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Anti-Synchronization of a Class of Chaotic Systems with Application to Lorenz System: A Unified Analysis of the Integer Order and Fractional Order

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Abstract: The paper proves a unified analysis for finite-time anti-synchronization of a class of integer-order and fractional-order chaotic systems. We establish an effective controller to ensure that the chaotic system with unknown parameters achieves anti-synchronization in finite time under our controller. Then, we apply our results to the integer-order and fractional-order Lorenz system, respectively. Finally, numerical simulations are presented to show the feasibility of the proposed control scheme. At the same time, through the numerical simulation results, it is show that for the Lorenz chaotic system, when the order is greater, the more quickly is anti-synchronization achieved.

1. Introduction

In recent years, the synchronization and anti-synchronization of chaotic systems have become a challenging and interesting problem due to the potential applications of chaotic systems in secure communication and control processing, chemical reactions, biological systems, etc. Therefore, the synchronization and anti-synchronization of integer-order and fractional-order chaotic systems has attracted wide attention.

Recently, many control schemes have been proposed to synchronize integer-order chaotic systems; for example, alternate output feedback control, nonlinear control, adaptive control, unknown input control, coupling scheme control, fuzzy control, sampled-data control, and so on [1–12]. Al-Sawalha discussed the anti-synchronization problem of two different hyper-chaotic systems in [13]. Chen et al. proposed a new sliding mode control strategy for a class of chaotic systems with different structures and dimensions in [14]. Chen and Li considered the master-slave anti-synchronization scheme of modified Chua's circuits with linear feedback control in [15].

On the other hand, fractional calculus is a powerful tool in physics and industries. It is shown that some fractional-order dynamical systems can cause chaos [16,17]. Synchronization of the fractional-order chaotic Lü system was investigated in [18]. After that, many control schemes were proposed for the synchronization and anti-synchronization of fractional-order chaotic systems such as active sliding mode control [19], linear state error feedback control [20], nonlinear control [21],

and so on. Huang and Cao [22] studied anti-synchronization of a fractional-order chaotic financial system. Moreover, the relationship between the order and synchronization (anti-synchronization) was demonstrated numerically. Since most of results were based on the assumption that the system parameters were known, in a realistic world, most of the system's parameters cannot be exactly known in advance. Al-Sawalha [23] proved that the third-order chaotic system can be anti-synchronized with the projection of a fourth-order chaotic system under the assumption that their parameters are unknown. Othmana [24] utilized an adaptive control scheme to study the dual anti-synchronization behavior between two chaotic systems with fully-uncertain parameters. Cai et al. [25] proved the problem of projective synchronization of the adaptive full-state mixed function for financial hyper-chaotic systems with uncertain parameters.

Most of the above-mentioned research focused on the long-time behavior (i.e., time tends to infinity); however, in real-world applications, time is often limited due to its life span. Therefore, the finite-time control has been indispensably proposed for the synchronization and anti-synchronization of chaotic systems (see, e.g., [26–36]). Ma and Dong [37] presented a finite-time adaptive synchronization strategy for a class of new hyper-chaotic systems with unknown slave system parameters. Finite-time anti-synchronization of two identical and two different variable-order fractional chaotic systems with unknown parameters was studied in [38]. Note that most of the research was on the integer-order and fractional-order chaotic systems being synchronized by different controllers. However, it is rare to study the finite-time anti-synchronization of integer- and fractional-order chaotic systems with uniform control, and the parameters of the chaotic system are unknown. Therefore, in this paper, we mainly study the finite-time anti-synchronization of the integer- and fractional-order chaotic systems with unknown parameters by a unified controller. The main contributions of this paper are summarized as follows.

(1)   We study the finite-time anti-synchronization of a class of integer- and fractional-order chaotic systems with unknown parameters under the control of a unified controller. By using Lyapunov stability theory and fractional derivative theory, it is proven that integer and fractional chaotic systems with unknown parameters can achieve finite-time anti-synchronization under the control of a unified controller.

(2)   The unified controller theory is applied to integer- and fractional-order Lorentz systems respectively to achieve finite-time anti-synchronization. The correctness of the unified controller theory is verified theoretically.

(3)   The correctness of the theoretical results is verified by simulation examples. The time when the Lorenz system of different orders reaches anti-synchronization is calculated. At the same time, it can be observed that the finite-time T of anti-synchronization decreases with the increase of system order.

The remainder of the paper is organized as follows. The next section designs an effective controller to ensure that the integer-order and fractional-order chaotic systems achieve anti-synchronization in finite time. In Section 3, we apply our results to the integer-order Lorenz system. In Section 4, we apply our results to the fractional-order Lorenz system. In Section 5, numerical simulations are proposed to verify the validity of the method. At last, we conclude this paper and give prospects for future work in Section 6.

2. Finite-Time Anti-Synchronization of the Integer-Order and Fractional-Order Chaotic Systems

2.1. Model Formulation

Lemma 1 ([39])**.** *Suppose that a continuous and positive definite function V(t)satisfies the inequality:*

$$\dot{V}(t) \leq -pV^{\varepsilon}(t), \ \forall t \geq t_0, \ V(t_0) \geq 0,$$

where $p > 0$ and $0 < \varepsilon < 1$ are two constants. For any given time t_0, $V(t)$ satisfies the following inequality: $V^{1-\varepsilon}(t) \leq V^{1-\varepsilon}(t_0) - p(1-\varepsilon)(t-t_0)$, $t_0 \leq t \leq t_1$ and $V(t) \equiv 0$ for all $t \geq t_1$ with t_1 given by $t_1 = t_0 + \frac{V^{1-\varepsilon}(t_0)}{p(1-\varepsilon)}$.

