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# On the Oscillation of Non-Linear Fractional Difference Equations with Damping

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**Abstract:** In studying the Riccati transformation technique, some mathematical inequalities and comparison results, we establish new oscillation criteria for a non-linear fractional difference equation with damping term. Preliminary details including notations, definitions and essential lemmas on discrete fractional calculus are furnished before proceeding to the main results. The consistency of the proposed results is demonstrated by presenting some numerical examples. We end the paper with a concluding remark.

**Keywords:** oscillation of solutions; non-linear fractional difference equation; damping term

**MSC:** 39A21; 26A33

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## 1. Introduction

In the investigations of qualitative properties for differential and difference equations, research on the oscillation of solutions has gained noticeable attention among many researchers over the last few decennium [1–3]. Recent years, in particular, have witnessed an explosive interest in the theory of fractional differential equations [4,5]. As a result, researchers have started the study of oscillation of fractional differential and difference equations. Despite the appearance of some recent results, investigations in the direction of oscillation of fractional differential and difference equations are still inert in the initial phases.

In his remarkable paper, Grace et al. [6] initiated the topic of oscillation of fractional differential equations and provided substantial results on the oscillation of non-linear fractional differential equations within Riemann-Liouville differential operator. The results are also stated when the Riemann-Liouville differential operator is replaced by Caputo's differential operator. Afterwards, several results have appeared and thus many types of fractional differential and difference equations have been investigated; the reader can consult the papers [6–22] where different approaches have been used to prove the main results. For the sake of completeness and comparison, we review some results in the sequel.

By the help of the newly defined discrete fractional calculus [23], the authors in [8] descretized the results of Grace et al., in [6] and obtained sufficient conditions for the oscillation of the non-linear fractional difference equation

$$\nabla^{\alpha} y(t) + f_1(t, y(t)) = r(t) + f_2(t, y(t)), \tag{1}$$

where $m-1<\alpha<m$, $m\in N$, and $\nabla^{\alpha}$ denotes the Riemann-Liouville or Caputo's difference operator of order $\alpha$. In [9], the previous results which were produced in [8] for Equation (1) have been improved and different oscillation criteria have been reported. In [10], Sagayaraj et al., discussed the oscillation of the non-linear fractional difference equation

$$\Delta(p(t)(\Delta^{\alpha}y(t))^{\gamma})+q(t)f(G(t))=0, \tag{2}$$

where $\alpha\in(0,1]$, $\gamma>0$ is a quotient of odd positive integers and $G$ is defined as

$$G(t)=\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}y(s). \tag{3}$$

The kernel $(t-s-1)^{(-\alpha)}$ in (3) will be specifically defined later in the context. In [11], the authors continued further and investigated the oscillation of the non-linear fractional difference equation with damping term

$$\Delta(c(t)(\Delta^{\alpha}y(t))^{\gamma})+p(t)(\Delta^{\alpha}y(t))^{\gamma}+q(t)f(G^{\gamma}(t))=0, \tag{4}$$

where $\alpha\in(0,1]$, $\gamma>0$ is a quotient of odd positive integers and $G$ is defined as in (3). In the paper [12], Li studied the oscillation of the non-linear fractional difference equation with damping and forcing terms

$$(1+p(t))\Delta(\Delta^{\alpha}y(t))+p(t)\Delta^{\alpha}y(t)+f(t,y(t))=g(t), \tag{5}$$

where $\alpha\in(0,1)$. In the papers [13,14], never the less, the authors discussed respectively the oscillation of the equations

$$\Delta(p(t)\Delta([r(t)(\Delta^{\alpha}y(t))]^{\gamma}))+F(t,G(t))=0, \tag{6}$$

and

$$\Delta(p(t)[\Delta(r(t)(\Delta^{\alpha}y(t))^{\gamma_1})]^{\gamma_2})+q(t)f(t,G(t))=0, \tag{7}$$

where $\alpha\in(0,1]$, $\gamma$, $\gamma_1$, $\gamma_2$ are the quotients of odd positive integers and $G$ is defined as in (3). The operators $\Delta$ and $\Delta^{\alpha}$ are nothing but the delta difference operator and the fractional difference operator of order $\alpha$, respectively. In addition and based on the techniques used in the proofs, the parameters and the non-linear terms defined in the above listed equations satisfy miscellaneous conditions of certain types.

In this paper, and motivated by the above mentioned work, we investigate the oscillatory behavior of the non-linear fractional difference equation with damping term of the form

$$\Delta\big(a(t)\Delta^{\alpha}y(t)\big)+p(t)\Delta^{\alpha}y(t)+q(t)f(G(t))=0, \tag{8}$$

where $t\in\mathbb{N}_{t_0+1-\alpha}$, $\Delta^{\alpha}$ denotes the Riemann-Liouville fractional difference operator of order $\alpha\in(0,1]$ and $G$ is defined as in (3). In view of Equation (8), one can figure out that this paper provides extension to some existing results in the literature. Besides, our approach is different and is based on the implementation of the Riccati transformation technique, some mathematical inequalities and comparison results.

To prove the main results, we make use of the following assumptions

(A1) $p$ is a non-negative sequence such that $1-p(t)>0$ for large $t$;
(A2) $q$ is a non-negative sequence;
(A3) $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and there exists a constant $K>0$ such that $f(x)/x\geq K$ for all $x\neq 0$;
(A4) $f(x)-f(y)=S(x,y)(x-y)$ for all $x,y\neq 0$, where $S$ is a non-negative function.

