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A Solution for Volterra Fractional Integral Equations by Hybrid Contractions

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Abstract: In this manuscript, we propose a solution for Volterra type fractional integral equations by using a hybrid type contraction that unifies both nonlinear and linear type inequalities in the context of metric spaces. Besides this main goal, we also aim to combine and merge several existing fixed point theorems that were formulated by linear and nonlinear contractions.

Keywords: contraction; hybrid contractions; volterra fractional integral equations; fixed point

JEL Classification: 47H10; 54H25; 46J10

1. Introduction and Preliminaries

In the last few decades, one of the most attractive research topics in nonlinear functional analysis is to solve fractional differential and fractional integral equations that can be reduced properly to standard differential equations and integral equations, respectively. In this paper, we aim to get a proper solution for Volterra type fractional integral equations by using a hybrid type contraction. For this purpose, we first initialize the new hybrid type contractions that combine linear and nonlinear inequalities.

We first recall the auxiliary functions that we shall use effectively: Let Ψ be the set of all nondecreasing functions $\Lambda : [0, \infty) \to [0, \infty)$ in a way that

(Λ_Σ) there are $k_0 \in \mathbb{N}$ and $\delta \in (0, 1)$ and a convergent series $\sum_{i=1}^{\infty} v_i$ such that $v_i \geq 0$ and

$$\Lambda^{i+1}(t) \leq \delta \Lambda^k(t) + v_i, \tag{1}$$

for $i \geq i_0$ and $t \geq 0$.

Each $\Lambda \in \Phi$ is called a (c)-comparison function (see [1,2]).

The following lemma demonstrate the usability and power of such auxiliary functions:

Lemma 1 ([2]). *If $\Lambda \in \Phi$, then*

(i) *The series $\sum_{k=1}^{\infty} \Lambda^k(\sigma)$ is convergent for $\sigma \geq 0$.*
(ii) *$(\Lambda^n(\sigma))_{n \in \mathbb{N}}$ converges to 0 as $n \to \infty$ for $\sigma \geq 0$;*
(iii) *Λ is continuous at 0;*
(iv) *$\Lambda(\sigma) < \sigma$, for any $\sigma \in (0, \infty)$.*

All the way through the paper, a pair (X, d) presents a **complete metric space** if it is not mentioned otherwise. In addition, the letter T presents a self-mapping on (X, d).

In what follows, we shall state the definition of a new hybrid contraction:

Definition 1. *A mapping $T : (X, d) \to (X, d)$ is called a hybrid contraction of type A, if there is Λ in Φ so that*

$$d(T\Omega, T\omega) \leq \Lambda\left(\mathcal{A}_T^p(\Omega, \omega)\right), \tag{2}$$

where $p \geq 0$ and $\sigma_i \geq 0, i = 1, 2, 3, 4$, such that $\sum_{i=1}^{4} \sigma_i = 1$ and

$$\mathcal{A}_T^p(\Omega,\omega) = \begin{cases} [\sigma_1(d(\Omega,\omega))^p + \sigma_2(d(\Omega,T\Omega))^p + \sigma_3(d(\omega,T\omega))^p + \sigma_4\left(\frac{d(\omega,T\Omega)+d(\Omega,T\omega)}{2}\right)^p]^{1/p}, \\ \qquad \text{for } p > 0, \quad \Omega,\omega \in X \\ (d(\Omega,\omega))^{\sigma_1}(d(\Omega,T\Omega))^{\sigma_2}(d(\omega,T\omega))^{\sigma_3}, \\ \qquad \text{for } p = 0, \quad \Omega,\omega \in X \backslash F_T(X), \end{cases} \tag{3}$$

where $F_T(X) = \{\varrho \in X : T\varrho = \varrho\}$.

Leu us underline some particular cases from Definition 1.

