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Refinements of Majorization Inequality Involving Convex Functions via Taylor's Theorem with Mean Value form of the Remainder

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Abstract: The aim of this paper is to establish some refined versions of majorization inequality involving twice differentiable convex functions by using Taylor theorem with mean-value form of the remainder. Our results improve several results obtained in earlier literatures. As an application, the result is used for deriving a new fractional inequality.

Keywords: majorization inequality; twice differentiable convex functions; refined inequality; Taylor theorem

MSC: 26A51; 26D15; 26D20

1. Introduction

The notion of majorization was introduced in the celebrated monograph [1] by Hardy, Littlewood and Pólya, which was used as a measure of the diversity of the components of an n-dimensional vector.

Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$ and $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ be two n-tuples. The n-tuple $\boldsymbol{\nu}$ is said to be majorized by $\boldsymbol{\vartheta}$ (in symbols $\boldsymbol{\nu} \prec \boldsymbol{\vartheta}$) if $\sum_{i=1}^{k} \nu_{[i]} \leq \sum_{i=1}^{k} \vartheta_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} \vartheta_i$, where $\nu_{[1]} \geq \nu_{[2]} \geq \cdots \geq \nu_{[n]}$ and $\vartheta_{[1]} \geq \vartheta_{[2]} \geq \cdots \geq \vartheta_{[n]}$ are rearrangements of $\boldsymbol{\nu}$ and $\boldsymbol{\vartheta}$ in a descending order.

The majorization has been found many applications in different fields of mathematics. A survey of the applications of majorization and relevant results can be found in the monograph of Marshall and Olkin [2]. Recently, the authors have given considerable attention to the generalizations and applications of the majorization and related inequalities, for details, we refer the reader to our papers [3–13].

In this paper we focus on a type of majorization inequality involving convex functions, which reveals the correlations among majorization, convex functions and inequalities. Now, let us recall briefly this type of majorization inequality.

The following classical majorization inequality can be found in the monographs of Marshall and Olkin [2] and Pečarić et al. [14].

Theorem 1. *Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ be two n-tuples, $\nu_i, \vartheta_i \in I$ $(i = 1, 2, \ldots, n)$, I is an interval. Then*

$$\sum_{i=1}^{n} \Psi(\nu_i) \leq \sum_{i=1}^{n} \Psi(\vartheta_i) \tag{1}$$

holds for every continuous convex function $\Psi : I \to \mathbb{R}$ if and only if $\boldsymbol{\nu} \prec \boldsymbol{\vartheta}$ holds.

Fuchs [15] gave a weighted generalization of the majorization theorem, as follows:

Theorem 2. *Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ be two decreasing n-tuples, $\nu_i, \vartheta_i \in I$ $(i = 1, 2, \ldots, n)$, I is an interval. Suppose $\ell_1, \ell_2, \ldots, \ell_n$ are real numbers such that $\sum_{i=1}^{k} \ell_i \nu_i \leq \sum_{i=1}^{k} \ell_i \vartheta_i$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} \ell_i \nu_i = \sum_{i=1}^{n} \ell_i \vartheta_i$. Then*

$$\sum_{i=1}^{n} \ell_i \Psi(\nu_i) \leq \sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) \tag{2}$$

holds for any continuous convex function $\Psi : I \to \mathbb{R}$.

Bullen, Vasić, and Stanković [16] presented a result similar to the above result, in which the condition of the tuples $\boldsymbol{\nu}$, $\boldsymbol{\vartheta}$ is relaxed and the condition of the function Ψ is intensified.

Theorem 3. *Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ be two decreasing n-tuples, $\nu_i, \vartheta_i \in I$ $(i = 1, 2, \ldots, n)$, I is an interval. Suppose $\ell_1, \ell_2, \ldots, \ell_n$ are real numbers such that $\sum_{i=1}^{k} \ell_i \nu_i \leq \sum_{i=1}^{k} \ell_i \vartheta_i$ for $k = 1, 2, \ldots, n$. If $\Psi : I \to \mathbb{R}$ is a continuous increasing convex function, then*

$$\sum_{i=1}^{n} \ell_i \Psi(\nu_i) \leq \sum_{i=1}^{n} \ell_i \Psi(\vartheta_i). \tag{3}$$

The aim of this paper is to establish the refinements of majorization inequalities of Theorems 1–3. To achieve this, we will first establish an equality by using Taylor theorem with mean-value form of the remainder, which enables us to deduce the refined versions of majorization inequalities mentioned above.