A solution $x$ of Equation (8) is said to be oscillatory if for every integer $N_0 > 0$, there exists $t \geq N_0$ such that $x(t)x(t+1) \leq 0$; otherwise, it is said to be non-oscillatory. An equation is oscillatory if all its solutions oscillate.

The structure of this paper is as follows: Section 2 is devoted to assembling some preliminaries and essential lemmas operated as infrastructure to prove the main results. Section 3 provides the main oscillation results for Equation (8). Three examples are provided in Section 4 to support the theory. We end the paper by concluding remark in Section 5.

## 2. Essential Preliminaries

In this section, we recall some basic notations, definitions and essential lemmas on discrete fractional calculus that are needed in the subsequent sections. These preliminaries operate as substantial infrastructure prior to proving the main results.

For arbitrary $\alpha$, we define

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)},$$

where we have the convention that division at pole yields zero, i.e., we assume that if $t-\alpha+1 \in \{0,-1,\ldots,-k,\ldots\}$, then $t^{(\alpha)} = 0$.

**Definition 1** ([24]). *Let $\alpha > 0$ and $\sigma(t) = t+1$ be the forward jumping operator. Then, the fractional sum of $f$ is defined by*

$$\Delta^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s), \qquad t \in \mathbb{N}_a = \{a, a+1, \ldots\}. \tag{9}$$

We observe herein that the operator $\Delta^{-\alpha}$ maps functions define on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$.

**Definition 2** ([24]). *Let $\alpha > 0$. The fractional difference of $f$ is defined by*

$$\begin{aligned} \Delta^{\alpha} f(t) &= \Delta^n \Delta^{-(n-\alpha)} f(t) \\ &= \frac{\Delta^n}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} (t-\sigma(s))^{(n-\alpha-1)} f(s), \qquad t \in \mathbb{N}_{a+(n-\alpha)}. \end{aligned}$$

The following lemma has technical structure that will facilitate proving process.

**Lemma 1** ([14]). *Let $y$ be a solution of (8) and let $G(t)$ be defined as in (3). Then*

$$\Delta(G(t)) = \Gamma(1-\alpha)\Delta^{\alpha} y(t).$$

**Lemma 2** ([25,26]). *Let the function $K(t,s,y) : \mathbb{N}_{t_0} \times \mathbb{N}_{t_0} \times \mathbb{R}^+ \to \mathbb{R}$ be such that the function $K(t,s,\cdot)$ is non-decreasing for each fixed $t$, $s$. Furthermore, let $h$ be a given sequence satisfying*

$$u(t) \geq (\leq)\, h(t) + \sum_{s=t_0}^{t-1} K(t,s,u(s))$$

*and*

$$v(t) = h(t) + \sum_{s=t_0}^{t-1} K(t,s,v(s))$$

*for $t \in \mathbb{N}_{t_0}$. Then, $u(t) \geq (\leq)\, v(t)$ for all $t \in \mathbb{N}_{t_0}$.*

**Lemma 3.** *Let $a(t) \equiv 1$ in (8). If $y$ is a non-oscillatory solution of (8) and*

$$\sum_{t=t_0}^{\infty} \prod_{s=t_0}^{t-1} [1 - p(s)] = \infty, \tag{10}$$

*then there is an integer $t_1 \geq t_0$ such that $y(t)\Delta^\alpha y(t) > 0$ for all $t \geq t_1$.*

**Proof.** Suppose that $y$ is a non-oscillatory solution of Equation (8). Without loss of generality, we may assume that $y$ is eventually positive. The proof of the case when $y$ is eventually negative is similar, hence is omitted. Then there exists $T \in [t_0, \infty)$ such that $y(t) > 0$ and $G(t) > 0$ for $t \geq T$.

We claim that $\Delta^\alpha y(t)$ is eventually positive. Suppose that there exists an integer $t_1 > T$ such that $\Delta^\alpha y(t_1) < 0$ or $\Delta^\alpha y(t_1) = 0$. In the former case; in view of (8), we have

$$\begin{aligned} \Delta^\alpha y(t_1)\big(\Delta(\Delta^\alpha y(t_1))\big) &= -p(t_1)\big(\Delta^\alpha y(t_1)\big)^2 - q(t_1) f(G(t_1))\Delta^\alpha y(t_1) \\ &\geq -p(t_1)\big(\Delta^\alpha y(t_1)\big)^2. \end{aligned}$$

It follows that

$$\Delta^\alpha y(t_1)[\Delta^\alpha y(t_1+1) - \Delta^\alpha y(t_1)] \geq -p(t_1)\,(\Delta^\alpha y(t_1))^2$$

or

$$\Delta^\alpha y(t_1)\Delta^\alpha y(t_1+1) \geq [1 - p(t_1)]\,(\Delta^\alpha y(t_1))^2.$$

Then by $(A1)$, we must have $\Delta^\alpha y(t_1+1) < 0$. Therefore, by induction, we obtain $\Delta^\alpha y(t) < 0$ for all $t \geq t_1$. In the latter case, however, in view of (8) and $(A3)$, we get

$$\Delta(\Delta^\alpha y(t_1)) = -q(t_1) f(G(t_1)) \leq 0$$

which implies that $\Delta^\alpha y(t_1+1) - \Delta^\alpha y(t_1) \leq 0$, and hence we have, $\Delta^\alpha y(t_1+1) \leq 0$.