1. For $p = 1$, $\sigma_4 = 0$ and $\mu_i = \kappa\sigma_i$, for $i = 1, 2, 3$, we get a contraction of Reich-Rus-Ćirić type:

$$d(T\Omega, T\omega) \leq \mu_1 d(\Omega, \omega) + \mu_2 d(\Omega, T\Omega) + \mu_3 d(\omega, T\omega),$$

for $\Omega, \omega \in X$, where $\kappa \in [0, 1)$, see [2–4].

2. In the statement above, for $\mu_i = \frac{1}{3}$, we find particular form Reich–Rus–Ćirić type contraction,

$$d(T\Omega, T\omega) \leq \frac{1}{3}\left[d(\Omega, \omega) + d(\Omega, T\Omega) + d(\omega, T\omega)\right],$$

for $\Omega, \omega \in X$.

3. If $p = 2$, and $\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{3}$, $\sigma_4 = 0$, we find the following condition,

$$d(T\Omega, T\omega) \leq \frac{\kappa}{\sqrt{3}}[d^2(\Omega, \omega) + d^2(\Omega, T\Omega) + d^2(\omega, T\omega)]^{1/2}$$

for all $\Omega, \omega \in X$, where $\kappa \in [0, 1)$.

4. If $p = 1$ and $\sigma_2 = \sigma_3 = \frac{1}{2}$, $\sigma_1 = \sigma_4 = 0$, we have a Kannan type contraction,

$$d(T\Omega, T\omega) \leq \frac{\kappa}{2}[d(\Omega, T\Omega) + d(\omega, T\omega)],$$

for all $\Omega, \omega \in X$, see [5].

5. If $p = 2$ and $\sigma_2 = \sigma_3 = \frac{1}{2}$, $\sigma_1 = \sigma_4 = 0$, we have

$$d(T\Omega, T\omega) \leq \frac{\kappa}{\sqrt{2}}[d^2(\Omega, T\Omega) + d^2(\omega, T\omega)]^{1/2}$$

for all $\Omega, \omega \in X$.

6. If $p = 0$ and $\sigma_1 = 0$, $\sigma_2 = \delta$, $\sigma_3 = 1-\delta$, $\sigma_4 = 0$, we get an interpolative contraction of Kannan type:
$$d(T\Omega, T\omega) \leq \kappa(d(\Omega, T\Omega))^{\delta}(d(\omega, T\omega))^{1-\delta},$$
for all $\Omega, \omega \in X \backslash F_T(X)$, where $\kappa \in [0,1)$, see [6].

7. If $p = 0$ and $\sigma_1 = \alpha$, $\sigma_2 = \beta$, $\sigma_3 = 1-\beta-\alpha$, $\sigma_4 = 0$ with $\alpha, \beta \in (0,1)$, then
$$d(T\Omega, T\omega) \leq \kappa(d(\Omega, \omega))^{\alpha}(d(\Omega, T\Omega))^{\beta}(d(\omega, T\omega))^{1-\beta-\alpha},$$
for all $\Omega, \omega \in X \backslash F_T(X)$. It is an interpolative contraction of Reich–Rus–Ćirić type [7] (for other related interpolate contraction type mappings, see [8–11]).

In this paper, we provide some fixed point results involving the hybrid contraction (18). At the end, we give a concrete example and we resolve a Volterra fractional type integral equation.

2. Main Results

Our essential result is

Theorem 1. *Suppose that a self-mapping T on (X, d) is a hybrid contraction of type A. Then, T possesses a fixed point ρ and, for any $\varsigma_0 \in X$, the sequence $\{T^n\varsigma_0\}$ converges to ρ if either*

($\mathcal{C}_1$) *T is continuous at ρ;*
($\mathcal{C}_2$) *or, $[\sigma_2^{1/p} + \frac{\sigma_4}{2}^{1/p}] < 1$;*
($\mathcal{C}_2$) *or, $[\sigma_3^{1/p} + \frac{\sigma_4}{2}^{1/p}] < 1$.*

Proof. We shall use the standard Picard algorithm to prove the claims in the theorem. Let $\{\varsigma_n\}$ be defined by the recursive relation $\varsigma_{n+1} = T\varsigma_n$, $n \geq 0$, by taking an arbitrary point $x \in X$ and renaming it as $x = \varsigma_0$. Hereafter, we shall assume that
$$\varsigma_n \neq \varsigma_{n+1} \Leftrightarrow d(\varsigma_n, \varsigma_{n+1}) > 0 \text{ for all } n \in \mathbb{N}_0.$$
Indeed, it is easy that the converse case is trivial and terminate the proof. More precisely, if there is n_0 so that $\varsigma_{n_0} = \varsigma_{n_0+1} = T\varsigma_{n_0}$, then ς_{n_0} turns to be a fixed point of T.

