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Article

q-Analogue of Differential Subordinations

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Abstract: In this article, we study differential subordnations in q-analogue. Some properties of analytic functions in q-analogue associated with cardioid domain and limacon domain are considered. In particular, we determine conditions on α such that $1+\alpha\frac{z\partial_q h(z)}{(h(z))^n}$ $(n=0,1,2,3)$ are subordinated by Janowski functions and $h(z) \prec 1+\frac{4}{3}z+\frac{2}{3}z^2$. We also consider the same implications such that $h(z) \prec 1+\sqrt{2}z+\frac{1}{2}z^2$. We apply these results on analytic functions to find sufficient conditions for q-starlikeness related with cardioid and limacon.

Keywords: q-calculus; differential subordination; Janowski function; cardioid domain; limacon domain

MSC: 30C45; 30C50

1. Introduction

We recall here some basic notions from the literature of Geometric Function Theory which are essential for clarity and understandings of the upcoming work. We start with the symbol $\mathcal{A}$ which represents the family of analytic functions in $\mathcal{D}=\{z:|z|<1\}$ and any function f in $\mathcal{A}$ satisfies the conditions $f(0)=0$ and $f'(0)-1=0$. That is, if f in $\mathcal{A}$, then it has the Taylor series expansion as:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n,\ \ z\in\mathcal{D}. \tag{1}$$

Also let $\mathcal{S}$ denote a subclass of $\mathcal{A}$ which contains univalent functions in $\mathcal{D}$. The notion of subordinations between analytic functions is represented by $f \prec g$ and is defined as; a function f is subordinated by function g, if we can find an analytic function w with the properties $w(0)=0$ and $|w(z)|<|z|$ such that $f(z)=g(w(z))$. Further, if g is univalent in $\mathcal{D}$, then we have:

$$f(z)\prec g(z) \Leftrightarrow f(0)=g(0)\ \ \&\ \ f(\mathcal{D})\subset g(\mathcal{D}). \tag{2}$$

Ma and Minda [1] studied the function φ which is analytic and normalized by $\varphi(0)=1$ and $\varphi'(0)>0$ with $Re\{\varphi(z)\}>0$ in $\mathcal{D}$. The function φ maps $\mathcal{D}$ onto regions which is starlike with respect to 1 and symmetric along the real axis. Further, they introduced the subclasses of starlike and convex functions respectively as

$$\begin{aligned}\mathcal{S}^*(\varphi) &= \left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\varphi(z),\ (z\in\mathcal{D})\right\},\\ \mathcal{C}(\varphi) &= \left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\varphi(z),\ \ (z\in\mathcal{D})\right\}.\end{aligned}$$

If we choose $\varphi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, then $\mathcal{S}^*[A,B] := \mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right)$ is the family of Janowski starlike functions, see [2]. Further by taking $A = 1 - 2\alpha$ with $0 \leq \alpha < 1$ and $B = -1$, we get the class $\mathcal{S}^*(\alpha) := \mathcal{S}^*[1-2\alpha, -1]$ of starlike functions of order α. Also, the notation $\mathcal{S}^* := \mathcal{S}^*(0)$ represents the familiar class of starlike functions. The subclass $\mathcal{S}^*_{\mathcal{B}} := \mathcal{S}^*(\sqrt{1+z})$ which motivates the researchers was investigated by Sokół et al. [3] , containing functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the Bernoulli lemniscate given by $|w^2 - 1| < 1$. If we choose $\varphi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, then the class $\mathcal{S}^*(\varphi)$ coincides with the class $\mathcal{S}^*_{\mathcal{C}}$ studied by Sharma et al. [4], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the cardioid given by $(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0$. If we choose $\varphi(z) = 1 + \sin z$, then we get the set $\mathcal{S}^*_{\sin}$, established by Cho et al. [5]. By selecting $\varphi(z) = 1 + \sqrt{2}z + \frac{1}{2}z^2$, we acheive an interesting class $\mathcal{S}^*_{\mathcal{L}}$ containing starlike functions associated with limacon given by $(4x^2 + 4y^2 - 8x - 5)^2 - 8(4x^2 + 4y^2 - 12x - 3) = 0$. The class $\mathcal{S}^*_{\mathcal{L}}$ was introduced in [6]. Further, by choosing some more particular function $\varphi(z)$, we get several interesting subclasses of starlike functions. For some details, see [7–10].

In some recent years, a more intensive approach has been shown by the researchers in quantum calculus (q-calculus) because of its wide spread applications in various branches of sciences particularly in Mathematics and Physics. Among contributors to the study, Jackson was the first who provided basic notions and established the theory of q-calculus [11,12]. The idea of derivative in q-analogue was used for the first time by Ismail et al. [13] to initiate and study the geometry of q-starlike functions. After that, a comprenhensive applications of q-calculus in the field of Geometric Function Theory was contributed by Srivastava in a book chapter (see, for details, [14] (pp. 347 *et seq.*)) and in the same chapter he also given the usage of q-hypergeometric functions in function theory. The concepts of q-starlikeness was further extended to certain subclasses of starlike functions in q-analogue by Agrawal and Sahoo in [15] (for the recent contributions on this topic, see the work done by Srivastava et al. [16–19]). Also, with help of Hadamard product, the q-analogue of Ruscheweyh operator has been introduced by Kanas and Răducanu [20] and further studied in [21–24]. Many researchers contributed in the development of the theory by introducing certain classes with the help of q-calculus. For some details about these contributions, see [25–28].