Definition 1 ([40]). *The αth-order Riemann–Liouville fractional integration of function $f(t)$ with respect to t is given by:*

$$D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{1}$$

where $\Gamma(\alpha)$ is the Gammafunction and t_0 is the initial time.

Definition 2 ([40]). *Let $m - 1 < \alpha < m$, $m \in N$; the Caputo fractional derivative of order α of function $f(t)$ with respect to t is defined as follows:*

$$_{t_0} D_t^\alpha f(t) = \frac{d^\alpha y}{dx^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau. \tag{2}$$

Throughout this paper, we employ the Caputo fractional derivative.

Lemma 2 ([40]). *For any constants C_1, C_2, if $f(t)$, $g(t) \in C[t_0, b]$ and $\alpha > 0$, then:*

$$\frac{d^\alpha}{dt^\alpha}(C_1 f(t) \pm C_2 g(t)) = C_1 \frac{d^\alpha}{dt^\alpha}(f(t)) \pm C_2 \frac{d^\alpha}{dt^\alpha}(g(t)). \tag{3}$$

Lemma 3 ([40]). *Suppose $f(t) \in C_a^\alpha([a,b])$, $D_a^\alpha(f(t)) \in C_a^\beta([a,b])$, $\alpha > 0$, $\beta > 0$, $n - 1 < \alpha < n$, $m - 1 < \beta < m$, then:*

$$_a^C D_t^\beta (_a^C D_t^\alpha f(t)) = _a^C D_t^\alpha (_a^C D_t^\beta f(t)) = _a^C D_t^{\alpha+\beta} f(t). \tag{4}$$

Lemma 4 ([41]). *Consider:*

$$_a^C D_t^\alpha x^n < \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha} {}_a^C D_t^\alpha x. \tag{5}$$

Considering the following chaotic system with uncertain parameters (the drive system):

$$D_t^\alpha x_1(t) = f_1(x_1(t)) + F_1(x_1(t))A, \tag{6}$$

where $\alpha \in (0,1]$, $x_1(t) = (x_{11}(t), \cdots, x_{1n}(t))^T \in R^n$ are the state vectors, $f_1 : R^n \to R^n$ are continuous vector functions, $F_1 : R^n \to R^{n \times n}$ are matrices functions, and $A = (a_1, a_2, ..., a_n)^T$ is the unknown parameter vector of the drive system. When $\alpha = 1$, System (6) reduces to the integer-order system in [23].

The response system is described by:

$$D_t^\alpha x_2(t) = f_2(x_2(t)) + F_2(x_2(t))B + u(x_1(t), x_2(t)), \tag{7}$$

where $\alpha \in (0,1]$, $x_2(t) = (x_{21}(t), \cdots, x_{2n}(t))^T \in R^n$ are the state vectors, $f_2 : R^n \to R^n$ are continuous vector functions, $F_2 : R^n \to R^{n \times n}$ are matrix functions, $B = (b_1, b_2..., b_n)^T$ is the unknown parameter vector of the response system, and $u = u(x_1(t), x_2(t)) \in R^n$ is a controller.

The errors between systems (6) and (7) are:

$$e(t) = x_2(t) + x_1(t). \tag{8}$$

Our goal is to design an appropriate controller u with a parameter vector such that the trajectory of the response system (7) could be anti-synchronous with the drive system (6) in finite time, and the unknown parameters are identified simultaneously. We say the response system (7) is anti-synchronous with the drive system (6) in finite time if:

$$\lim_{t \to T} \|e(t)\| = 0, \tag{9}$$

where $\|\cdot\|$ is the Euclidean norm.

2.2. Finite Time Anti-Synchronization: A Unified Analysis for Integer Order and Fractional Order

In this section, we will design a unified controller to achieve finite-time anti-synchronization for systems (6) and (7) with unknown parameters of integer- and fractional-order, and we prove that the controller is valid.

Theorem 1. *For any initial conditions, the two systems (6) and (7) are finite-time anti-synchronized if the control law equation is designed as follows:*

$$u = -f_1(x_1(t)) - f_2(x_2(t)) - F_1(x_1(t))A - F_2(x_2(t))B - ke^p - e, \tag{10}$$

where $0 < p < 1$, $e = e(t)$ and k is a gain matrix.

Proof. The error dynamical system between the drive system (6) and the response system (7) is:

$$D_t^\alpha e = -ke^p - e. \tag{11}$$

(i) If $\alpha = 1$, the Lyapunov function can be chosen as:

$$V(t, e) = \frac{1}{2}e^2. \tag{12}$$

The time derivative of V is:

$$\dot{V}(t, e) = \dot{e} \cdot e = (-ke^p - e) \cdot e = -ke^{1+p} - e^2 \leq -ke^{1+p}.$$

so $\dot{V}(t, e) \leq -2kV^{\frac{1+p}{2}}$ and $0 < p < 1$, then, according to Lemma 1, we have a constant $T = \frac{V(0)^{\frac{1-p}{2}}}{k(1-p)}$, when $t > T$, $e \equiv 0$. Therefore, the finite-time anti-synchronization of the chaotic systems (6) and (7) is achieved under the controller u.