Now, we shall examine the cases $p = 0$ and $p > 0$, separately. We first consider the case $p > 0$.

On account of the given condition (18), we find
$$d(\varsigma_{n+1}, \varsigma_n) \leq \Lambda\left(\mathcal{A}_T^p(\varsigma_n, \varsigma_{n-1})\right), \tag{4}$$
where
$$\begin{aligned}\mathcal{A}_T^p(\varsigma_n, \varsigma_{n-1}) &= [\sigma_1(d(\varsigma_n, \varsigma_{n-1}))^p + \sigma_2(d(\varsigma_n, \varsigma_{n+1}))^p + \sigma_3(d(\varsigma_{n-1}, \varsigma_n))^p \\ &\quad +\sigma_4\left(\frac{d(\varsigma_{n-1},\varsigma_{n+1})+d(\varsigma_n,\varsigma_n)}{2}\right)^p\Bigg]^{1/p} \\ &= [\sigma_1(d(\varsigma_n, \varsigma_{n-1}))^p + \sigma_2(d(\varsigma_n, \varsigma_{n+1}))^p + \sigma_3(d(\varsigma_{n-1}, \varsigma_n))^p \\ &\quad +\sigma_4\left(\tfrac{}{2}[d(\varsigma_{n-1}, \varsigma_n) + d(\varsigma_n, \varsigma_{n+1})]\right)^p\Bigg]^{1/p}.\end{aligned}$$
Suppose that $d(\varsigma_n, \varsigma_{n+1}) \geq d(\varsigma_{n-1}, \varsigma_n)$. With an elementary estimation in Label (4) from the right-hand side and keeping $\sum_{i=1}^4 \sigma_i = 1$ in mind, we find that
$$d(\varsigma_{n+1}, \varsigma_n) \leq \Lambda\left(d(\varsigma_{n+1}, \varsigma_n)\sqrt[p]{\sum_{i=1}^4 \sigma_i}\right) = \Lambda\left(d(\varsigma_{n+1}, \varsigma_n)\right) < d(\varsigma_{n+1}, \varsigma_n), \tag{5}$$

a contradiction. Attendantly, we find that $d(\varsigma_n, \varsigma_{n+1}) < d(\varsigma_{n-1}, \varsigma_n)$ and further

$$d(\varsigma_{n+1}, \varsigma_n) \leq \Lambda\left(d(\varsigma_{n-1}, \varsigma_n)\right) < d(\varsigma_{n-1}, \varsigma_n). \tag{6}$$

Inductively, from the inequalities above, we deduce

$$d(\varsigma_{n+1}, \varsigma_n) \leq \Lambda^n(d(\varsigma_1, \varsigma_0)), \text{ for all } n \in \mathbb{N}. \tag{7}$$

From Label (7) and using the triangular inequality, for all $k \geq 1$, we have

$$\begin{aligned} d(\varsigma_n, \varsigma_{n+k}) &\leq d(\varsigma_n, \varsigma_{n+1}) + \ldots + d(\varsigma_{n+k-1}, \varsigma_{n+k}) \\ &\leq \sum_{r=n}^{n+k-1} \Lambda^r(d(\varsigma_1, \varsigma_0)) \\ &\leq \sum_{r=n}^{+\infty} \Lambda^r(d(\varsigma_1, \varsigma_0)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, the constructive sequence $\{\varsigma_n\}$ is Cauchy in (X, d). Taking the completeness of the metric space (X, d) into account, we conclude the existence of $\rho \in X$ such that

$$\lim_{n\to\infty} d(\varsigma_n, \rho) = 0. \tag{8}$$

Now, we shall indicate that ρ is the requested fixed point of T under the given assumptions. Suppose that $(\mathcal{C}_1)$ holds, that is, T is continuous. Then,

$$\rho = \lim_{n\to\infty} \varsigma_{n+1} = \lim_{n\to\infty} T\varsigma_n = T(\lim_{n\to\infty} \varsigma_n) = T\rho.$$