2. Lemma

Lemma 1. *Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ be two n-tuples, $\nu_i, \vartheta_i \in (a, b)$ $(i = 1, 2, \ldots, n)$, and let $\ell_1, \ell_2, \ldots, \ell_n$ be real numbers. If $\Psi : [a, b] \to \mathbb{R}$ is a function such that $\Psi' \in C[a, b]$ and Ψ'' exists on (a, b), then there exists τ_i between ν_i and ϑ_i satisfying*

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) = \sum_{i=1}^{n} \Psi'(\nu_i) \ell_i (\vartheta_i - \nu_i) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2. \tag{4}$$

Proof. Using the Taylor's formula with the Lagrange remainder (mean-value form of the remainder) gives

$$\Psi(\vartheta_i) = \Psi(\nu_i) + \frac{\Psi'(\nu_i)}{1!}(\vartheta_i - \nu_i) + \frac{\Psi''(\tau_i)}{2!}(\vartheta_i - \nu_i)^2, \tag{5}$$

where $\nu_i, \vartheta_i \in (a, b)$, τ_i is a real number between ν_i and ϑ_i $(i = 1, 2, \ldots, n)$.

Multiplying both sides of (5) by ℓ_i and taking summation over i $(i = 1, 2, \ldots, n)$, we get

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) = \sum_{i=1}^{n} \ell_i \Psi(\nu_i) + \sum_{i=1}^{n} \Psi'(\nu_i) \ell_i (\vartheta_i - \nu_i) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2,$$

which is the desired equality (4). The proof of Lemma 1 is complete. □

3. Main Results

In this section, we establish some refinements of the majorization inequality.

Theorem 4. *Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ be two n-tuples, $\nu_i, \vartheta_i \in (a, b)$ $(i = 1, 2, \ldots, n)$. If $\boldsymbol{\nu} \prec \boldsymbol{\vartheta}$ and $\Psi : [a, b] \to \mathbb{R}$ is a twice differentiable convex function, then there exists a real number τ_i between $\nu_{[i]}$ and $\vartheta_{[i]}$ $(i = 1, 2, \ldots, n)$ such that*

$$\sum_{i=1}^{n} \Psi(\vartheta_i) - \sum_{i=1}^{n} \Psi(\nu_i) \geq \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} (\vartheta_{[i]} - \nu_{[i]})^2. \tag{6}$$

where $\nu_{[1]} \geq \nu_{[2]} \geq \cdots \geq \nu_{[n]}$ and $\vartheta_{[1]} \geq \vartheta_{[2]} \geq \cdots \geq \vartheta_{[n]}$ are rearrangements of $\boldsymbol{\nu}$ and $\boldsymbol{\vartheta}$ in a descending order.

Proof. Using Lemma 1 with $\ell_i = 1$, $\nu_i = \nu_{[i]}$, $\vartheta_i = \vartheta_{[i]}$ $(i = 1, 2, \ldots, n)$, one has

$$\sum_{i=1}^{n} \Psi(\vartheta_{[i]}) - \sum_{i=1}^{n} \Psi(\nu_{[i]}) = \sum_{i=1}^{n} \Psi'(\nu_{[i]})(\vartheta_{[i]} - \nu_{[i]}) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} (\vartheta_{[i]} - \nu_{[i]})^2,$$

that is

$$\sum_{i=1}^{n} \Psi(\vartheta_i) - \sum_{i=1}^{n} \Psi(\nu_i) = \sum_{i=1}^{n} \Psi'(\nu_{[i]})(\vartheta_{[i]} - \nu_{[i]}) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} (\vartheta_{[i]} - \nu_{[i]})^2, \tag{7}$$

where $\nu_i, \vartheta_i \in (a, b)$, τ_i is a real number between $\nu_{[i]}$ and $\vartheta_{[i]}$ $(i = 1, 2, \ldots, n)$.