If $\Delta^\alpha y(t_1+1) < 0$, then by the above observation, we have

$$\Delta^\alpha y(t) < 0 \text{ for } t \geq t_1 + 1.$$

If $\Delta^\alpha y(t_1+1) = 0$, then by induction, we may conclude that

$$\Delta^\alpha y(t_1+2) \leq 0.$$

By induction again, we end up with two situations: either $\Delta^\alpha y(t)$ is eventually negative or $\Delta^\alpha y(t) = 0$ for $t \geq t_1$. However, the latter case is impossible. Indeed, since $q$ is non-negative, we have an integer $T^* > t_1$, so that $q(T^*) > 0$. Then in view of (8), we have

$$\begin{aligned} 0 &= \Delta\big(\Delta^\alpha y(T^*)\big) + p(T^*)\Delta^\alpha y(T^*) + q(T^*) f(G(T^*)) \\ &= q(T^*) f(G(T^*)) > 0 \end{aligned}$$

which is a contradiction.

If we now define $u(t) = -\Delta^\alpha y(t)$ for $t \geq t_2 \geq t_1$ such that $\Delta^\alpha y(t_2) < 0$, then from (8), we have

$$\Delta\big(-u(t)\big) - p(t)u(t) - q(t) f(G(t)) \leq 0$$

or

$$\Delta\big(u(t)\big) + p(t)u(t) \geq 0 \quad \text{for} \quad t \geq t_2.$$

Clearly, we have

$$u(t+1) \geq [1-p(t)]u(t) \quad \text{for} \quad t \geq t_2.$$

Therefore, we get

$$u(t) \geq u(t_2+1) \prod_{s=t_2+1}^{t-1} [1-p(s)]$$

or

$$\Delta^{\alpha} y(t) \leq -u(t_2+1) \prod_{s=t_2+1}^{t-1} [1-p(s)].$$

Applying Lemma 1, we obtain

$$\frac{\Delta G(t)}{\Gamma(1-\alpha)} \leq -u(t_2+1) \prod_{s=t_2+1}^{t-1} [1-p(s)]$$

or

$$\Delta G(t) \leq \Gamma(1-\alpha)\Delta^{\alpha} y(t_2+1) \prod_{s=t_2+1}^{t-1} [1-p(s)].$$

Summing the last inequality from $t_2+1$ to $t-1$, we get

$$G(t) \leq G(t_2+1) + \Gamma(1-\alpha)\Delta^{\alpha} y(t_2+1) \sum_{l=t_2+1}^{t-1} \prod_{s=t_2+1}^{l-1} [1-p(s)]$$

for $t \geq t_2+1$. Hence, by (10), we have $G(t) \to -\infty$ as $t \to \infty$, which is a contradiction to the fact that $G(t) > 0$. Therefore, $\Delta^{\alpha} y(t)$ is eventually positive. □

**Lemma 4** ([27]). *Let $\lambda_1$ be a positive real number. Then the inequality*

$$\lambda_2 z - \lambda_1 z^2 \leq \frac{\lambda_2^2}{4\lambda_1}$$

*holds for all $\lambda_2, z \in \mathbb{R}$.*

## 3. Main Results

In this section, we study the oscillatory behavior of solutions of Equation (8) under certain conditions.

**Theorem 1.** *Let $a(t) \equiv 1$ in (8) and (10) holds. If there exists a positive sequence $g$ such that*

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[ Kq(s)g(s) - \frac{[\Delta g(s) - p(s)g(s)]^2}{4\Gamma(1-\alpha)g(s)} \right] = \infty, \tag{11}$$

*then Equation (8) is oscillatory.*

**Proof.** Suppose that $y$ is a non-oscillatory solution of Equation (8). Without loss of generality, we may assume that $y$ is an eventually positive solution of (8) such that $y(t) > 0$ for all large $t$. The proof of the case when $y$ is eventually negative is similar, hence is omitted.

In view of Lemma 3 and Equation (8), we have

$$y(t) > 0, \quad \Delta^{\alpha} y(t) > 0 \quad \text{and} \quad \Delta\big(\Delta^{\alpha} y(t)\big) \leq 0 \quad \text{for} \quad t \geq t_1 \tag{12}$$

for some $t_1 \geq t_0$.

Define the transformation

$$w(t) := g(t)\frac{\Delta^{\alpha}y(t)}{G(t)} \quad \text{for} \quad t \geq t_1. \tag{13}$$

Then $w(t) > 0$ and

$$\begin{aligned}\Delta w(t) &= g(t)\Delta\left(\frac{\Delta^{\alpha}y(t)}{G(t)}\right) + \frac{\Delta^{\alpha}y(t+1)}{G(t+1)}\Delta g(t)\\ &= g(t)\left\{\frac{\Delta(\Delta^{\alpha}y(t))}{G(t)} - \frac{\Delta^{\alpha}y(t+1)\Delta G(t)}{G(t)G(t+1)}\right\} + \frac{w(t+1)}{g(t+1)}\Delta g(t).\end{aligned}$$

In view of Equation (8), Lemma 1, and using the fact that $G(t+1) > G(t)$, we have

$$\begin{aligned}\Delta w(t) &< \frac{g(t)}{G(t)}\Big[-p(t)\Delta^{\alpha}y(t) - q(t)f(G(t))\Big] - \frac{g(t)\Delta^{\alpha}y(t+1)}{G(t+1)^2}\Gamma(1-\alpha)\Delta^{\alpha}y(t)\\ &\quad + \frac{w(t+1)}{g(t+1)}\Delta g(t)\\ &= -\frac{p(t)\Delta^{\alpha}y(t)g(t)}{G(t)} - q(t)g(t)\frac{f(G(t))}{G(t)} - g(t)\Gamma(1-\alpha)\frac{\Delta^{\alpha}y(t)}{\Delta^{\alpha}y(t+1)}\frac{w^2(t+1)}{g^2(t+1)}\\ &\quad + \frac{w(t+1)}{g(t+1)}\Delta g(t).\end{aligned} \tag{14}$$