(ii) If $0 < \alpha < 1$, the drive system (6) and the response system (7) are fractional-order systems.

$$[e_1, e_2, e_3] \begin{bmatrix} \frac{d^\alpha e_1}{dt^\alpha} \\ \frac{d^\alpha e_2}{dt^\alpha} \\ \frac{d^\alpha e_3}{dt^\alpha} \end{bmatrix} = [e_1, e_2, e_3] \begin{bmatrix} -e_1 - ke_1^p \\ -e_2 - ke_2^p \\ -e_3 - ke_3^p \end{bmatrix}$$
$$= -e_1^2 - ke_1^{1+p} - e_2^2 - ke_2^{1+p} - e_3^2 - ke_3^{1+p} \leq -k(e_1^2 + e_2^2 + e_3^2)^{\frac{1+p}{2}}. \tag{13}$$

By using Lemma 4, we have:

$$[e_1, e_2, e_3] \begin{bmatrix} \frac{d^\alpha e_1}{dt^\alpha} \\ \frac{d^\alpha e_2}{dt^\alpha} \\ \frac{d^\alpha e_3}{dt^\alpha} \end{bmatrix} = e_1 \frac{d^\alpha e_1}{dt^\alpha} + e_2 \frac{d^\alpha e_2}{dt^\alpha} + e_3 \frac{d^\alpha e_3}{dt^\alpha} > \frac{\Gamma(2)}{\Gamma(2+\alpha)} \frac{d^\alpha (e_1^2 + e_2^2 + e_3^2)^{\frac{1+\alpha}{2}}}{dt^\alpha}. \tag{14}$$

It can be obtained form (13) and (14) that:

$$\frac{\Gamma(2)}{\Gamma(2+\alpha)}\frac{d^\alpha(e_1^2+e_2^2+e_3^2)^{\frac{1+\alpha}{2}}}{dt^\alpha} < -k(e_1^2+e_2^2+e_3^2)^{\frac{1+p}{2}}. \tag{15}$$

Consequently, we obtain:

$$\frac{d^\alpha(e_1^2+e_2^2+e_3^2)^{\frac{1+\alpha}{2}}}{dt^\alpha}(e_1^2+e_2^2+e_3^2)^{-\frac{1+p}{2}} < -k\frac{\Gamma(2+\alpha)}{\Gamma(2)}, \tag{16}$$

which implies:

$$\frac{d^\alpha(e_1^2+e_2^2+e_3^2)^{\alpha-\frac{1+p}{2}}}{dt^\alpha} < -k\frac{\Gamma(2+\alpha)}{\Gamma(2)}\frac{\Gamma(\alpha+\frac{1-p}{2})}{\Gamma(\frac{1-p}{2})}. \tag{17}$$

Integrating both sides of (31) over $[0,T]$ and setting $H = e_1^2+e_2^2+e_3^2$, it follows that:

$$H^{\alpha-\frac{1+p}{2}}(T) - H^{\alpha-\frac{1+p}{2}}(0) < -k\frac{\Gamma(2+\alpha)}{\Gamma(2)}\frac{\Gamma(\alpha+\frac{1-p}{2})}{\Gamma(\frac{1-p}{2})}\frac{1}{\Gamma(1+\alpha)}T^\alpha.$$

Thus, when:

$$T < \left(H^{\alpha-\frac{1+p}{2}}(0)\frac{\Gamma(2)}{k\Gamma(2+\alpha)}\frac{\Gamma(\frac{1-p}{2})\Gamma(1+\alpha)}{\Gamma(\alpha+\frac{1-p}{2})}\right)^{\frac{1}{\alpha}},$$

$H(T) = 0$, i.e., the system (6) and the system (7) are anti-synchronized in a finite time. $\square$

3. Application to the Lorenz System of Integer Order

In this section, we apply our results to integer-order Lorentz systems to show that the designed controller achieves finite-time anti-synchronization control. The drive Lorenz system [42] is given by:

$$\begin{cases} \dot{x}_1 = \sigma(y_1 - x_1), \\ \dot{y}_1 = \rho x_1 - x_1 z_1 - y_1, \\ \dot{z}_1 = x_1 y_1 - \gamma z_1. \end{cases} \tag{18}$$

The response of Lorenz system [42] is given by:

$$\begin{cases} \dot{x}_2 = \sigma_1(y_2 - x_2) + u_1, \\ \dot{y}_2 = \rho_1 x_2 - x_2 z_2 - y_2 + u_2, \\ \dot{z}_2 = x_2 y_2 - \gamma_1 z_2 + u_3, \end{cases} \tag{19}$$

where $\sigma, \rho, \gamma, \sigma_1, \rho_1$, and γ_1 are unknown systems parameters and $u = (u_1, u_2, u_3)^T$ is the controller function to be determined. In the following, an effective controller is designed to achieve the finite-time anti-synchronization Lorenz system with fully-unknown parameters.

Adding (18) to (19), we get:

$$\begin{cases} \dot{e}_1 = \sigma_1(y_2 - x_2) + \sigma(y_1 - x_1) + u_1, \\ \dot{e}_2 = \rho_1 x_2 + \rho x_1 - x_2 z_2 - x_1 z_1 - e_2 + u_2, \\ \dot{e}_3 = x_2 y_2 + x_1 y_1 - \gamma_1 z_2 - \gamma z_1 + u_3, \end{cases} \tag{20}$$

where $e_1 = x_2 + x_1, e_2 = y_2 + y_1, e_3 = z_2 + z_1$.

Our goal is to find an effective controller $u = (u_1, u_2, u_3)^T$, such that the drive system (18) anti-synchronizes with the response system (19) asymptotically in finite time, i.e., $\lim_{t \to T}\|e\| = 0$, where

$e = [e_1, e_2, e_3]^T$. The following theorem shows that the drive system (18) and the response system (19) can achieve anti-synchronization in finite time.

Theorem 2. *The controller is designed as:*

$$
\begin{cases}
u_1 = -\sigma_1(y_2 - x_2) - \sigma(y_1 - x_1) - e_1 - ke_1^p, \\
u_2 = -\rho_1 x_2 - \rho x_1 + x_2 z_2 + x_1 z_1 - ke_2^p, \\
u_3 = \gamma_1 z_2 + \gamma z_1 - x_2 y_2 - x_1 y_1 - e_3 - ke_3^p,
\end{cases}
\tag{21}
$$

where k is a positive constant and $0 < p < 1$.