Now, we suppose that $(\mathcal{C}_2)$ holds, that is, $[\sigma_2^{1/p} + \frac{\sigma_4}{2}^{1/p}] < 1$.

$$\begin{aligned} 0 < d(T\rho, \rho) &\leq d(T\rho, \varsigma_{n+1}) + d(\varsigma_{n+1}, \rho) \\ &= d(T\rho, T\varsigma_{n+1}) + d(\varsigma_{n+1}, \rho) \\ &\leq \Lambda\left(\mathcal{A}_T^p(\rho, \varsigma_n)\right) + d(\varsigma_{n+1}, \rho), \\ &< \mathcal{A}_T^p(\rho, \varsigma_n) + d(\varsigma_{n+1}, \rho), \end{aligned} \tag{9}$$

where

$$\mathcal{A}_T^p(\rho, \varsigma_n) = \left[\sigma_1(d(\rho, \varsigma_n))^p + \sigma_2(d(\rho, T\rho))^p + \sigma_3(d(\varsigma_n, \varsigma_{n+1}))^p + \sigma_4\left(\tfrac{d(\varsigma_n, T\rho) + d(\rho, \varsigma_{n+1})}{2}\right)^p\right]^{1/p}.$$

As $n \to \infty$, we have

$$0 < d(T\rho, \rho) \leq \Delta d(T\rho, \rho),$$

where $\Delta := [\sigma_2^{1/p} + \frac{\sigma_4}{2}^{1/p}]$. Since $\Delta := [\sigma_2^{1/p} + \frac{\sigma_4}{2}^{1/p}] < 1$, which is a contradiction, that is, $T\rho = \rho$.

We skip the details of the case $(\mathcal{C}_3)$ since it is verbatim of the proof of the case $(\mathcal{C}_2)$. Indeed, the only the difference follows from the fact that $\mathcal{A}_T^p(\rho, \varsigma_n) \neq \mathcal{A}_T^p(\varsigma_n, \rho)$ since σ_2 not need to be equal to σ_3.

As a last step, we shall consider the case $p = 0$. Here, Label (18) and Label (3) become

$$d(T\Omega, T\omega) \leq \Lambda\left((d(\Omega, \omega))^{\sigma_1}(d(\Omega, T\Omega))^{\sigma_2}(d(\omega, T\omega))^{\sigma_3}[\frac{d(T\Omega, \omega) + d(\Omega, T\omega)}{2}]^{1-\sigma_1-\sigma_2-\sigma_3}\right) \tag{10}$$

for all $\Omega, \omega \in X \backslash F_T(X)$, where $\kappa \in [0,1)$ and $\sigma_1, \sigma_2, \sigma_3 \in (0,1)$. Set $\Omega = \theta_n$ and $\omega = \theta_{n-1}$ in the inequality (10), we find that

$$\begin{aligned} d(\theta_{n+1}, \theta_n) = d(T\theta_n, T\theta_{n-1}) & \leq \Lambda\Big([d(\theta_n, \theta_{n-1})]^{\sigma_1}[d(\theta_n, T\theta_n)]^{\sigma_2} \cdot [d(\theta_{n-1}, T\theta_{n-1})]^{\sigma_3} \\ & \cdot \left[\tfrac{1}{2}(d(\theta_n, \theta_n) + d(\theta_{n-1}, \theta_{n+1}))\right]^{1-\sigma_1-\sigma_2-\sigma_3}\Big) \\ & \leq \Lambda\Big([d(\theta_n, \theta_{n-1})]^{\sigma_1} \cdot [d(\theta_n, \theta_{n+1})]^{\sigma_2} \cdot [d(\theta_{n-1}, \theta_n)]^{\sigma_3} \\ & \cdot \left[\tfrac{1}{2}(d(\theta_{n-1}, \theta_n) + d(\theta_n, \theta_{n+1}))\right]^{1-\sigma_1-\sigma_2-\sigma_3}\Big). \end{aligned} \tag{11}$$