Let

$$A_k = \sum_{i=1}^{k} \vartheta_{[i]}, \quad B_k = \sum_{i=1}^{k} \nu_{[i]} \quad (k = 1, 2, \ldots, n), \quad A_0 = B_0 = 0.$$

Considering the first term in the right hand side of (7), we have

$$\begin{aligned}
\sum_{i=1}^{n} \Psi'(\nu_{[i]})(\vartheta_{[i]} - \nu_{[i]}) &= \sum_{i=1}^{n} \Psi'(\nu_{[i]})(A_i - A_{i-1} - B_i + B_{i-1}) \\
&= \sum_{i=1}^{n} \Psi'(\nu_{[i]})(A_i - B_i) - \sum_{i=1}^{n} \Psi'(\nu_{[i]})(A_{i-1} - B_{i-1}) \\
&= \Psi'(\nu_{[n]})(A_n - B_n) + \sum_{i=1}^{n-1} (\Psi'(\nu_{[i]}) - \Psi'(\nu_{[i+1]}))(A_i - B_i).
\end{aligned}$$

It follows from $\boldsymbol{\nu} \prec \boldsymbol{\vartheta}$ that $A_n - B_n = 0$ and $A_i - B_i \geq 0$ for $i = 1, 2, \ldots, n-1$.

Additionally, since Ψ is a continuous convex function on $[a, b]$, we deduce from $\nu_{[i]} \geq \nu_{[i+1]}$ $(i = 1, 2, \ldots, n-1)$ that

$$\Psi'(\nu_{[i]}) - \Psi'(\nu_{[i+1]}) \geq 0 \ \text{ for } \ i = 1, 2, \ldots, n-1.$$

Hence

$$\sum_{i=1}^{n} \Psi'(\nu_{[i]})(\vartheta_{[i]} - \nu_{[i]}) \geq 0,$$

which, along with the equality (7), leads to the required inequality (6). This completes the proof of Theorem 4. □

Remark 1. *The inequality of Theorem 4 is a refinement of the inequality of Theorem 1, since the term $\sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} (\vartheta_{[i]} - \nu_{[i]})^2$ in inequality (6) is nonnegative.*

In the following, we provide two refinements of majorization inequality by keeping one of the tuples decreasing (increasing).

Theorem 5. *Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_n)$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \dots, \vartheta_n)$ be two n-tuples, $\nu_i, \vartheta_i \in (a, b)$ $(i = 1, 2, \dots, n)$, let $\Psi : [a, b] \to \mathbb{R}$ be a twice differentiable convex function, and let $\ell_1, \ell_2, \dots, \ell_n$ be real numbers such that $\sum_{i=1}^{k} \ell_i \nu_i \le \sum_{i=1}^{k} \ell_i \vartheta_i$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^{n} \ell_i \nu_i = \sum_{i=1}^{n} \ell_i \vartheta_i$.*

(i) If $\boldsymbol{\nu}$ is a decreasing n-tuple, then there exists a real number τ_i between $\nu_{[i]}$ and $\vartheta_{[i]}$ $(i = 1, 2, \dots, n)$ such that

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) \ge \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2. \tag{8}$$

(ii) If $\boldsymbol{\vartheta}$ is a increasing n-tuple, then there exists another real number σ_i between $\nu_{[i]}$ and $\vartheta_{[i]}$ $(i = 1, 2, \dots, n)$ such that

$$\sum_{i=1}^{n} \ell_i \Psi(\nu_i) - \sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) \ge \sum_{i=1}^{n} \frac{\Psi''(\sigma_i)}{2} \ell_i (\vartheta_i - \nu_i)^2. \tag{9}$$

Proof. (i) It follows from Lemma 1 that

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) = \sum_{i=1}^{n} \Psi'(\nu_i) \ell_i (\vartheta_i - \nu_i) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2, \tag{10}$$

where $\nu_i, \vartheta_i \in (a, b)$, τ_i is a real number between ν_i and ϑ_i $(i = 1, 2, \dots, n)$. Let

$$A_k = \sum_{i=1}^{k} \ell_i \vartheta_i, \quad B_k = \sum_{i=1}^{k} \ell_i \nu_i \ \ (k = 1, 2, \dots, n), \quad A_0 = B_0 = 0.$$