On the other hand, in view of (8) and (12), we have

$$\Delta^{\alpha}y(t) \geq \Delta^{\alpha}y(t+1). \tag{15}$$

By virtue of (15) and $(A3)$, (14) becomes

$$\begin{aligned}\Delta w(t) &\leq -\frac{p(t)g(t)\Delta^{\alpha}y(t+1)}{G(t)} - Kq(t)g(t) - \frac{g(t)\Gamma(1-\alpha)}{g^2(t+1)}w^2(t+1)\\ &\quad + \frac{w(t+1)}{g(t+1)}\Delta g(t).\end{aligned}$$

Using $G(t+1) > G(t)$ and (13), we get

$$\begin{aligned}\Delta w(t) &\leq -\frac{p(t)g(t)\Delta^{\alpha}y(t+1)}{G(t+1)} - Kq(t)g(t) - \frac{g(t)\Gamma(1-\alpha)}{g^2(t+1)}w^2(t+1) + \frac{w(t+1)}{g(t+1)}\Delta g(t)\\ &= -p(t)g(t)\frac{w(t+1)}{g(t+1)} - Kq(t)g(t) - \frac{g(t)\Gamma(1-\alpha)}{g^2(t+1)}w^2(t+1) + \frac{w(t+1)}{g(t+1)}\Delta g(t)\\ &= -Kq(t)g(t) + \left[\frac{\Delta g(t) - p(t)g(t)}{g(t+1)}\right]w(t+1) - \frac{g(t)\Gamma(1-\alpha)}{g^2(t+1)}w^2(t+1)\end{aligned} \tag{16}$$

which implies that

$$\Delta w(t) \leq -Kq(t)g(t) + \frac{[\Delta g(t) - p(t)g(t)]^2}{4\Gamma(1-\alpha)g(s)}. \tag{17}$$

Summing the above inequality from $t_1$ to $t-1$, we obtain

$$w(t_1) - w(t) = \sum_{s=t_1}^{t-1}\Delta w(s) \leq \sum_{s=t_1}^{t-1}\left[-Kq(s)g(s) + \frac{[\Delta g(s) - p(s)g(s)]^2}{4\Gamma(1-\alpha)g(s)}\right].$$

Thus, we have

$$\sum_{s=t_1}^{t-1}\left[Kq(s)g(s)-\frac{[\Delta g(s)-p(s)g(s)]^2}{4\Gamma(1-\alpha)g(s)}\right]\leq w(t_1)-w(t)<w(t_1).$$

Taking limit supremum of the both sides of the last inequality as $t\to\infty$, we get

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[Kq(s)g(s)-\frac{[\Delta g(s)-p(s)g(s)]^2}{4\Gamma(1-\alpha)g(s)}\right]\leq w(t_1)<\infty$$

which is a contradiction to our assumption (11). This completes the proof. □

**Theorem 2.** *Assume that $S(x,y)\geq\xi>0$ for $x,y\neq0$. If there exists a positive sequence $\phi(t)$ such that*

$$\sum_{s=t_0}^{\infty}\frac{1}{a(s)\phi(s)}=\infty, \tag{18}$$

$$\sum_{s=t_0}^{\infty}\phi(s+1)q(s)=\infty, \tag{19}$$

$$a(t)\Delta\phi(t)\geq p(t)\phi(t+1), \quad t\geq t_0, \tag{20}$$

$$\sum_{s=t_0}^{\infty}\frac{\phi(s+1)p^2(s)}{a(s)}<\infty \tag{21}$$

*and*

$$\sum_{s=t_0}^{\infty}\frac{a(s)[\Delta\phi(s)]^2}{\phi(s+1)}<\infty, \tag{22}$$

*then Equation (8) is oscillatory.*

**Proof.** Suppose that $y$ is a non-oscillatory solution of Equation (8). Without loss of generality, we may assume that $y$ is an eventually positive solution of (8) such that $y(t)>0$ for all large $t$. The proof of the case when $y$ is eventually negative is similar, and thus is omitted.

Define $\omega(t):=\phi(t)a(t)\Delta^{\alpha}y(t)$. Then we have

$$\Delta\omega(t)=\phi(t+1)\Delta(a(t)\Delta^{\alpha}y(t))+a(t)\Delta^{\alpha}y(t)\Delta\phi(t).$$

From Equation (8), we have

$$\frac{\Delta\omega(t)}{f(G(t+1))}=-\frac{\phi(t+1)q(t)f(G(t))}{f(G(t+1))}-\frac{\phi(t+1)p(t)\Delta^{\alpha}y(t)}{f(G(t+1))}+\frac{a(t)\Delta^{\alpha}y(t)\Delta\phi(t)}{f(G(t+1))}$$

and

$$\Delta\left(\frac{\omega(t)}{f(G(t))}\right)=\frac{\Delta\omega(t)f(G(t))-\omega(t)S(G(t+1),G(t))\Delta G(t)}{f(G(t))f(G(t+1))}$$

that is

$$\begin{aligned}\Delta\left(\frac{\omega(t)}{f(G(t))}\right)=&-\frac{\phi(t+1)q(t)f(G(t))}{f(G(t+1))}-\frac{\phi(t+1)p(t)\Delta^{\alpha}y(t)}{f(G(t+1))}+\frac{a(t)\Delta^{\alpha}y(t)\Delta\phi(t)}{f(G(t+1))}\\&-\frac{\Gamma(1-\alpha)\omega(t)S(G(t+1),G(t))\Delta^{\alpha}y(t)}{f(G(t))f(G(t+1))}.\end{aligned} \tag{23}$$