Proof. The error dynamical system between the drive system (18) and the response system (19) is:

$$
\begin{cases}
\dot{e}_1 = -ke_1^p - e_1, \\
\dot{e}_2 = -ke_2^p - e_2, \\
\dot{e}_3 = -ke_3^p - e_3.
\end{cases}
\tag{22}
$$

If a Lyapunov function candidate is chosen as:

$$
V(t, e) = \frac{1}{2} e^2.
\tag{23}
$$

where:

$$
V(t, e) = [V_1(t, e_1), V_2(t, e_2), V_3(t, e_3)]^T.
$$

The time derivative of V is:

$$
\dot{V}_i(t, e_i) = \dot{e}_i e_i = (-ke_i^p - e_i)e_1 = -ke_i^{1+p} - e_i^2 \le -ke_i^{1+p} \le -2kV_i^{\frac{1+p}{2}}, \quad i = 1, 2, 3,
$$

so $0 < \frac{1+p}{2} < 1$, then according to Lemma 1, we have a constant $T = \frac{V(0)^{\frac{1-p}{2}}}{k(1-p)}$, when $t > T$, $e \equiv 0$. Therefore, it can be seen that the error system (20) is stable for a limited time, and we realize finite-time anti-synchronization between Lorenz systems. $\square$

4. Application to the Lorenz System of Fractional Order

In this section, our goal is to apply our control scheme to finite-time anti-synchronization of the fractional-order Lorenz system. The fractional-order Lorenz system is given as follows [43]:

$$
\begin{cases}
\frac{d^\alpha x_1}{dt^\alpha} = \sigma(y_1 - x_1), \\
\frac{d^\alpha y_1}{dt^\alpha} = \rho x_1 - x_1 z_1 - y_1, \\
\frac{d^\alpha z_1}{dt^\alpha} = x_1 y_1 - \gamma z_1.
\end{cases}
\tag{24}
$$

If the controller is $u(t) = [u_1(t), u_2(t), u_3(t)]^T$, we can get the controlled fractional-order Lorenz system as follows:

$$
\begin{cases}
\frac{d^\alpha x_2}{dt^\alpha} = \sigma_1(y_2 - x_2) + u_1, \\
\frac{d^\alpha y_2}{dt^\alpha} = \rho_1 x_2 - x_2 z_2 - y_2 + u_2, \\
\frac{d^\alpha z_2}{dt^\alpha} = x_2 y_2 - \gamma_1 z_2 + u_3.
\end{cases}
\tag{25}
$$

Consequently, the system (24) and (25) yields the fractional-order error system as follows:

$$
\begin{cases}
\frac{d^\alpha e_1}{dt^\alpha} = \sigma_1(y_2 - x_2) + \sigma(y_1 - x_1) + u_1, \\
\frac{d^\alpha e_2}{dt^\alpha} = \rho_1 x_2 + \rho x_1 - x_2 z_2 - x_1 z_1 - e_2 + u_2, \\
\frac{d^\alpha e_3}{dt^\alpha} = x_2 y_2 + x_1 y_1 - \gamma_1 z_2 - \gamma z_1 + u_3,
\end{cases}
\tag{26}
$$

where $e_1 = x_2 + x_1$, $e_2 = y_2 + y_1$, $e_3 = z_2 + z_1$. Now, our goal is to design a suitable controller $u(t) = [u_1(t), u_2(t), u_3(t)]^T$ to guarantee that the controlled system (25) and the drive system (24) are anti-synchronized in finite-time under the situation of uncertain parameters.

Theorem 3. *If the controller satisfies:*

$$
\begin{cases}
u_1 = -\sigma_1 y_2 + \sigma x_2 - \sigma y_1 + \sigma_1 x_2 - e_1 - k e_1^p, \\
u_2 = x_2 z_2 + x_1 z_1 - \rho_1 x_2 - \rho x_1 - k e_2^p, \\
u_3 = -x_2 y_2 - x_1 y_1 + \gamma_1 z_2 + \gamma z_2 - e_3 - k e_3^p,
\end{cases}
\tag{27}
$$

where $0 < p < 1, k > 0$, the controlled system (24) and the drive system (25) will be anti-synchronized in finite time T.

Proof.

$$
[e_1, e_2, e_3]
\begin{bmatrix}
\frac{d^\alpha e_1}{dt^\alpha} \\
\frac{d^\alpha e_2}{dt^\alpha} \\
\frac{d^\alpha e_3}{dt^\alpha}
\end{bmatrix}
= [e_1, e_2, e_3]
\begin{bmatrix}
-e_1 - k e_1^p \\
-e_2 - k e_2^p \\
-e_3 - k e_3^p
\end{bmatrix}
\tag{28}
$$
$$
= -e_1^2 - k e_1^{1+p} - e_2^2 - k e_2^{1+p} - e_3^2 - k e_3^{1+p} \leq -k(e_1^2 + e_2^2 + e_3^2)^{\frac{1+p}{2}}.
$$

By using Lemma 4, we can obtain:

$$
[e_1, e_2, e_3]
\begin{bmatrix}
\frac{d^\alpha e_1}{dt^\alpha} \\
\frac{d^\alpha e_2}{dt^\alpha} \\
\frac{d^\alpha e_3}{dt^\alpha}
\end{bmatrix}
= e_1 \frac{d^\alpha e_1}{dt^\alpha} + e_2 \frac{d^\alpha e_2}{dt^\alpha} + e_3 \frac{d^\alpha e_3}{dt^\alpha} > \frac{\Gamma(2)}{\Gamma(2+\alpha)} \frac{d^\alpha (e_1^2 + e_2^2 + e_3^2)^{\frac{1+\alpha}{2}}}{dt^\alpha}.
\tag{29}
$$