Then, we have $A_i \ge B_i$ $(i = 1, 2, \dots, n-1)$, $A_n = B_n$, and

$$\begin{aligned} \sum_{i=1}^{n} \Psi'(\nu_i) \ell_i (\vartheta_i - \nu_i) &= \sum_{i=1}^{n} \Psi'(\nu_i)(A_i - A_{i-1} - B_i + B_{i-1}) \\ &= \Psi'(\nu_n)(A_n - B_n) + \sum_{i=1}^{n-1} (\Psi'(\nu_i) - \Psi'(\nu_{i+1}))(A_i - B_i). \\ &= \sum_{i=1}^{n-1} (\Psi'(\nu_i) - \Psi'(\nu_{i+1}))(A_i - B_i). \end{aligned}$$

Noting that Ψ is a continuous convex function on $[a, b]$, and $\boldsymbol{\nu}$ is a decreasing n-tuple, we obtain $\Psi'(\nu_i) - \Psi'(\nu_{i+1}) \ge 0$ for $i = 1, 2, \dots, n-1$.

Hence

$$\sum_{i=1}^{n} \Psi'(\nu_i) \ell_i (\vartheta_i - \nu_i) \ge 0,$$

which, together with inequality (10), leads to the required inequality (8).

(ii) Similarly, we can prove the inequality (9) under the condition that $\boldsymbol{\vartheta}$ is an increasing n-tuple. The proof of Theorem 5 is complete. □

Remark 2. *The inequality (8) of Theorem 5 is a refinement of the inequality (2) of Theorem 2 in the case when $\ell_1, \ell_2, \dots, \ell_n$ are positive numbers.*

Theorem 6. *Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_n)$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \dots, \vartheta_n)$ be two n-tuples, $\nu_i, \vartheta_i \in (a, b)$ $(i = 1, 2, \dots, n)$, let $\Psi : [a, b] \to \mathbb{R}$ be a twice differentiable and increasing convex function, and let $\ell_1, \ell_2, \dots, \ell_n$ be real numbers such that $\sum_{i=1}^{k} \ell_i \nu_i \le \sum_{i=1}^{k} \ell_i \vartheta_i$ for $k = 1, 2, \dots, n$. If $\boldsymbol{\nu}$ is a decreasing n-tuple, then there exists a real number τ_i between $\nu_{[i]}$ and $\vartheta_{[i]}$ $(i = 1, 2, \dots, n)$ such that*

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) \ge \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2. \tag{11}$$

Proof. Let

$$A_k = \sum_{i=1}^{k} \ell_i \vartheta_i, \quad B_k = \sum_{i=1}^{k} \ell_i \nu_i \ \ (k = 1, 2, \dots, n), \quad A_0 = B_0 = 0.$$

By Lemma 1, for any $\nu_i, \vartheta_i \in (a, b)$ $(i = 1, 2, \dots, n)$, there exists a real number between ν_i and ϑ_i such that

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) = \sum_{i=1}^{n} \Psi'(\nu_i) \ell_i (\vartheta_i - \nu_i) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2$$

$$= \sum_{i=1}^{n} \Psi'(\nu_i)(A_i - A_{i-1} - B_i + B_{i-1}) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2$$

$$= \Psi'(\nu_n)(A_n - B_n) + \sum_{i=1}^{n-1} (\Psi'(\nu_i) - \Psi'(\nu_{i+1}))(A_i - B_i) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2.$$

Since Ψ is a continuous convex function on $[a, b]$, and $\boldsymbol{\nu}$ is a decreasing n-tuple, we obtain $\Psi'(\nu_i) - \Psi'(\nu_{i+1}) \geq 0$ for $i = 1, 2, \dots, n-1$. In addition, since Ψ is an increasing function on $[a, b]$, we get $\Psi'(\nu_n) \geq 0$. Now, by using the assumption conditions $A_i \geq B_i$ $(k = 1, 2, \dots, n)$, we conclude that

$$\Psi'(\nu_n)(A_n - B_n) + \sum_{i=1}^{n-1} (\Psi'(\nu_i) - \Psi'(\nu_{i+1}))(A_i - B_i) \geq 0.$$