Let $q \in (0,1)$ and $z \in \mathcal{D}$ with $z \neq 0$. Then the q-derivative of f is defined by

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}. \tag{3}$$

By the virtue of (1) and (3) , we easily calculated that for $n \in \mathbb{N}$ and $z \in \mathcal{D}$

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n,q]\, a_n z^{n-1}, \tag{4}$$

where

$$[n,q] := \frac{1-q^n}{1-q} = 1 + \sum_{k=1}^{n-1} q^l, \text{ and } [0,q] = 0.$$

Using the definition of q-derivative, Seoudy and Aouf [29] introduced the class $\mathcal{S}^*_q(\varphi)$. Also, for this class, the familiar Fekete-Szegö problem was obtained by the authors. This class is defined as;

$$\mathcal{S}^*_q(\varphi) = \left\{ f \in \mathcal{A} : \frac{z\partial_q f(z)}{f(z)} \prec \varphi(z),\ (z \in \mathcal{D}) \right\}.$$

By choosing particular functions instead of the function φ, we obtain several interesting subclasses of starlike functions associated with different image domains. We define few of them as follows.

$$\begin{aligned}
\mathcal{S}^*_{\mathcal{B}_q} &= \left\{ f \in \mathcal{A} : \frac{z\partial_q f(z)}{f(z)} \prec \phi_{\mathcal{B}}(z), \ (z \in \mathcal{D}) \right\}, \\
\mathcal{S}^*_{\mathcal{C}_q} &= \left\{ f \in \mathcal{A} : \frac{z\partial_q f(z)}{f(z)} \prec \phi_{\mathcal{C}}(z), \ (z \in \mathcal{D}) \right\}, \\
\mathcal{S}^*_{\mathcal{L}_q} &= \left\{ f \in \mathcal{A} : \frac{z\partial_q f(z)}{f(z)} \prec \phi_{\mathcal{L}}(z), \ (z \in \mathcal{D}) \right\},
\end{aligned}$$

where the particular functions $\phi_{\mathcal{B}}(z)$, $\phi_{\mathcal{C}}(z)$ and $\phi_{\mathcal{L}}(z)$ are given by

$$\begin{aligned}
\phi_{\mathcal{B}}(z) &= \sqrt{1+z}, \\
\phi_{\mathcal{C}}(z) &= 1 + \frac{4}{3}z + \frac{2}{3}z^2, \\
\phi_{\mathcal{L}}(z) &= 1 + \sqrt{2}z + \frac{1}{2}z^2.
\end{aligned}$$

Recently, Ali et al. [30] have studied some differential subordinations. More precisely they studied the differental subordination $1 + \alpha z p'(z)/p^n(z) \prec \sqrt{1+z}$ and found that $p(z) \prec \sqrt{1+z}$, where $n = 0, 1, 2$ for some particular range of α. Similar kind of differential subordinations are also discussed by various authors. They used these results to find sufficient conditions for starlike functions, see [31–36]. Motivated by the above work, we introduce and investigate some q-differential subordinations. In particular, we determine conditions on α so that $1 + \alpha \frac{z\partial_q h(z)}{(h(z))^n}$ are subordinated by Janowski functions and $h(z)$ is subordinated by $1 + \frac{4}{3}z + \frac{2}{3}z^2$, where $n = 0, 1, 2, 3$. Similar results are also obtained for $h(z) \prec 1 + \sqrt{2}z + \frac{1}{2}z^2$, $z \in \mathcal{D}$. We use these results to find sufficient conditions for q-starlike functions associated with cardioid and limacon.

To prove our main results we need the following.

Lemma 1 ([37] (q-Jack's Lemma))**.** *Let w be analytic in $\mathcal{D}$ with $w(0) = 0$. If w attains its maximum value on the circle $|z| = 1$ at $z_0 = re^{i\theta}$, for $\theta \in [-\pi, \pi]$, then for $0 < q < 1$*

$$z_0 \partial_q w(z_0) = m w(z_0),$$

where m is real and $m \geq 1$.

2. Differential Subordination Related with Cardioid

Theorem 1. *Assume that*

$$|\alpha| \geq \frac{3(A-B)}{2(1-q)(1-|B|)}, \ -1 < B < A \leq 1 \tag{5}$$

and h is an analytic function defined on $\mathcal{D}$ with $h(0) = 1$ satisfying

$$1 + \alpha z \partial_q h(z) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathcal{D}. \tag{6}$$

In addition, we suppose that

$$1 + \alpha z \partial_q h(z) = \frac{1+Aw(z)}{1+Bw(z)}, \quad z \in \mathcal{D},$$

where w is an analytic in $\mathcal{D}$ with $w(0) = 0$. Then

$$h(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad z \in \mathcal{D}.$$

Proof. We define a function

$$p(z) = 1 + \alpha z \partial_q h(z), \tag{7}$$

where p is analytic and $p(0)=1$. Consider

$$h(z)=1+\frac{4}{3}w(z)+\frac{2}{3}w^{2}(z). \tag{8}$$

To prove our result, it will be enough to show that $|w(z)|<1$. Using (7) and (8), we obtain

$$p(z)=1+\frac{\alpha}{3}z\partial_q w(z)\left\{4(1+w(z))-2(1-q)z\partial_q w(z)\right\}.$$