It can be obtained from (28) and (29) that:

$$
\frac{\Gamma(2)}{\Gamma(2+\alpha)} \frac{d^\alpha (e_1^2 + e_2^2 + e_3^2)^{\frac{1+\alpha}{2}}}{dt^\alpha} < -k(e_1^2 + e_2^2 + e_3^2)^{\frac{1+p}{2}}.
\tag{30}
$$

which implies:

$$
\frac{d^\alpha (e_1^2 + e_2^2 + e_3^2)^{\alpha - \frac{1+p}{2}}}{dt^\alpha} < -k \frac{\Gamma(2+\alpha)}{\Gamma(2)} \frac{\Gamma(\alpha + \frac{1-p}{2})}{\Gamma(\frac{1-p}{2})}.
\tag{31}
$$

Integrating both sides of (31) over $[0, T]$ and setting $H = e_1^2 + e_2^2 + e_3^2$, it follows that:

$$
H^{\alpha - \frac{1+p}{2}}(T) - H^{\alpha - \frac{1+p}{2}}(0) < -k \frac{\Gamma(2+\alpha)}{\Gamma(2)} \frac{\Gamma(\alpha + \frac{1-p}{2})}{\Gamma(\frac{1-p}{2})} \frac{1}{\Gamma(1+\alpha)} T^\alpha.
$$

Thus, when:

$$
T < \left(H^{\alpha - \frac{1+p}{2}}(0) \frac{\Gamma(2)}{k\Gamma(2+\alpha)} \frac{\Gamma(\frac{1-p}{2})\Gamma(1+\alpha)}{\Gamma(\alpha + \frac{1-p}{2})} \right)^{\frac{1}{\alpha}},
\tag{32}
$$

$H(T) = 0$, i.e., the system (24) and the system (25) are anti-synchronized in a finite time. □

Remark 1. *In this paper, a unified controller was used to control the integer- and fractional-order Lorentz systems to achieve finite-time anti-synchronization. At the same time, our method can also be extended to general control problems of integer- and fractional-order chaotic systems with unknown parameters in finite-time anti-synchronization.*

5. Numerical Simulations

In this section, we will verify the correctness of the main results by several simulation examples of Lorenz systems of integer- and fractional-order with the same conditions. The simulation method adopted was the Adams–Bashforth–Morton predictive-correction scheme [44] with a step size of 0.01. The systems' parameters were set to $\sigma = 26$, $\rho = 22$, $\gamma = \frac{8}{3}$; In addition, for which condition $p = 0.9$ and $k = 0.6$. The initial conditions of the systems were taken as $x_1(0) = 0.1$, $y_1(0) = 0.6$, $z_1(0) = 0.9$, $x_2(0) = 0.5$, $y_2(0) = 1.6$, $z_2(0) = 1.9$. According to Lemma 1, the estimated time for integer-order Lorenz systems (18) and (19) to achieve finite-time anti-synchronization is 18.9501. Figure 1 shows the state trajectory curves. When the order of fractional Lorentz system is a = 0.96 and a = 0.9 respectively, the finite time anti-synchronization time of system (24) and system (25) is estimated by using inequality (32), which is 19.2284 and 19.6110. The simulation results are depicted in Figures 2 and 3. Figure 4 shows that the time *T* of finite-time anti-synchronization in the Lorentz system varied with the increase of system order from 0.89 to one. Figures 5–7 further show the effect of order on finite time *T*.
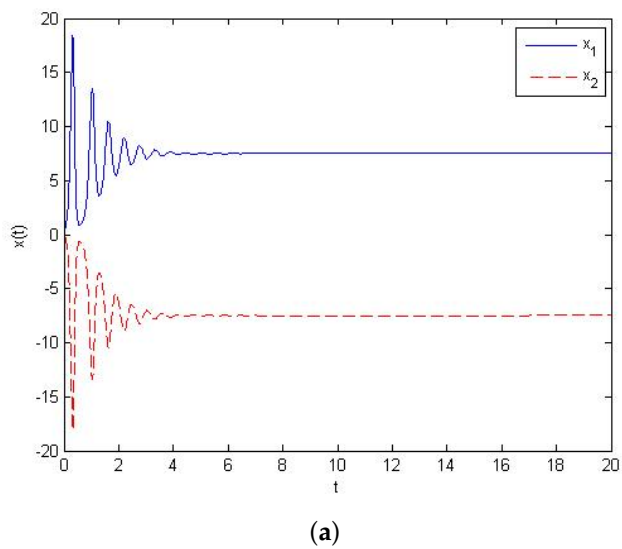