Suppose that $d(\theta_{n-1}, \theta_n) < d(\theta_n, \theta_{n+1})$ for some $n \geq 1$. Thus,

$$\frac{1}{2}(d(\theta_{n-1}, \theta_n) + d(\theta_n, \theta_{n+1})) \leq d(\theta_n, \theta_{n+1}).$$

Consequently, inequality (11) yields that

$$[d(\theta_n, \theta_{n+1})]^{\sigma_1+\sigma_3} \leq \Lambda\left([d(\theta_{n-1}, \theta_n)]^{\sigma_1+\sigma_3}\right) < [d(\theta_{n-1}, \theta_n)]^{\sigma_1+\sigma_3}. \tag{12}$$

Thus, we conclude that $d(\theta_{n-1}, \theta_n) \geq d(\theta_n, \theta_{n+1})$, which is a contradiction. Thus, we have

$$d(\theta_n, \theta_{n+1}) \leq d(\theta_{n-1}, \theta_n) \quad \text{for all } \ n \geq 1.$$

Hence, $\{d(\theta_{n-1}, \theta_n)\}$ is a non-increasing sequence with positive terms. On account of the simple observation below,

$$\frac{1}{2}(d(\theta_{n-1}, \theta_n) + d(\theta_n, \theta_{n+1})) \leq d(\theta_{n-1}, \theta_n), \quad \text{for all } \ n \geq 1$$

together with an elementary elimination, the inequality (11) implies that

$$d(\theta_n, \theta_{n+1}) \leq \Lambda(d(\theta_{n-1}, \theta_n)) < d(\theta_{n-1}, \theta_n) \tag{13}$$

for all $n \in \mathbb{N}$. Since the inequality (13) is equivalent to Label (6), by following the corresponding lines, we derive that the iterated sequence $\{\theta_n\}$ is Cauchy and converges to $\theta^* \in X$ that is, $\lim\limits_{n\to\infty} d(\theta_n, \theta^*) = 0$. Suppose that $\theta^* \neq T\theta^*$. Since $\theta_n \neq T\theta_n$ for each $n \geq 0$, by letting $x = \theta_n$ and $y = \theta^*$ in (18), we have

$$\begin{aligned} d(\theta_{n+1}, T\theta^*) = d(T\theta_n, T\theta^*) \leq \Lambda\Big(& [d(\theta_n, \theta^*)]^{\sigma_1} \cdot [d(\theta_n, T\theta_n)]^{\sigma_2} \cdot [d(\theta^*, T\theta^*)]^{\sigma_3} \\ & \cdot \left[\frac{1}{2}(d(\theta_{n+1}, T\theta^*) + d(\theta^*, T\theta_{n+1}))\right]^{1-\sigma_2-\sigma_1-\sigma_3}\Big). \end{aligned} \tag{14}$$

Letting $n \to \infty$ in the inequality (14), we get $d(\theta^*, T\theta^*) = 0$, which is a contradiction. That is, $T\theta^* = \theta^*$. $\square$

Corollary 1. *Let T be a self-mapping on (X, d). Suppose that there is $\kappa \in [0,1)$ such that*

$$d(T\Omega, T\omega) \leq \kappa \mathcal{A}_T^p(\Omega, \omega), \tag{15}$$

where $p \geq 0$. Then, there is a fixed point ρ of T if either

($\mathcal{C}_1$) T is continuous at such point ρ;