Also

$$\begin{aligned}
&\left|\frac{p(z)-1}{A-Bp(z)}\right|= \\
&\left|\frac{1+\frac{\alpha}{3}z\partial_q w(z)\left\{4+4w(z)-2(1-q)z\partial_q w(z)\right\}-1}{A-B\left[1+\frac{\alpha}{3}z\partial_q w(z)\left\{4+4w(z)-2(1-q)z\partial_q w(z)\right\}\right]}\right| \\
&=\left|\frac{\alpha z\partial_q w(z)\left\{4+4w(z)-2(1-q)z\partial_q w(z)\right\}}{3(A-B)+\alpha Bz\partial_q w(z)\left[4+4w(z)-2(1-q)z\partial_q w(z)\right]}\right|.
\end{aligned}$$

Suppose that there exists a point $z_0\in\mathcal{D}$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1.$$

Then by using Lemma 1, there exists a number $m\ge 1$ such that $z_0\partial_q w(z_0)=mw(z_0)$. Suppose $w(z_0)=e^{i\theta}$, $\theta\in[-\pi,\pi]$. Then for $z_0\in\mathcal{D}$, we have

$$\begin{aligned}
\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| &=\left|\frac{\alpha mw(z_0)\left\{4+4w(z_0)-2(1-q)mw(z_0)\right\}}{3(A-B)+\alpha Bmw(z_0)\left[4+4w(z_0)-2(1-q)mw(z_0)\right]}\right| \\
&\ge\frac{m|\alpha|\left|4+4e^{i\theta}-2(1-q)me^{i\theta}\right|}{3(A-B)+\alpha m|B|\left|4+4e^{i\theta}-2(1-q)me^{i\theta}\right|} \\
&=\frac{m|\alpha|\sqrt{16+(4-2(1-q)m)^{2}+8(4-2(1-q)m)\cos\theta}}{3(A-B)+m|\alpha||B|\sqrt{16+(4-2(1-q)m)^{2}+8(4-2(1-q)m)\cos\theta}}.
\end{aligned}$$

Consider the function

$$\Psi(\theta)=\frac{m|\alpha|\sqrt{16+(4-2(1-q)m)^{2}+8(4-2(1-q)m)\cos\theta}}{3(A-B)+m|\alpha||B|\sqrt{16+(4-2(1-q)m)^{2}+8(4-2(1-q)m)\cos\theta}},$$

for $\theta\in[-\pi,\pi]$. It is clear that Ψ is an even function, therefore we find the minimum value of Ψ when $\theta\in[0,\pi]$. Now

$$\Psi'(\theta)=\frac{-12|\alpha|m(A-B)\left\{4-2(1-q)m\right\}\sin\theta}{\begin{array}{c}\left(\sqrt{16+(4-2(1-q)m)^{2}+8(4-2(1-q)m)\cos\theta}\right)\\ \left(3(A-B)+|\alpha||B|\sqrt{16+(4-2(1-q)m)^{2}+8(4-2(1-q)m)\cos\theta}\right)^{2}\end{array}}.$$

It is easy to see that $\Psi'(\theta)=0$ when $\theta=0,\ \pi$. Similarly, we can see that $\Psi''(\theta)>0$, when $\theta=\pi$. Hence $\Psi(\theta)\ge\Psi(\pi)$. Consider the function

$$\begin{aligned}\Phi(m) &= \frac{|\alpha|\, m\sqrt{16+(4-2(1-q)\,m)^2-8(4-2(1-q)\,m)}}{3(A-B)+|\alpha|\,|B|\sqrt{16+(4-2(1-q)\,m)^2-8(4-2(1-q)\,m)}}\\ &= \frac{2|\alpha|(1-q)\,m^2}{3(A-B)+2|\alpha|\,|B|(1-q)\,m^2}\end{aligned}$$

Then

$$\Phi'(m)=\frac{12|\alpha|(1-q)(A-B)}{\{3(A-B)+2m^2|\alpha|\,|B|(1-q)\}^2}>0.$$

which shows that Φ is an increasing function and it has its minimum value at $m=1$, so

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq \frac{2|\alpha|(1-q)}{3(A-B)+2|\alpha||B|(1-q)}.$$

Now by (5), we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right|\geq 1,$$

which contradicts (6). Hence $|w(z)|<1$ and so we get the desired result. □

If we put $p(z)=\frac{z\partial_q f(z)}{f(z)}$ in Theorem 1, we get the following result.

Corollary 1. *Let* $|\alpha|\geq\frac{3(A-B)}{2(1-q)(1-|B|)}$, $-1<B<A\leq 1$ *and* $f\in\mathcal{A}$ *satisfy the subordination*

$$1+\alpha z\partial_q\left(\frac{z\partial_q f(z)}{f(z)}\right)\prec\frac{1+Az}{1+Bz},\quad z\in\mathcal{D}. \tag{9}$$

Then $f\in\mathcal{S}^*_{\mathcal{C}_q}$

If we choose $A=1$ and $B=0$ in Corollary 1, then we obtain the following result.