Then, summing both sides of (23) from $t_0$ to $t-1$, we obtain

$$\frac{\omega(t_0)}{f(G(t_0))} = \frac{\omega(t)}{f(G(t))} + \sum_{s=t_0}^{t-1}\left\{\frac{\phi(s+1)\,q(s)\,f(G(s))}{f(G(s+1))} + \frac{\phi(s+1)\,p(s)\,\Delta^{\alpha}y(s)}{f(G(s+1))} - \frac{a(s)\,\Delta^{\alpha}y(s)\,\Delta\phi(s)}{f(G(s+1))} + \frac{\Gamma(1-\alpha)\,\omega(s)\,S(G(s+1),G(s))\,\Delta^{\alpha}y(s)}{f(G(s))\,f(G(s+1))}\right\}$$

or

$$\frac{\phi(t_0)\,a(t_0)\,\Delta^{\alpha}y(t_0)}{f(G(t_0))} = \frac{\phi(t)\,a(t)\,\Delta^{\alpha}y(t)}{f(G(t))} + \sum_{s=t_0}^{t-1}\left\{\frac{\phi(s+1)\,q(s)\,f(G(s))}{f(G(s+1))} + \frac{\phi(s+1)\,p(s)\,\Delta^{\alpha}y(s)}{f(G(s+1))} - \frac{a(s)\,\Delta^{\alpha}y(s)\,\Delta\phi(s)}{f(G(s+1))} + \frac{\Gamma(1-\alpha)\,\phi(s)\,a(s)\,S(G(s+1),G(s))\,(\Delta^{\alpha}y(s))^2}{f(G(s))\,f(G(s+1))}\right\}. \tag{24}$$

Now, by Schwartz's inequality we have the following inequalities:

$$\left(\sum_{s=t_0}^{t-1}\frac{\phi(s+1)\,p(s)\,\Delta^{\alpha}y(s)}{f(G(s+1))}\right)^2 \le \beta_1^2\sum_{s=t_0}^{t-1}\frac{a(s)\,\phi(s+1)\,(\Delta^{\alpha}y(s))^2}{\{f(G(s+1))\}^2} \tag{25}$$

and

$$\left(\sum_{s=t_0}^{t-1}\frac{a(s)\,\Delta^{\alpha}y(s)\,\Delta\phi(s)}{f(G(s+1))}\right)^2 \le \beta_2^2\sum_{s=t_0}^{t-1}\frac{a(s)\,\phi(s+1)\,(\Delta^{\alpha}y(s))^2}{\{f(G(s+1))\}^2}, \tag{26}$$

where

$$\beta_1^2 = \sum_{s=t_0}^{t-1}\frac{\phi(s+1)\,p^2(s)}{a(s)} > 0 \quad\text{and}\quad \beta_2^2 = \sum_{s=t_0}^{t-1}\frac{a(s)\,(\Delta\phi(s))^2}{\phi(s+1)} > 0.$$

In view of the above (25) and (26) inequalities and $S(x,y) \ge \xi$, the summations in (24) are bounded. Hence (24) turns out the inequality

$$\frac{\phi(t)\,a(t)\,\Delta^{\alpha}y(t)}{f(G(t))} + \sum_{s=t_0}^{t-1}\frac{\phi(s+1)\,q(s)\,f(G(s))}{f(G(s+1))} - (\beta_1+\beta_2)\left(\sum_{s=t_0}^{t-1}\frac{a(s)\,\phi(s+1)\,(\Delta^{\alpha}y(s))^2}{\{f(G(s+1))\}^2}\right)^{1/2} + \xi\sum_{s=t_0}^{t-1}\left\{\frac{\Gamma(1-\alpha)\,\phi(s)\,a(s)\,(\Delta^{\alpha}y(s))^2}{f(G(s))\,f(G(s+1))}\right\} \le \frac{\phi(t_0)\,a(t_0)\,\Delta^{\alpha}y(t_0)}{f(G(t_0))}.$$

Then (19), (21) and (22) imply that

$$\lim_{t\to\infty}\frac{\phi(t)\,a(t)\,\Delta^{\alpha}y(t)}{f(G(t))} = -\infty$$

which leads to $\Delta^{\alpha}y(t) < 0$.

