_1 \triangleleft_3 x_0) *_1 w_1 + v_1 \triangleleft_1 (w_1 \triangleright_3 x_0) + v_1 *_1 (w_1 \triangleleft_3 x_0)$ ;
- $v_0 \triangleright_2 [x_1, y_1]_1 = [v_0 \triangleright_2 x_1, y_1]_1 + (v_0 \triangleleft_2 x_1) \triangleright_1 y_1 + [x_1, v_0 \triangleright_2 y_1]_1 - (v_0 \triangleleft_2 y_1) \triangleright_1 x_1$ ;
- $v_0 \triangleleft_2 (v_1 \triangleright_1 x_1) + v_0 *_2 (v_1 \triangleleft_1 x_1) - v_1 \triangleleft_1 (v_0 \triangleright_2 x_1) + (v_0 \triangleleft_2 x_1) *_1 v_1 - (v_0 *_2 v_1) \triangleleft_1 x_1 = 0$ ;
- $\mathfrak{L}_{x_0}(v_1 \triangleright_3 y_0) = v_1 \triangleright_3 [x_0, y_0]_0 + (v_1 \triangleleft_3 y_0) \triangleright_3 x_0 + \mathfrak{L}_{y_0}(v_1 \triangleright_3 x_0) - (v_1 \triangleleft_3 x_0) \triangleright_3 y_0$ ;
- $\mathfrak{L}_{v_0 \triangleright_0 x_0} x_1 = (v_0 \triangleleft_2 x_1) \triangleright_3 x_0 - (v_0 \triangleleft_0 x_0) \triangleright_2 x_1 - \mathfrak{L}_{x_0}(v_0 \triangleright_2 x_1) + v_0 \triangleright_2 \mathfrak{L}_{x_0} x_1$ ;
- $v_1 \triangleleft_3 (v_0 \triangleright_0 x_0) - v_0 \triangleleft_2 (v_1 \triangleright_3 x_0) = (v_0 \triangleleft_0 x_0) *_2 v_1 - (v_0 *_2 v_1) \triangleleft_3 x_0 + v_0 *_2 (v_1 \triangleleft_3 x_0)$ ;
- $(v_0 *_0 w_0) \triangleleft_2 x_1 - v_0 \triangleleft_2 (w_0 \triangleright_2 x_1) - v_0 *_2 (w_0 \triangleleft_2 x_1) + w_0 \triangleleft_2 (v_0 \triangleright_2 x_1) + w_0 *_2 (v_0 \triangleleft_2 x_1) = 0$ ;
- $\mu(v_1) \triangleright_2 x_1 = v_1 \triangleright_1 x_1$ ;
- $\mu(v_1) \triangleleft_2 x_1 = v_1 \triangleleft_1 x_1$ .

In this case, the associated unified product  $\mathfrak{g} \bowtie_{\Omega(\mathfrak{g}, V)} V$  is called the bicrossed product  $\mathfrak{g} \bowtie V$  of  $(\mathfrak{g}, V, \triangleleft_k, \triangleright_k; k = 0, 1, 2, 3)$  of the Lie 2-algebras  $\mathfrak{g}$  and  $V$ . The brackets and derivation on  $\mathfrak{g} \bowtie V$  are given by:

$$\{(x_i, v_i), (y_i, w_i)\}_i := ([x_i, y_i]_i + v_i \triangleright_i y_i - w_i \triangleright_i x_i, v_i *_i w_i + v_i \triangleleft_i y_i - w_i \triangleleft_i x_i),$$

$$\mathfrak{L}_{(x_0, v_0)}(x_1, v_1) := (\mathfrak{L}_{x_0} x_1 + v_0 \triangleright_2 x_1 - v_1 \triangleright_3 x_0, v_0 *_2 v_1 + v_0 \triangleleft_2 x_1 - v_1 \triangleleft_3 x_0)$$

for  $i = 0, 1$  and  $x_i, y_i \in \mathfrak{g}_i, v_i, w_i \in V_i$ .

**Theorem 4.** Suppose that  $\mathfrak{g} := ((\mathfrak{g}_0, [\cdot, \cdot]_0), (\mathfrak{g}_1, [\cdot, \cdot]_1), \phi, \mathfrak{L})$ ,  $V := ((V_0, *_0), (V_1, *_1), \mu, *_2)$  are two Lie 2-algebras and  $\mathfrak{c} : \mathfrak{c}_1 \xrightarrow{\epsilon} \mathfrak{c}_0$  is a 2-vector space such that diagram (1) satisfies. Assume that  $(\{\cdot, \cdot\}_i, \epsilon, \mathfrak{L}^{\mathfrak{c}}; i = 0, 1)$  is a strict Lie 2-algebra structure on  $\mathfrak{c}$  such that  $\mathfrak{g}$  and  $V$  are sub-Lie 2-algebras in  $(\mathfrak{c}, \{\cdot, \cdot\}_i, \epsilon, \mathfrak{L}^{\mathfrak{c}}; i = 0, 1)$ . Then Lie 2-algebra  $(\mathfrak{c}, \{\cdot, \cdot\}_i, \epsilon, \mathfrak{L}^{\mathfrak{c}}; i = 0, 1)$  is isomorphic to  $\mathfrak{g} \bowtie V$  of  $\mathfrak{g}$  and  $V$ .

**Proof.** It follows from Theorem 2.  $\square$

### 5. Conclusions

This paper contains information on how to construct a strict Lie 2-algebra from one strict Lie 2-algebra by another strict Lie 2-algebra. A Lie 2-algebra can be obtained from a Lie 2-group. A strict Lie 2-group is usually called a crossed module of Lie groups. It is also an internal object in the category of Lie groups. Similarly, one can study extension structures of a Lie 2-group by another Lie 2-group. One natural avenue of further exploration is to consider the relations between these two kinds of extension, and, furthermore, the relations between the cohomological groups.

Please note that this paper is limited to consideration of extensions of algebras. The paper has neglected the possible topology and geometry of the Lie 2-groups, more generalization 2-gerbes, of which the fields are flat connections—classical, in the sense familiar to physicists.

Another avenue is to discuss algebraic 2-groups over a field of characteristic non-zero  $p$ . Then, the Frobenius maps give the corresponding Lie algebra  $[p]$ -maps. A kind of Lie algebra over a field of characteristic  $p$  with a  $[p]$ -map is called restricted Lie algebra. Hence one can study a similar question about strict restricted Lie 2-algebras and algebraic 2-groups.

A strict Lie 2-algebra is similar to another extension algebra of Lie algebras, namely the two-term  $L_\infty$  algebra. It is well known that an  $L_\infty$  algebra is the same thing as a semi-free graded-commutative differential graded algebra. Suppose  $M$  is a smooth manifold. Then the algebra  $\Omega(M)$  of differential forms is a graded-commutative differential graded algebra. One can construct a semi-free graded-commutative differential graded algebra  $CE(L)$  from any  $L_\infty$  algebra  $L$ , which is called Chevalley–Eilenberg algebra. It is a complicated but straightforward exercise to check that the nilpotence  $d_{CE}^2 = 0$  for Chevalley–Eilenberg algebra differential is exactly equivalent to the homotopy Jacobi identities. Using this ideal, the authors hope to seek similar graded-commutative differential graded algebras to reduce a large number of the equations in Theorem 1. All of these need to be further explored.

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