Corollary 2. *Let* $|\alpha|\geq\frac{3}{(1-q)}$, $-1<B<A\leq 1$*and* $f\in\mathcal{A}$ *satisfy the subordination*

$$1+\alpha z\partial_q\left(\frac{z\partial_q f(z)}{f(z)}\right)\prec 1+z,\quad z\in\mathcal{D}. \tag{10}$$

Then $f\in\mathcal{S}^*_{\mathcal{C}_q}$.

Theorem 2. *Assume that*

$$|\alpha|\geq\frac{(A-B)}{2(1-q)(1-|B|)},\ -1<B<A\leq 1 \tag{11}$$

and h is an analytic function defined on $\mathcal{D}$ *with* $h(0)=1$ *satisfying*

$$1+\alpha\frac{z\partial_q h(z)}{h(z)}\prec\frac{1+Az}{1+Bz},\quad z\in\mathcal{D}. \tag{12}$$

In addition, we suppose that

$$1+\alpha\frac{z\partial_q h(z)}{h(z)}=\frac{1+Aw(z)}{1+Bw(z)},\quad z\in\mathcal{D},$$

where w is an analytic in $\mathcal{D}$ *with* $w(0)=0$*. Then*

$$h(z)\prec 1+\frac{4}{3}z+\frac{2}{3}z^2,\quad z\in\mathcal{D}.$$

Proof. We define a function

$$p(z) = 1 + \alpha \frac{z\partial_q h(z)}{h(z)}, \tag{13}$$

where p is analytic and $p(0) = 1$. Consider

$$h(z) = 1 + \frac{4}{3} w(z) + \frac{2}{3} w^2(z). \tag{14}$$

Using (13) and (14), we obtain

$$p(z) = 1 + \frac{\alpha z \partial_q w(z) \left\{4 + 4w(z) - 2(1-q) z\partial_q w(z)\right\}}{3\left(1 + \frac{4}{3}w(z) + \frac{2}{3}w^2(z)\right)}.$$

Therefore

$$\left| \frac{p(z) - 1}{A - Bp(z)} \right| =$$

$$\left| \frac{\alpha z\partial_q w(z) \left\{4 + 4w(z) - 2(1-q) z\partial_q w(z)\right\}}{\left(3 + 4w(z) + 2(w(z))^2\right)(A - B) + \alpha B \left[\left(4 + 4w(z) - 2(1-q) z\partial_q w(z)\right) z\partial_q w(z)\right]} \right|.$$

Hence by applying Lemma 1, we conclude that

$$\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq \frac{2|\alpha|(1-q)}{(A-B) + 2|\alpha B|(1-q)}.$$

By using (11) , we have

$$\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq 1,$$

which contradicts (12). Hence we get the desired result. □

If we put $p(z) = \frac{z\partial_q f(z)}{f(z)}$ in the above theorem, we get the following result.

Corollary 3. *Let* $|\alpha| \geq \frac{(A-B)}{2(1-q)(1-|B|)}$, $-1 < B < A \leq 1$*and* $f \in \mathcal{A}$ *such that*

$$1 + \alpha z \left(\frac{f(z)}{z\partial_q f(z)} \right) \partial_q \left(\frac{z\partial_q f(z)}{f(z)} \right) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{D}. \tag{15}$$

Then $f \in \mathcal{S}^*_{\mathcal{C}_q}$.

Theorem 3. *Assume that*

$$|\alpha| \geq \frac{(A-B)}{6(1-q)(1-|B|)}, \; -1 < B < A \leq 1 \tag{16}$$

and h *is an analytic function defined on* $\mathcal{D}$ *with* $h(0) = 1$ *satisfying*

$$1 + \alpha \frac{z\partial_q h(z)}{h^2(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{D}. \tag{17}$$

In addition, we suppose that

$$1 + \alpha \frac{z\partial_q h(z)}{h^2(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad z \in \mathcal{D},$$

where w is an analytic in $\mathcal{D}$ with $w(0) = 0$. Then

$$h(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad z \in \mathcal{D}.$$

Proof. We define a function

$$p(z) = 1 + \alpha \frac{z\partial_q h(z)}{(h(z))^2}, \tag{18}$$

where p is analytic and $p(0) = 1$. Consider

$$h(z) = 1 + \frac{4}{3}w(z) + \frac{2}{3}w^2(z)$$

After some simple calculations, we obtain

$$p(z) = 1 + \frac{3\alpha z\partial_q w(z)\left\{4 + 4w(z) - 2(1-q)z\partial_q w(z)\right\}}{(3 + 4w(z) + 2w^2(z))^2}.$$

Therefore

$$\left|\frac{p(z) - 1}{A - Bp(z)}\right| =$$

$$\left|\frac{3\alpha z\partial_q w(z)\left\{4 + 4w(z) - 2(1-q)z\partial_q w(z)\right\}}{(3 + 4w(z) + 2w^2(z))^2(A - B) - 3\alpha Bz\partial_q w(z)\left[(4 + 4w(z) - 2(1-q)z\partial_q w(z))\right]}\right|.$$

Hence by applying Lemma 1, we conclude that

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq \frac{6|\alpha|(1-q)}{(A-B)+6|\alpha B|(1-q)}.$$

Now by using (16), we have

$$\left|\frac{p(z_0) - 1}{A - Bp(z_0)}\right| \geq 1,$$

which contradicts (17). Hence we get the required result. □

If we put $p(z) = \frac{z\partial_q f(z)}{f(z)}$ in the above theorem, we get the following result.