Therefore, we have

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) \geq \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2.$$

The Theorem 6 is proved. □

Remark 3. *The inequality (11) of Theorem 6 is a refinement of the inequality (3) of Theorem 3 in the case when $\ell_1, \ell_2, \dots, \ell_n$ are positive numbers.*

Theorem 7. *Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_n)$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \dots, \vartheta_n)$ be two n-tuples, $\nu_i, \vartheta_i \in (a, b)$ $(i = 1, 2, \dots, n)$, let $\Psi : [a, b] \to \mathbb{R}$ be a twice differentiable convex function, and let $\ell_1, \ell_2, \dots, \ell_n$ be positive numbers. If $\boldsymbol{\nu}$ and $\boldsymbol{\vartheta} - \boldsymbol{\nu}$ are monotonic in the same sense, then there exists a real number τ_i between $\nu_{[i]}$ and $\vartheta_{[i]}$ $(i = 1, 2, \dots, n)$ such that*

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) \geq \frac{1}{\ell_1 + \ell_2 + \cdots + \ell_n} \sum_{i=1}^{n} \ell_i \Psi'(\nu_i) \sum_{i=1}^{n} \ell_i (\vartheta_i - \nu_i) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2. \tag{12}$$

Proof. Since Ψ is convex function, and tuple $\boldsymbol{\nu}$ and tuple $\boldsymbol{\vartheta} - \boldsymbol{\nu}$ are monotonic in the same sense, we conclude that $\Psi'(\boldsymbol{\nu})$ and $\boldsymbol{\vartheta} - \boldsymbol{\nu}$ are monotonic in the same sense.

Using the Chebyshev's inequality for weights $\ell_1, \ell_2, \dots, \ell_n$, we obtain

$$(\sum_{i=1}^{n} \ell_i) \sum_{i=1}^{n} \ell_i \Psi'(\nu_i)(\vartheta_i - \nu_i) \geq \sum_{i=1}^{n} \ell_i \Psi'(\nu_i) \sum_{i=1}^{n} \ell_i (\vartheta_i - \nu_i).$$

On the other hand, by Lemma 1, for any $\nu_i, \vartheta_i \in (a, b)$ $(i = 1, 2, \dots, n)$, there exists a real number τ_i between ν_i and ϑ_i such that

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) = \sum_{i=1}^{n} \Psi'(\nu_i) \ell_i (\vartheta_i - \nu_i) + \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2.$$

Hence, we get

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) \geq \frac{1}{\ell_1 + \ell_2 + \cdots + \ell_n} \sum_{i=1}^{n} \ell_i \Psi'(\nu_i) \sum_{i=1}^{n} \ell_i (\vartheta_i - \nu_i)$$

$$+ \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2.$$

This proves the required inequality (12) in Theorem 7. □

Applying an additional condition $\sum_{i=1}^{n} \ell_i \nu_i \leq \sum_{i=1}^{n} \ell_i \vartheta_i$ to inequality (12), we obtain the following result.

Corollary 1. *Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ be two n-tuples, $\nu_i, \vartheta_i \in (a, b)$ $(i = 1, 2, \ldots, n)$, let $\Psi : [a, b] \to \mathbb{R}$ be a twice differentiable and increasing convex function, and let $\ell_1, \ell_2, \ldots, \ell_n$ be positive numbers. If $\boldsymbol{\nu}$ and $\boldsymbol{\vartheta} - \boldsymbol{\nu}$ are monotonic in the same sense, and $\sum_{i=1}^{n} \ell_i \nu_i \leq \sum_{i=1}^{n} \ell_i \vartheta_i$, then there exists a real number τ_i between $\nu_{[i]}$ and $\vartheta_{[i]}$ $(i = 1, 2, \ldots, n)$ such that*

$$\sum_{i=1}^{n} \ell_i \Psi(\vartheta_i) - \sum_{i=1}^{n} \ell_i \Psi(\nu_i) \geq \sum_{i=1}^{n} \frac{\Psi''(\tau_i)}{2} \ell_i (\vartheta_i - \nu_i)^2. \tag{13}$$

4. An Application

In this section we establish a new fractional inequality to illustrate the application of our results.