(**a**)
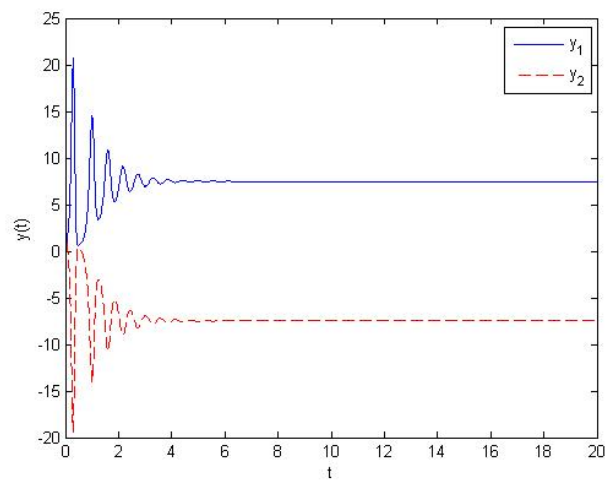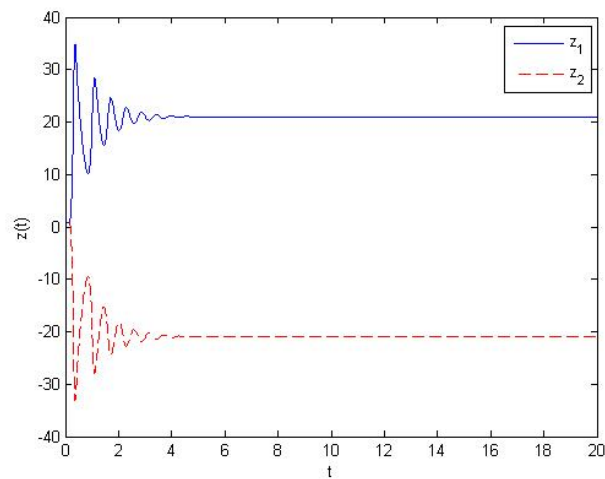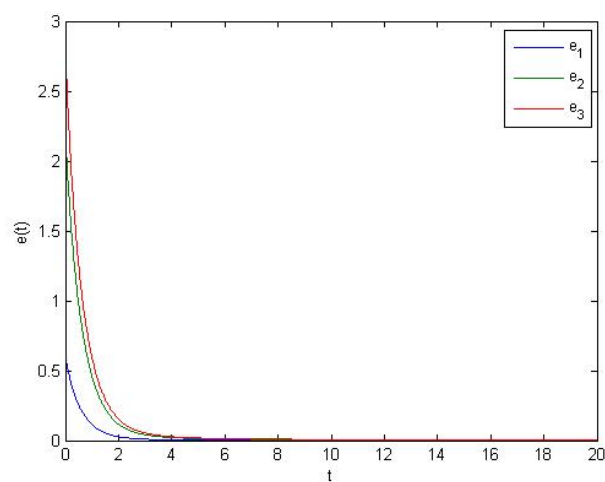

(**b**)

Figure 1. *Cont.*

(c)