Corollary 4. *Let $|\alpha| \geq \frac{(A-B)}{6(1-q)(1-|B|)}$, $-1 < B < A \leq 1$and $f \in \mathcal{A}$ satisfy*

$$1 + \alpha z\left(\frac{f(z)}{z\partial_q f(z)}\right)^2 \partial_q\left(\frac{z\partial_q f(z)}{f(z)}\right) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{D}. \tag{19}$$

Then $f \in \mathcal{S}^*_{\mathcal{C}_q}$.

Theorem 4. *Assume that*

$$|\alpha| \geq \frac{(A - B)}{18(1-q)(1 - |B|)}, \; -1 < B < A \leq 1 \tag{20}$$

and h is an analytic function defined on $\mathcal{D}$ with $h(0) = 1$ satisfying

$$1 + \alpha \frac{z\partial_q h(z)}{h^3(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{D}. \tag{21}$$

In addition, we suppose that

$$1+\alpha\frac{z\partial_q h(z)}{h^3(z)}=\frac{1+Aw(z)}{1+Bw(z)},\quad z\in\mathcal{D},$$

where w is an analytic in $\mathcal{U}$ with $w(0)=0$. Then

$$h(z)\prec 1+\frac{4}{3}z+\frac{2}{3}z^2,\quad z\in\mathcal{D}.$$

Proof. Here we define a function

$$p(z)=1+\alpha\frac{z\partial_q h(z)}{(h(z))^3},\tag{22}$$

where p is analytic and $p(0)=1$. Consider

$$h(z)=1+\tfrac{4}{3}w(z)+\tfrac{2}{3}w^2(z).$$

After some simplifications, we obtain

$$p(z)=1+\frac{\alpha z\partial_q w(z)\left\{4+4w(z)-2(1-q)z\partial_q w(z)\right\}}{3\left(1+\frac{4}{3}w(z)+\frac{2}{3}w^2(z)\right)^3}.$$

Therefore

$$\left|\frac{p(z)-1}{A-Bp(z)}\right|=$$

$$\left|\frac{9\alpha z\partial_q w(z)\left\{4+4w(z)-2(1-)z\partial_q w(z)\right\}}{\left(3+4w(z)+2(w(z))^2\right)^3(A-B)+9\alpha Bz\partial_q w(z)\left[4+4w(z)-2(1-q)z\partial_q w(z)\right]}\right|.$$

Hence by Lemma 1, we conclude that

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right|\geq\frac{18|\alpha|(1-q)}{(A-B)+18|\alpha B|(1-q)}.$$

Now by using (20) , we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right|\geq 1,$$

which is contradiction. We complete the required proof. □

If we put $p(z)=\frac{z\partial_q f(z)}{f(z)}$ in the above theorem, we get the following result.

Corollary 5. *Let $|\alpha|\geq\frac{(A-B)}{18(1-q)(1-|B|)}$, $-1<B<A\leq 1$ and $f\in\mathcal{A}$ such that*

$$1+\alpha z\left(\frac{f(z)}{z\partial_q f(z)}\right)^3\partial_q\left(\frac{z\partial_q f(z)}{f(z)}\right)\prec\frac{1+Az}{1+Bz},\quad z\in\mathcal{D}.\tag{23}$$

Then $f\in\mathcal{S}^*_{\mathcal{C}_q}$.

Theorem 5. *Assume that*

$$|\alpha|\geq\frac{(A-B)}{2.3^{n-1}(1-q)(1-|B|)},\ -1<B<A\leq 1\tag{24}$$

and h is an analytic function defined on $\mathcal{D}$ with $h(0) = 1$ satisfying

$$1 + \alpha \frac{z\partial_q h(z)}{(h(z))^n} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{D}. \tag{25}$$

In addition, we suppose that

$$1 + \alpha \frac{z\partial_q h(z)}{(h(z))^n} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad z \in \mathcal{D},$$

where w is an analytic in $\mathcal{D}$ with $w(0) = 0$. Then

$$h(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad z \in \mathcal{D}.$$

Proof. The proof of Theorem 5 is similar to the theorems proved above and so here we choose to omit the details. □

If we put $p(z) = \frac{z\partial_q f(z)}{f(z)}$ in the above theorem, we get the following result.

Corollary 6. *Let $|\alpha| \geq \frac{(A-B)}{2.3^{n-1}(1-q)(1-|B|)}$, $-1 < B < A \leq 1$and $f \in \mathcal{A}$ such that*

$$1 + \alpha z \left(\frac{f(z)}{z\partial_q f(z)}\right)^n \partial_q \left(\frac{z\partial_q f(z)}{f(z)}\right) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{D}. \tag{26}$$

Then $f \in \mathcal{S}^*_{\mathcal{C}_q}$.