($\mathcal{C}_2$) or, $[\sigma_2^{1/p} + \frac{\sigma_4}{2}^{1/p}] < 1$;

$(\mathcal{C}_2)$ or, $[\sigma_3^{1/p} + \frac{\sigma_4}{2}^{1/p}] < 1$;

Definition 2. *A self-mapping T is called on (X, d) a hybrid contraction of type B, if there is $\Lambda \in \Phi$ such that*

$$d(T\Omega, T\omega) \leq \Lambda\left(\mathcal{W}_T^p(\Omega, \omega)\right), \tag{16}$$

where $p \geq 0$, $a = (\sigma_1, \sigma_2, \sigma_3)$, $\sigma_i \geq 0, i = 1, 2, 3$ such that $\sigma_1 + \sigma_2 + \sigma_3 = 1$ and

$$\mathcal{W}_T^p(\Omega, \omega) = \begin{cases} [\sigma_1(d(\Omega, \omega))^p + \sigma_2(d(\Omega, T\Omega))^p + \sigma_3(d(\omega, T\omega))^p]^{1/p}, & p > 0, \Omega, \omega \in X, \\ (d(\Omega, \omega))^{\sigma_1}(d(\Omega, T\Omega))^{\sigma_2}(d(\omega, T\omega))^{\sigma_3}, & p = 0, \ \Omega, \omega \in X \backslash F_T(X). \end{cases} \tag{17}$$

Notice that a hybrid contraction of type A and a hybrid contraction of type B are also called a weighted contraction of type A and type B, respectively.

As corollaries of Theorem 1, we also have the following.

Corollary 2. *Let T be a self-mapping on (X, d). Suppose that either T is a hybrid contraction of type B, or there is $\kappa \in [0, 1)$ so that*

$$d(T\Omega, T\omega) \leq \kappa \mathcal{W}_T^p(\Omega, \omega), \tag{18}$$

where $p \geq 0$. Then, there is a fixed point ρ of T if either

(i) *T is continuous at such point ρ;*
(ii) *or, $\sigma_2 < 1$;*
(iii) *or, $\sigma_3 < 1$.*

Corollary 3. *Let T be a self-mapping on (X, d). Suppose that:*

$$d(T\Omega, T\omega) \leq \kappa d^{\sigma_1}(\Omega, \omega) \cdot d^{\sigma_2}(\Omega, T\Omega) \cdot d^{\sigma_3}(\omega, T\omega), \tag{19}$$

for all $\Omega, \omega \in X \backslash F_T(X)$, where $\kappa \in [0, 1)$, $\sigma_1, \sigma_2, \sigma_3 \geq 0$ and $\sigma_1 + \sigma_2 + \sigma_3 = 1$. Then, there is a fixed point ρ of T.

Proof. Put in Corollary 2, $p = 0$ and $a = (\sigma_1, \sigma_2, \sigma_3)$. □

Remark 1. *Using Corollary 3, we get Theorem 2 in [7] (for metric spaces).*

Corollary 4. *Let T be a self-mapping on (X, d) such that*

$$d(T\Omega, T\omega) \leq \kappa \sqrt[3]{d(\Omega, \omega) \cdot d(\Omega, T\Omega) \cdot d(\omega, T\omega)}, \tag{20}$$

for all $\Omega, \omega \in X \backslash F_T(X)$, where $\kappa \in [0, 1)$. Then, there is a fixed point ρ of T.

Proof. Put in Corollary 2, $p = 0$ and $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. □

Corollary 5. *Let T be a self-mapping on (X, d) such that*

$$d(T\Omega, T\omega) \leq \frac{\kappa}{3}[d(\Omega, \omega) + d(\Omega, T\Omega) + d(\omega, T\omega)], \tag{21}$$

for all $\Omega, \omega \in X$, where $\kappa \in [0, 1)$.

Then, there is a fixed point ρ of T.