Now, we consider (24) for $T \ge t_0$, i.e.,

$$\begin{aligned}\frac{\phi(t)\,a(t)\,\Delta^{\alpha}y(t)}{f(G(t))} &+ \sum_{s=T}^{t-1}\frac{\Gamma(1-\alpha)\,\phi(s)\,a(s)\,S(G(s+1),G(s))\,(\Delta^{\alpha}y(s))^2}{f(G(s))\,f(G(s+1))}\\ &= \frac{\phi(t_0)\,a(t_0)\,\Delta^{\alpha}y(t_0)}{f(G(t_0))} - \sum_{s=t_0}^{t-1}\frac{\phi(s+1)\,q(s)\,f(G(s))}{f(G(s+1))}\\ &\quad+\sum_{s=t_0}^{T-1}\frac{\{a(s)\,\Delta\phi(s)-\phi(s+1)\,p(s)\}\,\Delta^{\alpha}y(s)}{f(G(s+1))} + \sum_{s=T}^{t-1}\frac{\{a(s)\,\Delta\phi(s)-\phi(s+1)\,p(s)\}\,\Delta^{\alpha}y(s)}{f(G(s+1))}\\ &\quad-\sum_{s=t_0}^{T-1}\frac{\Gamma(1-\alpha)\,\phi(s)\,a(s)\,S(G(s+1),G(s))\,(\Delta^{\alpha}y(s))^2}{f(G(s))\,f(G(s+1))}.\end{aligned} \tag{27}$$

Taking into account (20) and $\Delta^{\alpha} y(t) < 0$, we have

$$\frac{\phi(t)\, a(t)\, \Delta^{\alpha} y(t)}{f(G(t))} + \sum_{s=T}^{t-1} \frac{\Gamma(1-\alpha)\, \phi(s)\, a(s)\, S(G(s+1), G(s))\, (\Delta^{\alpha} y(s))^{2}}{f(G(s))\, f(G(s+1))} \leq -c$$

for some $c \geq t_1 \geq T$.

Define $u(t) := -\phi(t)\, a(t)\, \Delta^{\alpha} y(t)$. Then $u(t)$ satisfies the inequality

$$u(t) \geq c f(G(t)) + \sum_{s=T}^{t-1} \frac{\Gamma(1-\alpha)\, f(G(t))\, S(G(s+1), G(s))\, (-\Delta^{\alpha} y(s))}{f(G(s))\, f(G(s+1))} u(s).$$

Letting

$$K(t,s,x) = \frac{\Gamma(1-\alpha)\, f(G(t))\, S(G(s+1), G(s))\, (-\Delta^{\alpha} y(s))}{f(G(s))\, f(G(s+1))} x$$

and $h(t) = c f(G(t))$, we apply Lemma 2 to get

$$u(t) \geq v(t),$$

where $v(t)$ satisfies

$$v(t) = c f(G(t)) + \sum_{s=T}^{t-1} \frac{\Gamma(1-\alpha)\, f(G(t))\, S(G(s+1), G(s))\, (-\Delta^{\alpha} y(s))}{f(G(s))\, f(G(s+1))} v(s). \tag{28}$$

From (28), we obtain

$$\begin{aligned} \Delta\left[\frac{v(t)}{f(G(t))}\right] &= \Delta\left[c + \sum_{s=T}^{t-1} \frac{\Gamma(1-\alpha)\, S(G(s+1), G(s))\, (-\Delta^{\alpha} y(s))}{f(G(s))\, f(G(s+1))} v(s)\right] \\ &= \frac{\Gamma(1-\alpha)\, S(G(t+1), G(t))\, (-\Delta^{\alpha} y(t))}{f(G(t))\, f(G(t+1))} v(t). \end{aligned}$$

Namely we have that

$$\Delta\left[\frac{v(t)}{f(G(t))}\right] = \frac{\Delta v(t)}{f(G(t))} - \frac{v(t)\, \Gamma(1-\alpha)\, S(G(s+1), G(s))\, \Delta^{\alpha} y(s)}{f(G(s))\, f(G(s+1))}.$$

Hence we get $\Delta v(t) = 0$ and then $u(t) \geq v(t) = v(t_1) = c f(G(t_1))$. So

$$\Delta^{\alpha} y(t) \leq -\frac{c f(G(t_1))}{a(t)\, \phi(t)}$$

that is

$$\Delta G(t) \leq -\frac{c \Gamma(1-\alpha)\, f(G(t_1))}{a(t)\, \phi(t)}. \tag{29}$$

Summing the both sides of (29), from $t_1$ to $t-1$, we get

$$G(t) \leq -c \Gamma(1-\alpha)\, f(G(t_1)) \sum_{s=t_1}^{t-1} \frac{1}{a(s)\, \phi(s)}.$$

By (18), we have $G(t) \to -\infty$ which contradicts the $G(t) > 0$. The proof is complete. □

Let $H(t,s)$ be a positive sequence such that $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ and $\Delta_2 H(t,s) = H(t,s+1) - H(t,s) < 0$ for $t \geq s \geq t_0$.

**Theorem 3.** *Let $a(t) \equiv 1$ in (8) and (10) holds. If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} \left[ Kq(s)g(s)H(t,s) - \frac{h^2(t,s)g^2(s+1)}{4\Gamma(1-\alpha)g(s)H(t,s)} \right] = \infty, \tag{30}$$

*where*

$$h(t,s) = \Delta_2 H(t,s) + \left( \frac{\Delta g(s) - p(s)g(s)}{g(s+1)} \right) H(t,s)$$

*and g is as in Theorem 1, then Equation (8) is oscillatory.*

**Proof.** Suppose that $y$ is a non-oscillatory solution of Equation (8). Without loss of generality, we may assume that $y$ is an eventually positive solution of (8) such that $y(t) > 0$ for all large $t$. The proof of the case when $y$ is eventually negative is similar, hence is omitted.

In view of Lemma 3 and Equation (8), there exist some $t_1 \geq t_0$ such that (12) holds. Proceeding as in the proof of Theorem 1, one can reach inequality (16).