3. Differential Subordination Related with Limacon

Theorem 6. *Assume*

$$|\alpha| \geq \frac{2(A-B)}{\sqrt{8 + (1+q)\left(1 + q - 4\sqrt{2}\right)}(1 - |B|)}, \ -1 < B < A \leq 1 \tag{27}$$

and h is an analytic function defined on $\mathcal{D}$ with $h(0) = 1$ satisfying

$$1 + \alpha\partial_q h(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{D}. \tag{28}$$

In addition, we suppose that

$$1 + \alpha\partial_q h(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad z \in \mathcal{D}$$

with $w(0) = 0$, then

$$h(z) \prec 1 + \sqrt{2}z + \frac{1}{2}z^2, \quad z \in \mathcal{D}.$$

Proof. We define a function

$$p(z) = 1 + \alpha z\partial_q h(z), \tag{29}$$

where p is analytic and $p(0) = 1$. Consider

$$h(z) = 1 + \sqrt{2}w(z) + \frac{1}{2}w^2(z). \tag{30}$$

To prove our result, it will be enough to show that $|w(z)| < 1$. Using (29) and (30), we obtain

$$p(z) = 1 + \frac{\alpha z \partial_q w(z)}{2}\left\{2\sqrt{2} + 2w(z) - (1-q)\, z\partial_q w(z)\right\}.$$

Also

$$\left|\frac{p(z)-1}{A - Bp(z)}\right| = \left|\frac{1 + \frac{\alpha z\partial_q w(z)}{2}\left\{2\sqrt{2} + 2w(z) - (1-q)\, z\partial_q w(z)\right\} - 1}{A - B\left[1 + \frac{\alpha z\partial_q w(z)}{2}\left\{2\sqrt{2} + 2w(z) - (1-q)\, z\partial_q w(z)\right\}\right]}\right|$$
$$= \left|\frac{\alpha z\partial_q w(z)\left\{2\sqrt{2} + 2w(z) - (1-q)\, z\partial_q w(z)\right\}}{2(A-B) + \alpha B z\partial_q w(z)\left[2\sqrt{2} + 2w(z) - (1-q)\, z\partial_q w(z)\right]}\right|.$$

Suppose that there exists a point $z_0 \in \mathcal{D}$ such that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1.$$

Then by using Lemma 1, there exists a number $m \ge 1$ such that $z_0\partial_q w(z_0) = mw(z_0)$. Suppose $w(z_0) = e^{i\theta}$, $\theta \in [-\pi, \pi]$. Then for $z_0 \in \mathcal{D}$, we have

$$\left|\frac{p(z_0)-1}{A - Bp(z_0)}\right| = \left|\frac{\alpha m w(z_0)\left\{2\sqrt{2} + 2w(z_0) - (1-q)\, m w(z_0)\right\}}{2(A-B) + \alpha B m w(z_0)\left[2\sqrt{2} + 2w(z_0) - (1-q)\, m w(z_0)\right]}\right|$$
$$\ge \frac{m|\alpha|\left|2\sqrt{2} + 2e^{i\theta} - (1-q)\, m e^{i\theta}\right|}{2(A-B) + m|\alpha B|\left|2\sqrt{2} + 2e^{i\theta} - (1-q)\, m e^{i\theta}\right|}$$
$$= \frac{m|\alpha|\sqrt{8 + (2-(1-q)\,m)^2 + 4\sqrt{2}\left((2-(1-q)\,m)\right)\cos\theta}}{2(A-B) + m|\alpha|B\sqrt{8 + (2-(1-q)\,m)^2 + 4\sqrt{2}\left((2-(1-q)\,m)\right)\cos\theta}}.$$

Consider the function

$$\Theta_1(\theta) = \frac{m|\alpha|\sqrt{8 + (2-(1-q)\,m)^2 + 4\sqrt{2}\left((2-(1-q)\,m)\right)\cos\theta}}{2(A-B) + m|\alpha B|\sqrt{8 + (2-(1-q)\,m)^2 + 4\sqrt{2}\left((2-(1-q)\,m)\right)\cos\theta}}.$$

For $\theta \in [-\pi, \pi]$. It is clear that Θ_1 is an even function, therefore we find the minimum value of Θ_1 when $\theta \in [0, \pi]$. Now

$$\Theta_1'(\theta) = -\frac{4\sqrt{2}m|\alpha|(A-B)\, a\sin\theta}{\left(\sqrt{8 + a^2 + 4\sqrt{2}a\cos\theta}\right)\left(2(A-B) + m|\alpha B|\sqrt{8 + a^2 + 4\sqrt{2}a\cos\theta}\right)^2},$$

where $a = 2 - (1-q)\,m$. It is easy to see that $\Theta_1'(\theta) = 0$ when $\theta = 0,\ \pi$. Similarly, we can see that $\Theta_1''(\theta) > 0$, when $\theta = \pi$. Hence $\Theta_1(\theta) \ge \Theta_1(\pi)$. Now consider the function

$$\Lambda_1(m) = \frac{m|\alpha|\sqrt{8 + (2-(1-q)\,m)^2 - 4\sqrt{2}\,(2-(1-q)\,m)}}{2(A-B) + m|\alpha B|\sqrt{8 + (2-(1-q)\,m)^2 - 4\sqrt{2}\,(2-(1-q)\,m)}}.$$

Now

$$\Lambda_1'(m) = \frac{4|\alpha|(A-B)\left\{6-4\sqrt{2}+3bm\left(\sqrt{2}-1\right)+b^2m^2\right\}}{\sqrt{8+(2-bm)^2+4\sqrt{2}(bm-2)}\left(\begin{array}{c}2(A-B)+\\ m|\alpha B|\sqrt{8+(2-bm)^2-4\sqrt{2}(2-bm)}\end{array}\right)^2} > 0,$$

where $b = 1-q$. This shows that Λ is an increasing function and it has its minimum value at $m = 1$, so

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq \frac{|\alpha|\sqrt{8+(2-(1-q))^2-4\sqrt{2}(2-(1-q))}}{2(A-B)+|\alpha B|\sqrt{8+(2-(1-q))^2-4\sqrt{2}(2-(1-q))}}.$$

Now by (27), we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$

which contradicts (28). Hence $|w(z)| < 1$ and so we get the desired result. □

If we put $p(z) = \frac{z\partial_q f(z)}{f(z)}$ in the above theorem, we get the following result.