(i) *T is continuous at such point $\rho \in X$;*
(ii) *or, $b < 3$.*

Proof. Put in Corollary 2, $p = 1$ and $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
□

Corollary 6. *Let T be a self-mapping on (X, d) such that*

$$d(T\Omega, T\omega) \leq \frac{\kappa}{\sqrt{3}}[d^2(\Omega, \omega) + d^2(\Omega, T\Omega) + d^2(\omega, T\omega)]^{1/2}, \tag{22}$$

for all $\Omega, \omega \in X$, where $\kappa \in [0, 1)$, then T has a fixed point in X. The sequence $\{T^n\varsigma_0\}$ converges to ρ.

(i) *T is continuous at such point $\rho \in X$;*
(ii) *or, $b^2 < 3$.*

Proof. Put in Corollary 2, $p = 2$ and $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. □

Corollary 2 is illustrated by the following.

Example 1. *Choose $X = \{\tau_1, \tau_2, \tau_3, \tau_4\} \cup [0, \infty)$ (where τ_1, τ_2, τ_3 and τ_4 are negative reals). Take*

1. *$d(\Omega, \omega) = |\Omega - \omega|$ for $(\Omega, \omega) \in [0, \infty) \times [0, \infty)$;*
2. *$d(\Omega, \omega) = 0$ for $(\Omega, \omega) \in \{a, b, c, d\} \times [0, \infty)$ or $(\Omega, \omega) \in [0, \infty) \times \{\tau_1, \tau_2, \tau_3, \tau_4\}$;*
3. *for $(\Omega, \omega) \in \{\tau_1, \tau_2, \tau_3, \tau_4\} \times \{\tau_1, \tau_2, \tau_3, \tau_4\}$,*

$d(\Omega, \omega)$	τ_1	τ_2	τ_3	τ_4
τ_1	0	1	2	4
τ_2	1	0	1	3
τ_3	2	1	0	2
τ_4	4	3	2	0

Consider $T : \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ \tau_3 & \tau_4 & \tau_3 & \tau_4 \end{pmatrix}$ *and* $T\Omega = \frac{\Omega}{8}$ *for* $\Omega \in [0, \infty)$.

For $\Omega \in [0, \infty)$, the main theorem is satisfied straightforwardly. Thus, we examine the case $\Omega \in \{a, b, c, d\}$. Note that there is no $\kappa \in [0, 1)$ such that

$$d(T\tau_1, T\tau_2) \leq \frac{\kappa}{3}\left[d(\tau_1, \tau_2) + d(\tau_1, T\tau_1) + d(\tau_2, T\tau_2)\right],$$

namely, we have,

$$2 \leq \frac{\kappa}{3}\left[1 + 2 + 3\right].$$

Thus, Corollary 5 is not applicable.

Using (20), we have

$$d(T\tau_1, T\tau_2) \leq \kappa\sqrt[3]{d(\tau_1, \tau_2) \cdot d(\tau_1, T\tau_1) \cdot d(\tau_2, T\tau_2)},$$

i.e., $2 \leq \kappa\sqrt[3]{1 \cdot 2 \cdot 3}$, so $\kappa \geq \frac{2}{\sqrt[3]{6}} > 1$. Hence, Corollary 4 is not applicable.

Corollary 6 is applicable. In fact, for $\Omega, \omega \in X$, we have for $\kappa = \sqrt{\frac{6}{7}}$,

$$d(T\Omega, T\omega) \leq \frac{\kappa}{\sqrt{3}}[d^2(\Omega, \omega) + d^2(\Omega, T\Omega) + d^2(\omega, T\omega)]^{1/2}.$$

Here, $\{0, \tau_3, \tau_4\}$ is the set of fixed points of T.

3. Application on Volterra Fractional Integral Equations

The fractional Schrodinger equation (FSE) is known as the fundamental equation of the fractional quantum mechanics. As compared to the standard Schrodinger equation, it contains the fractional Laplacian operator instead of the usual one. This change brings profound differences in the behavior of wave function. Zhang et al. [12] investigated analytically and numerically the propagation of optical beams in the FSE with a harmonic potential. In addition, Zhang et al. [13] suggested a real physical system (the honeycomb lattice) as a possible realization of the FSE system, through utilization of the Dirac–Weyl equation, while Zhang et al. [14] investigated the dynamics of waves in the FSE with a $\mathcal{PT}$-symmetric potential. Still in fractional calculus, in this section, we study a nonlinear Volterra fractional integral equation.