Multiplying both sides of (16) by $H(t,s)$, and then summing up with respect to $s$ from $t_1$ to $t-1$, we get

$$\sum_{s=t_1}^{t-1} Kq(s)g(s)H(t,s) \leq -\sum_{s=t_1}^{t-1} \Delta w(s)H(t,s) + \sum_{s=t_1}^{t-1} \left[ \frac{\Delta g(s) - p(s)g(s)}{g(s+1)} \right] w(s+1)H(t,s) - \sum_{s=t_1}^{t-1} \frac{g(s)\Gamma(1-\alpha)}{g^2(s+1)} w^2(s+1)H(t,s). \tag{31}$$

Using summation by parts formula, we get

$$-\sum_{s=t_1}^{t-1} \Delta w(s)H(t,s) = H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s).$$

Therefore, (31) becomes

$$\begin{aligned}\sum_{s=t_1}^{t-1} Kq(s)g(s)H(t,s) &\leq H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} \left\{ \Delta_2 H(t,s) + \left[ \frac{\Delta g(s) - p(s)g(s)}{g(s+1)} \right] H(t,s) \right\} w(s+1) \\ &\quad - \sum_{s=t_1}^{t-1} \frac{g(s)\Gamma(1-\alpha)}{g^2(s+1)} w^2(s+1)H(t,s) \\ &= H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} \left[ h(t,s)w(s+1) - \frac{g(s)\Gamma(1-\alpha)}{g^2(s+1)} w^2(s+1)H(t,s) \right].\end{aligned} \tag{32}$$

Taking $b = h(t,s)$, $X = w(t+1)$ and

$$a = \frac{g(s)\Gamma(1-\alpha)}{g^2(s+1)} H(t,s),$$

and using Lemma 4, (32) turns out

$$\sum_{s=t_1}^{t-1} Kq(s)g(s)H(t,s) \leq H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} \frac{h^2(t,s)g^2(s+1)}{4\Gamma(1-\alpha)g(s)H(t,s)}$$

which yields to the inequality

$$\sum_{s=t_1}^{t-1} \left[ Kq(s)g(s)H(t,s) - \frac{h^2(t,s)g^2(s+1)}{4\Gamma(1-\alpha)g(s)H(t,s)} \right] \leq H(t,t_1)w(t_1) < H(t,t_0)w(t_1)$$

for $t > t_1 > t_0$. Thus

$$\begin{aligned}
&\sum_{s=t_0}^{t-1}\left[Kq(s)g(s)H(t,s)-\frac{h^2(t,s)g^2(s+1)}{4\Gamma(1-\alpha)g(s)H(t,s)}\right]\\
&\quad=\sum_{s=t_0}^{t_1-1}\left[Kq(s)g(s)H(t,s)-\frac{h^2(t,s)g^2(s+1)}{4\Gamma(1-\alpha)g(s)H(t,s)}\right]\\
&\qquad+\sum_{s=t_1}^{t-1}\left[Kq(s)g(s)H(t,s)-\frac{h^2(t,s)g^2(s+1)}{4\Gamma(1-\alpha)g(s)H(t,s)}\right]\\
&\quad<\sum_{s=t_0}^{t_1-1}\left[Kq(s)g(s)H(t,s)-\frac{h^2(t,s)g^2(s+1)}{4\Gamma(1-\alpha)g(s)H(t,s)}\right]+H(t,t_0)w(t_1)\\
&\quad<\sum_{s=t_0}^{t_1-1}Kq(s)g(s)H(t,s)+H(t,t_0)w(t_1)\\
&\quad<H(t,t_0)\sum_{s=t_0}^{t_1-1}Kq(s)g(s)+H(t,t_0)w(t_1).
\end{aligned}$$

Consequently, we have

$$\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left[Kq(s)g(s)H(t,s)-\frac{h^2(t,s)g^2(s+1)}{4\Gamma(1-\alpha)g(s)H(t,s)}\right]<\sum_{s=t_0}^{t_1-1}Kq(s)g(s)+w(t_1). \tag{33}$$

Taking limit supremum of both sides of (33) as $t\to\infty$, we get

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left[Kq(s)g(s)H(t,s)-\frac{h^2(t,s)g^2(s+1)}{4\Gamma(1-\alpha)g(s)H(t,s)}\right]\le\sum_{s=t_0}^{t_1-1}Kq(s)g(s)+w(t_1)<\infty$$

which contradicts with (30). This completes the proof. □

By virtue of Theorem 3, we can deduce alternative conditions for the oscillation of all solutions of (8). This can happen by choosing different forms of the sequences $g(t)$ and $H(t,s)$. For instance, if we set $g(t)=1$ for all $t\ge t_0$ and $H(t,s)=(t-s)^\lambda,\ \lambda\ge 1,\ t\ge s\ge t_0$. then, by using the inequality

$$x^\gamma-y^\gamma\le\gamma x^{\gamma-1}(x-y);\qquad x,y\ge 0,\ \gamma\ge 1,$$

we get

$$h(t,s)\le-\left[\lambda(t-s-1)^{\lambda-1}+p(s)(t-s)^\lambda\right],$$

and hence we formulate the following result.

**Corollary 1.** *If condition (30) in Theorem 3 is replaced by*

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^\lambda}\sum_{s=t_0}^{t-1}\left\{K(t-s)^\lambda q(s)-\frac{\left[\lambda(t-s-1)^{\lambda-1}+p(s)(t-s)^\lambda\right]^2}{4\Gamma(1-\alpha)(t-s)^\lambda}\right\}=\infty, \tag{34}$$

*then Equation (8) is oscillatory.*

## 4. Examples

To confirm our theoretical results, we present herein some numerical examples.