Corollary 7. *Let* $|\alpha| \geq \frac{2(A-B)}{\sqrt{8+(1+q)\left(1+q-4\sqrt{2}\right)}(1-|B|)}$, $-1 < B < A \leq 1$ *and* $f \in \mathcal{A}$ *satisfy*

$$1+\alpha\partial_q\left(\frac{z\partial_q f(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathcal{D}. \tag{31}$$

Then $f \in \mathcal{S}^*_{\mathcal{L}_q}$.

Theorem 7. *Assume*

$$|\alpha| \geq \frac{(A-B)\sqrt{28-16\sqrt{2}}}{(1-|B|)\sqrt{8+(1+q)\left(1+q-4\sqrt{2}\right)}}, \ -1 < B < A \leq 1 \tag{32}$$

and h is an analytic function defined on $\mathcal{D}$ with $h(0) = 1$ satisfying

$$1+\alpha\frac{z\partial_q h(z)}{h(z)} \prec \frac{1+Az}{1+Bz}. \tag{33}$$

In addition, we suppose that

$$1+\alpha\frac{z\partial_q h(z)}{h(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \quad z \in \mathcal{D},$$

where w is an analytic in $\mathcal{D}$ with $w(0) = 0$, then

$$h(z) \prec 1+\sqrt{2}z+\frac{1}{2}z^2.$$

Proof. We define a function

$$p(z) = 1+\alpha\frac{z\partial_q h(z)}{h(z)}, \tag{34}$$

where p is analytic and $p(0) = 1$. Consider

$$h(z) = 1+\sqrt{2}w(z)+\frac{1}{2}w^2(z). \tag{35}$$

Using (34) and (35), we obtain

$$p(z) = 1 + \frac{\alpha z \partial_q w(z)\left\{2\sqrt{2}+2w(z)-(1-q)z\partial_q w(z)\right\}}{2\left(\frac{2+2\sqrt{2}w(z)+w^2(z).}{2}\right)}.$$

Therefore

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| =$$

$$\left|\frac{\alpha z \partial_q w(z)\left\{2\sqrt{2}+2w(z)-(1-q)z\partial_q w(z)\right\}}{\left(2+2\sqrt{2}w(z)+w^2(z)\right)(A-B)+\alpha B z \partial_q w(z)\left[2\sqrt{2}+2w(z)-(1-q)z\partial_q w(z)\right]}\right|.$$

Then using the similar method as in Theorem 7, we obtain

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq \frac{|\alpha|\sqrt{8+(2-(1-q))^2-4\sqrt{2}(2-(1-q))}}{\sqrt{28-16\sqrt{2}}(A-B)+|\alpha B|\sqrt{8+(2-(1-q))^2-4\sqrt{2}(2-(1-q))}}.$$

By using (32) , we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$

which contradicts (33). Hence we get the desired result. □

If we put $p(z) = \frac{z\partial_q f(z)}{f(z)}$ in the above theorem, we get the following result.

Corollary 8. *Let* $|\alpha| \geq \frac{(A-B)\sqrt{28-16\sqrt{2}}}{(1-|B|)\sqrt{8+(1+q)\left(1+q-4\sqrt{2}\right)}}$, $-1 < B < A \leq 1$ *and* $f \in \mathcal{A}$ *satisfy*

$$1+\alpha z\left(\frac{f(z)}{z\partial_q f(z)}\right)\partial_q\left(\frac{z\partial_q f(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz}. \tag{36}$$

Then $f \in \mathcal{S}^*_{\mathcal{L}_q}$.

Theorem 8. *Assume*

$$|\alpha| \geq \frac{(12-8\sqrt{2})(A-B)}{(1-|B|)\sqrt{8+(1+q)\left(1+q-4\sqrt{2}\right)}},\ -1 < B < A \leq 1 \tag{37}$$

and h *is an analytic function defined on* $\mathcal{D}$ *with* $h(0) = 1$ *satisfying*

$$1+\alpha\frac{z\partial_q h(z)}{(h(z))^2} \prec \frac{1+Az}{1+Bz}.$$

In addition, we suppose that

$$1+\alpha\frac{z\partial_q h(z)}{(h(z))^2} = \frac{1+Aw(z)}{1+Bw(z)}, \quad z \in \mathcal{D},$$

where w *is an analytic in* $\mathcal{D}$ *with* $w(0) = 0$, *then*

$$h(z) \prec 1+\sqrt{2}z+\tfrac{1}{2}z^2.$$

Proof. We define a function

$$p(z) = 1+\alpha\frac{z^2\partial_q h(z)}{(h(z))^2}, \tag{38}$$

where p is analytic and $p(0) = 1$. Consider

$$h(z) \quad = \quad 1 + \sqrt{2}w(z) + \tfrac{1}{2}w^2(z).$$

After some simple calculations, we obtain

$$p(z) \quad = \quad 1 + \frac{\alpha z\partial_q w(z)\left\{2\sqrt{2}+2w(z)-(1-q)z\partial_q w(z)\right\}}{2\left(\frac{2+2\sqrt{2}w(z)+w^2(z).}{2}\right)^2}.$$