Set $0 < \tau < 1$ and $J = [\sigma_0, \sigma_0 + a]$ in $\mathbb{R}$ $(a > 0)$. Denote by $X = C(J, \mathbb{R})$ the set of continuous real-valued functions on J.

Now, particularly, we cosnider the following nonlinear Volterra fractional integral equation (in short, VFIE)

$$\xi(t) = \mathcal{F}(t) + \frac{1}{\Gamma(\tau)} \int_{\sigma_0}^{t} (t-s)^{\tau-1} h(s, \xi(s)) ds, \tag{23}$$

for all $t \in J$, where Γ is the gamma function, $\mathcal{F} : J \to \mathbb{R}$ and $h : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The VFIE (23) has been investigated in the literature on fractional calculus and its applications, see [15–17].

In the following result, under some assumptions, we ensure the existence of a solution for the VFIE (23).

Theorem 2. *Suppose that*

(H1) *There are constants $M > 0$ and $N > 0$ such that*

$$|h(t, u) - h(t, v)| \leq \frac{M|u - v|}{N + |u - v|} \tag{24}$$

for all $u, v \in \mathbb{R}$;

(H2) *Such M and N verify that*

$$\frac{Ma}{\Gamma(\tau + 1)} \leq N. \tag{25}$$

Then, the VFIE (23) has a solution in X.

Proof. For $\xi, \eta \in X$, consider the metric

$$d(\xi, \eta) = \sup_{t \in J} |\xi(t) - \eta(t)|.$$

Take the operator

$$T\xi(t) = \mathcal{F}(t) + \frac{1}{\Gamma(\tau)} \int_{\sigma_0}^{t} (t-s)^{\tau-1} h(s, \xi(s)) ds, \quad t \in J. \tag{26}$$

□

Clearly, T is well defined. Let $\xi, \eta \in X$, then for each $t \in J$,

$$\begin{aligned}|T\xi(t) - T\eta(t)| &= \frac{1}{\Gamma(\tau)}\int_{\sigma_0}^{t}(t-s)^{\tau-1}(h(s,\xi(s)) - h(s,\eta(s)))ds \\ &\le \frac{1}{\Gamma(\tau)}\int_{\sigma_0}^{t}(t-s)^{\tau-1}|h(s,\xi(s)) - h(s,\eta(s))|ds \\ &\le \frac{Ma}{\Gamma(\tau+1)}\frac{M|\xi(s)-\eta(s)|}{N+|\xi(s)-\eta(s))\|} \\ &\le \frac{Ma}{\Gamma(\tau+1)}\frac{M\|\xi-\eta\|}{N+\|\xi-\eta)\|}.\end{aligned}$$

We deduce that

$$\|T\xi - T\eta\| \le \frac{Ma}{\Gamma(\tau+1)}\frac{M\|\xi-\eta\|}{N+\|\xi-\eta)\|} = \Lambda(\|\xi-\eta)\|), \tag{27}$$

where $\Lambda(t) = \frac{La}{\Gamma(\tau+1)}\frac{Mt}{N+t}$ for $t \ge 0$. By hypothesis $(H2)$, $\Lambda \in \Phi$. Then,

$$d(T\xi, T\eta) \le \Lambda\left(\mathcal{F}_T^p(\xi,\eta)\right), \tag{28}$$

for $p > 0$, with $\sigma_2 = \sigma_2 = \sigma_4 = 0$ and $\sigma_1 = 1$. Applying Theorem 1, T has a fixed point in X, so the VFIE (23) has a solution in X.

4. Conclusions

The obtained results unify several existing results in a single theorem. We list some of the consequences, but it is clear that there are more consequences of our main results. Regarding the length of the paper, we skip them.

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