**Example 1.** *Consider the fractional difference equation with damping term*

$$\Delta\left(\Delta^\alpha y(t)\right) + \frac{1}{t}\Delta^\alpha y(t) + t^3\left(\sum_{s=2}^{t-1+\alpha}(t-s-1)^{(-\alpha)}y(s)\right)^3 = 0, \tag{35}$$

*where* $\alpha = 1/3$ and $t \in \mathbb{N}_{8/3}$. *This corresponds to* (8) *with* $a(t) = 1$, $p(t) = t^{-1}$, $q(t) = f(t) = t^3$ *and* $f(t)/t \geq K > 0$.

*It follows that*

$$\sum_{t=2}^{\infty}\prod_{s=2}^{t-1}[1-p(s)] = \sum_{t=2}^{\infty}\frac{1}{t-1} > \sum_{t=2}^{\infty}\frac{1}{t} = \infty.$$

*If we set* $g(t) = t^2$, *then*

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left\{Kq(s)g(s) - \frac{[\Delta g(s) - p(s)g(s)]^2}{4\Gamma(1-\alpha)g(s)}\right\} = \limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[Ks^5 - \frac{(s+1)^2}{4\Gamma(2/3)s^2}\right] = \infty.$$

*Thus, by Theorem* 1, *Equation* (1) *is oscillatory.*

**Example 2.** *Consider the fractional difference equation with damping term*

$$\Delta\left(\Delta^\alpha y(t)\right) + \frac{1}{t+1}\Delta^\alpha y(t) + \sum_{s=1}^{t-1+\alpha}(t-s-1)^{(-\alpha)}y(s) = 0, \tag{36}$$

*where* $\alpha = 2/3$ and $t \in \mathbb{N}_{4/3}$. *This corresponds to* (8) *with* $a(t) = 1$, $p(t) = (t+1)^{-1}$, $q(t) = 1$, $f(t) = t$ *and* $f(t)/t = 1 = K$. *Then*

$$\sum_{t=1}^{\infty}\prod_{s=1}^{t-1}[1-p(s)] = \sum_{t=1}^{\infty}\frac{1}{t} = \infty.$$

*If we take* $H(t,s) = (t-s)^2$, *then*

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{(t-t_0)^\lambda}\sum_{s=t_0}^{t-1}\left\{K(t-s)^\lambda q(s) - \frac{\left[\lambda(t-s-1)^{\lambda-1} + p(s)(t-s)^\lambda\right]^2}{4\Gamma(1-\alpha)(t-s)^\lambda}\right\}\\
&= \limsup_{t\to\infty}\frac{1}{(t-1)^2}\sum_{s=1}^{t-1}\left\{(t-s)^2 - \frac{\left[2(t-s-1) + (s+1)^{-1}(t-s)^2\right]^2}{4\Gamma(1/3)(t-s)^2}\right\}\\
&= \infty.
\end{aligned}$$

*Thus, by Corollary* 1, *Equation* (2) *is oscillatory.*

**Example 3.** *Consider the following fractional difference equation*

$$\Delta\left(\frac{1}{t^2}\Delta^\alpha y(t)\right) + \frac{1}{t^3+t^2}\Delta^\alpha y(t) + t\sum_{s=1}^{t-1+\alpha}(t-s-1)^{(-\alpha)}y(s) = 0, \tag{37}$$

*where* $\alpha = 1/2$, $t \in \mathbb{N}_{3/2}$.*This corresponds to* (8) *with* $a(t) = t^{-2}$, $p(t) = (t^3+t^2)^{-1}$, $q(t) = t$ *and* $f(t) = t$. *Let* $\phi(t) = t$. *Then the conditions of Theorem* 2 *become*

$$\sum_{s=t_0}^{\infty}\frac{1}{a(s)\,\phi(s)} = \sum_{s=t_0}^{\infty}\frac{1}{s^{-1}} = \infty,$$

$$\sum_{s=t_0}^{\infty}\phi(s+1)\,q(s) = \sum_{s=t_0}^{\infty}(s+1)\,s = \infty,$$

$$a(t)\,\Delta\phi(t) \geq p(t)\,\phi(t+1),$$

$$\sum_{s=t_0}^{\infty} \frac{\phi(s+1)\,p^2(s)}{a(s)} = \sum_{s=t_0}^{\infty} \frac{(s+1)\left(s^3+s^2\right)^{-2}}{s^{-2}} = \sum_{s=t_0}^{\infty} \frac{s^3+s^2}{s^6+2s^5+s^4} < \infty,$$

*and*

$$\sum_{s=t_0}^{\infty} \frac{s^{-2}}{s+1} = \sum_{s=t_0}^{\infty} \frac{1}{s^3+s^2} < \infty$$

*are satisfied. Thus, Equation* (37) *is oscillatory by Theorem* 2*.*

**Remark 1.** *We claim that no result in the literature can comment on the oscillatory behavior of solutions of Equation (37).*

## 5. A Concluding Remark

This paper is devoted to establishing oscillation criteria for the solutions of a class of non-linear fractional difference equations with damping term. In particular, we employed the Riccati transformation technique, some mathematical inequalities and comparison results, to prove three oscillation theorems for the proposed equation. To examine the validity of the proposed results in this paper, we presented three numerical examples that demonstrate consistency to the theoretical results. Unlike the existing results in the literature, we claim that the new oscillation criteria not only provide an extension to previous work but also are proved under less restrictive conditions. Consequently, one can clearly observe that our approach can also be applied to study the oscillation of other types of fractional difference equations.

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