Theorem 8. *Let ξ_1, ξ_2, ξ_3 be positive numbers and $\xi_1 \geq \xi_2 \geq \xi_3$. Then we have the inequality*

$$\frac{1}{2\xi_1} + \frac{1}{2\xi_2} + \frac{1}{2\xi_3} - \frac{1}{\xi_1 + \xi_2} - \frac{1}{\xi_1 + \xi_3} - \frac{1}{\xi_2 + \xi_3}$$

$$\geq \frac{(\xi_1 - \xi_2)^2}{2\xi_1(\xi_1 + \xi_2)^2} + \frac{(2\xi_2 - \xi_1 - \xi_3)^2}{2\xi_2(\xi_1 + \xi_3)^2} + \frac{(\xi_2 - \xi_3)^2}{2\xi_3(\xi_2 + \xi_3)^2}. \tag{14}$$

Proof. From the given condition $\xi_1 \geq \xi_2 \geq \xi_3$, it is easy to check that

$$\xi_1 + \xi_2 \geq \xi_1 + \xi_3 \geq \xi_2 + \xi_3, \quad 2\xi_1 \geq 2\xi_2 \geq 2\xi_3$$

and

$$(\xi_1 + \xi_2, \xi_1 + \xi_3, \xi_2 + \xi_3) \prec (2\xi_1, 2\xi_2, 2\xi_3).$$

Using Theorem 4 and taking $\boldsymbol{\nu} = (\xi_1 + \xi_2, \xi_1 + \xi_3, \xi_2 + \xi_3)$, $\boldsymbol{\vartheta} = (2\xi_1, 2\xi_2, 2\xi_3)$, $\Psi(x) = \frac{1}{x}$, $x \in (0, +\infty)$ in (6), we obtain that there exists a real number τ_i between $\nu_{[i]}$ and $\vartheta_{[i]}$ $(i = 1, 2, 3)$ such that

$$\frac{1}{2\xi_1} + \frac{1}{2\xi_2} + \frac{1}{2\xi_3} - \frac{1}{\xi_1 + \xi_2} - \frac{1}{\xi_1 + \xi_3} - \frac{1}{\xi_2 + \xi_3}$$

$$\geq \frac{1}{\tau_1^3}(\xi_1 - \xi_2)^2 + \frac{1}{\tau_2^3}(2\xi_2 - \xi_1 - \xi_3)^2 + \frac{1}{\tau_3^3}(\xi_2 - \xi_3)^2. \tag{15}$$

Further, by (5) we find that τ_1, τ_2, τ_3 satisfy

$$\frac{1}{2\xi_1} - \frac{1}{\xi_1 + \xi_2} = -\frac{\xi_1 - \xi_2}{(\xi_1 + \xi_2)^2} + \frac{1}{\tau_1^3}(\xi_1 - \xi_2)^2,$$

$$\frac{1}{2\xi_2} - \frac{1}{\xi_1 + \xi_3} = -\frac{2\xi_2 - \xi_1 - \xi_3}{(\xi_1 + \xi_3)^2} + \frac{1}{\tau_2^3}(2\xi_2 - \xi_1 - \xi_3)^2,$$

$$\frac{1}{2\xi_3} - \frac{1}{\xi_2 + \xi_3} = -\frac{\xi_3 - \xi_2}{(\xi_2 + \xi_3)^2} + \frac{1}{\tau_3^3}(\xi_3 - \xi_2)^2.$$

From the above equations, we have

$$\tau_1^3 = 2\xi_1(\xi_1 + \xi_2)^2,\ \tau_2^3 = 2\xi_2(\xi_1 + \xi_3)^2,\ \tau_3^3 = 2\xi_3(\xi_3 + \xi_2)^2. \tag{16}$$

Combining (15) and (16) leads to the desired inequality (14). The proof of Theorem 8 is complete. □

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