(d)
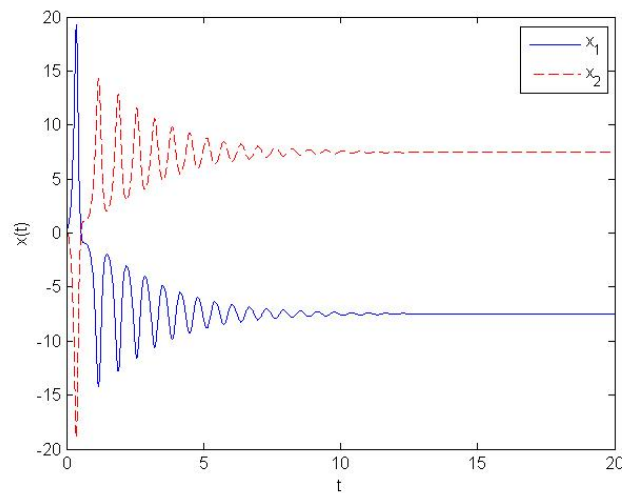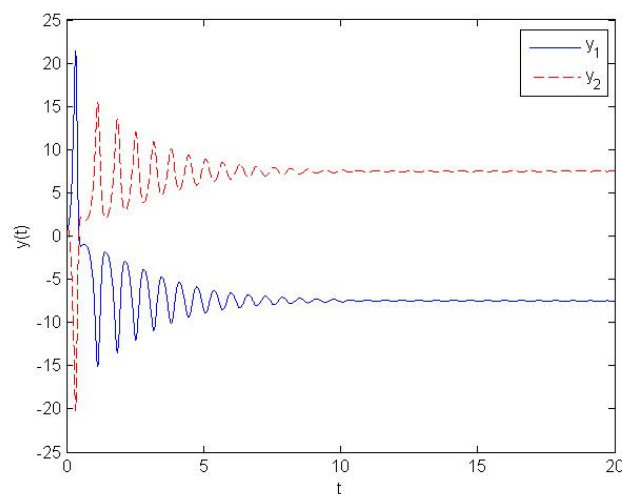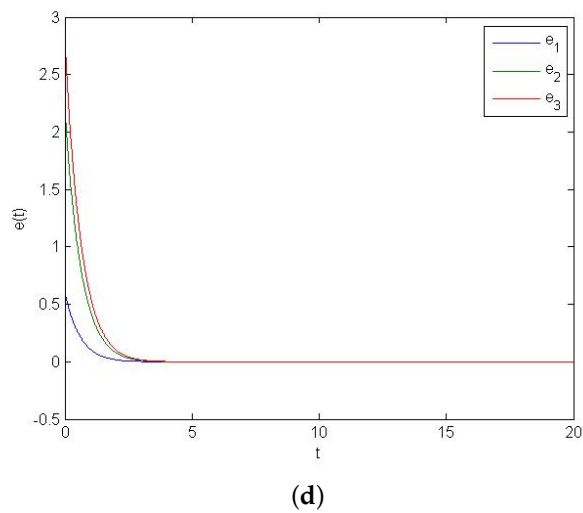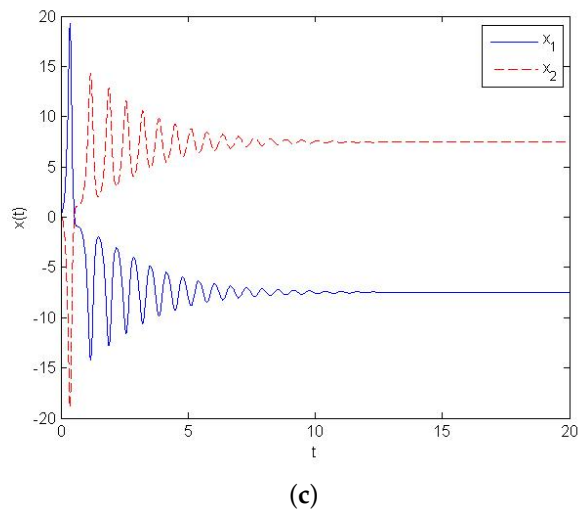
Figure 1. State trajectories of Lorenz Systems (18) and (19): (**a**) signals x_1 and x_2; (**b**) signals y_1 and y_2; (**c**) signals z_1 and z_2; (**d**) state trajectories of the error signals e_1, e_2, e_3.
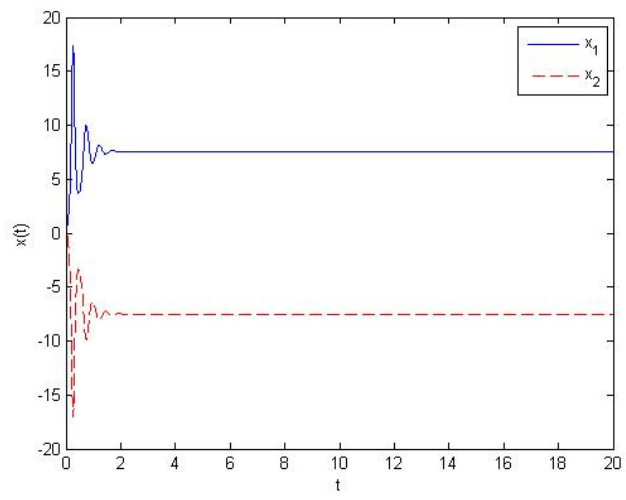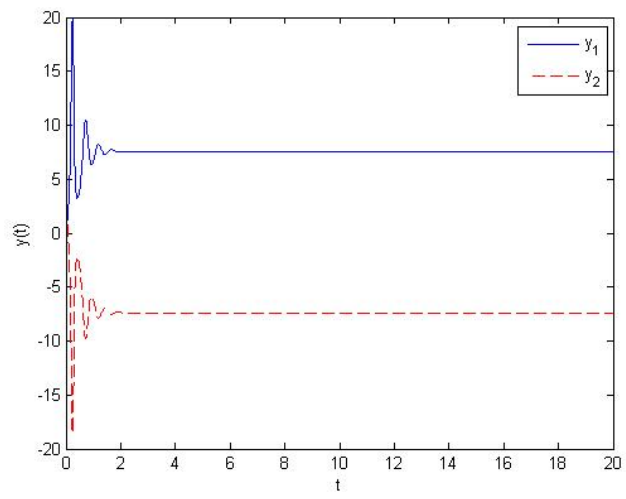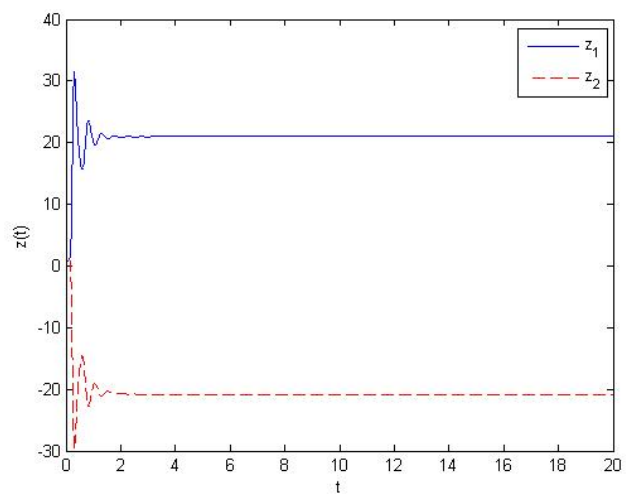

(a)

Figure 2. *Cont.*

(**b**)

(**c**)

(**d**)

Figure 2. State trajectories of fractional-order Lorenz Systems (24) and (25) when order $\alpha = 0.96$: (**a**) signals x_1 and x_2; (**b**) signals y_1 and y_2; (**c**) signals z_1 and z_2; (**d**) state trajectories of the error signals e_1, e_2, e_3.
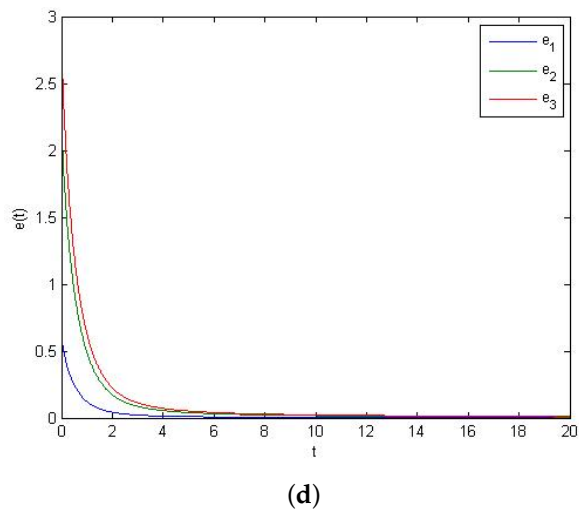
(**a**)

(**b**)

(**c**)

Figure 3. *Cont.*

(d)

Figure 3. State trajectories of fractional-order Lorenz Systems (24) and (25) when order $\alpha = 0.9$: (**a**) signals x_1 and x_2; (**b**) signals y_1 and y_2; (**c**) signals z_1 and z_2; (**d**) state trajectories of the error signals e_1, e_2, e_3.
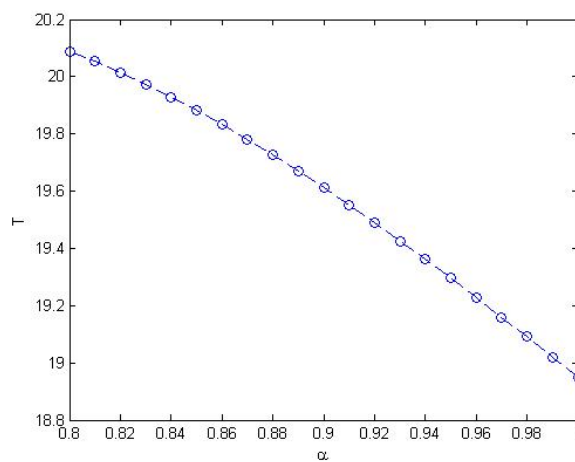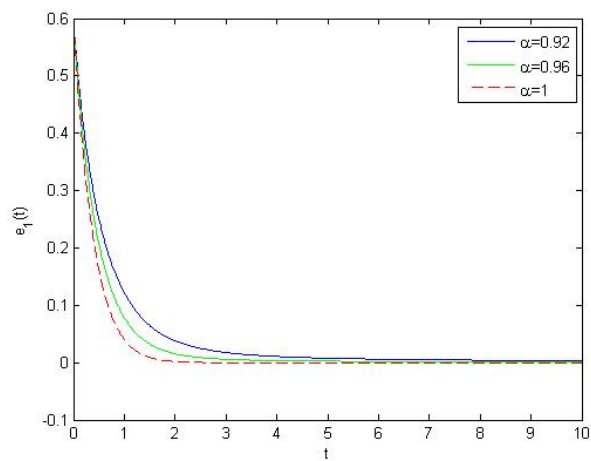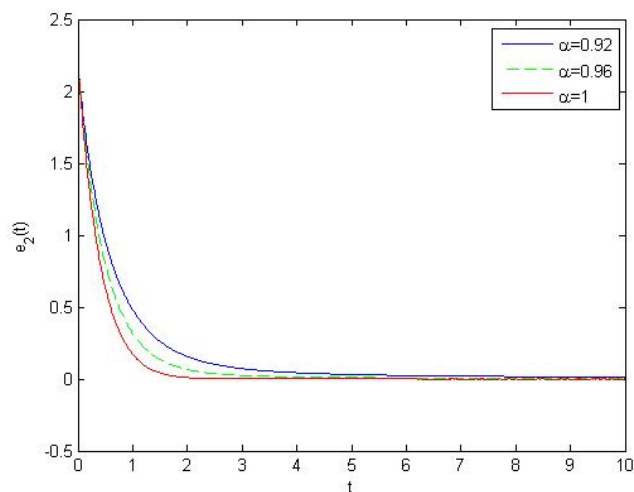

Figure 4. Diagram of the relationship between the different orders of the Lorenz systems and the anti-synchronization of the system in finite-time T.

Figure 5. The error state curves between the drive system and response system with different orders $q = 0.92, 0.96, 1$.

Figure 6. The error state curves between the drive system and response system with different orders q = 0.92, 0.96, 1.
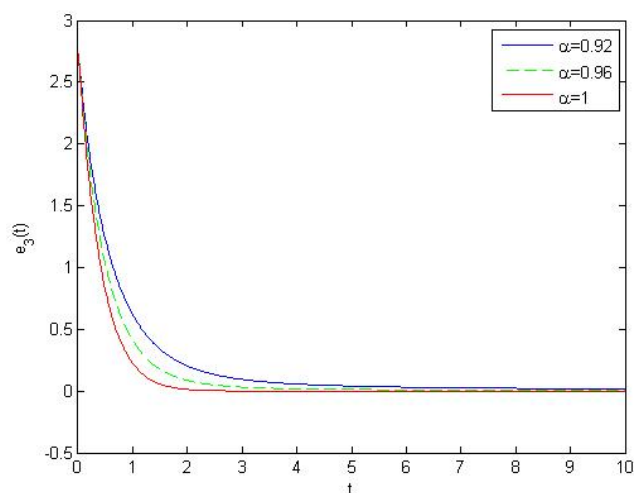
Figure 7. The error state curves between the drive system and response system with different orders q = 0.92, 0.96, 1.

Remark 2. *The simulation results showed that when the order of the Lorentz chaotic system was larger, the anti-synchronization time was shorter. In other words, the larger the system order α, the smaller the finite-time T to achieve finite-time anti-synchronization.*

Remark 3. *All the simulation methods in this paper were based on the Adams–Bashforth–Morton prediction-correction scheme [44]. However, the simulation method can also be some other methods, and the anti-synchronization would be unaffected. For example, the integer-order system can use the standard ode45 solver [45], and the fractional-order system can be simulated by some integration methods, such as the Pade methods [46].*

6. Conclusions

In this paper, the Lyapunov function and fractional derivative theory were used to prove that the integer- and fractional-order chaotic system with unknown parameters can be controlled by the unified controller to achieve finite-time anti-synchronization. The validity of the above conclusions was verified by controlling the integer- and fractional-order unknown parameter Lorentz system with

a unified controller to achieve finite-time anti-synchronization. At the same time, the reliability of the above theory and the influence of the order of the Lorentz system with unknown parameters on the anti-synchronization time *T* were verified by numerical simulation. In addition, future work is outlined as follows: how to design a unified controller to realize finite-time synchronization (anti-synchronization) of time-delay integer- and fractional-order chaotic systems with unknown parameters and the relationship between finite-time, time-delay, and system order to achieve synchronization (anti-synchronization).

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