Therefore

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| =$$

$$\left|\frac{\alpha z\partial_q w(z)\left\{2\sqrt{2}+2w(z)-(1-q)z\partial_q w(z)\right\}}{\frac{\left(2+2\sqrt{2}w(z)+w^2(z)\right)^2}{2}(A-B)+\alpha Bz\partial_q w(z)\left[\left(2\sqrt{2}+2w(z)-(1-q)z\partial_q w(z)\right)\right]}\right|.$$

Using similar method as in Theorem 7, we obtain

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \quad \geq \quad \frac{|\alpha|\sqrt{8+(2-(1-q))^2-4\sqrt{2}(2-(1-q))}}{\left(12-8\sqrt{2}\right)(A-B)+|\alpha B|\sqrt{8+(2-(1-q))^2-4\sqrt{2}(2-(1-q))}}.$$

Now by using (37) , we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$

which is contradiction. Hence we get the required result. □

If we put $p(z) = \frac{z\partial_q f(z)}{f(z)}$ in the above theorem, we get the following result.

Corollary 9. *Let* $|\alpha| \geq \frac{\left(12-8\sqrt{2}\right)(A-B)}{(1-|B|)\sqrt{8+(1+q)\left(1+q-4\sqrt{2}\right)}}$, $-1 < B < A \leq 1$ *and* $f \in \mathcal{A}$ *such that*

$$1 + \alpha z\left(\frac{f(z)}{z\partial_q f(z)}\right)^2 \partial_q\left(\frac{z\partial_q f(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz}.$$

Then $f \in \mathcal{S}^*_{\mathcal{L}_q}$.

Theorem 9. *Assume*

$$|\alpha| \quad \geq \quad \frac{\left(28-16\sqrt{2}\right)^{\frac{3}{2}}(A-B)}{4(1-|B|)\sqrt{8+(1+q)\left(1+q-4\sqrt{2}\right)}}, \; -1 < B < A \leq 1 \tag{39}$$

and h *is an analytic function defined on* $\mathcal{D}$ *with* $h(0) = 1$ *satisfying*

$$1 + \alpha\frac{z^3\partial_q h(z)}{(h(z))^3} \quad \prec \quad \frac{1+Az}{1+Bz}. \tag{40}$$

In addition, we suppose that

$$1 + \alpha\frac{z^3\partial_q h(z)}{(h(z))^3} = \frac{1+Aw(z)}{1+Bw(z)}, \quad z \in \mathcal{D},$$

where w *is an analytic in* $\mathcal{D}$ *with* $w(0) = 0$, *then*

$$h(z) \quad \prec \quad 1 + \sqrt{2}z + \tfrac{1}{2}z^2.$$

Proof. Here we define a function
$$p(z) = 1 + \alpha \frac{z^3 \partial_q h(z)}{(h(z))^3}, \tag{41}$$
where p is analytic and $p(0) = 1$. Consider
$$h(z) \quad = \quad 1 + \sqrt{2} w(z) + \tfrac{1}{2} w^2(z).$$

After some simplifications, we obtain
$$p(z) \quad = \quad 1 + \frac{\alpha z \partial_q w(z) \left\{2\sqrt{2} + 2w(z) - (q-1) z \partial_q w(z)\right\}}{2\left(\frac{2+2\sqrt{2}w(z)+w^2(z).}{2}\right)^3}.$$

Therefore
$$\left|\frac{p(z)-1}{A - Bp(z)}\right| =$$
$$\left|\frac{\alpha z \partial_q w(z) \left\{2\sqrt{2} + 2w(z) - (1-q) z \partial_q w(z)\right\}}{\frac{\left(2+2\sqrt{2}w(z)+w^2(z)\right)^3}{4}(A-B) + \alpha B z \partial_q w(z)\left[2\sqrt{2} + 2w(z) - (1-q) z \partial_q w(z)\right]}\right|.$$

Therefore by using Lemma 1, we obtain
$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \quad \geq \quad \frac{|\alpha|\sqrt{\left(12-8\sqrt{2}\right)+(1-q)^2+4\left(\sqrt{2}-1\right)(1-q)}}{\frac{\left(28-16\sqrt{2}\right)^{\frac{3}{2}}}{4}(A-B) + |\alpha||B|\left[\sqrt{\left(12-8\sqrt{2}\right)+(1-q)^2+4\left(\sqrt{2}-1\right)(1-q)}\right]}.$$

Next by using (39) , we have
$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$
which is contradiction. Hence, we get the required proof. □

If we put $p(z) = \frac{z \partial_q f(z)}{f(z)}$ in the above theorem, we get the following result.

Corollary 10. *Let* $|\alpha| \geq \frac{\left(28-16\sqrt{2}\right)^{\frac{3}{2}}(A-B)}{4(1-|B|)\sqrt{8+(1+q)\left(1+q-4\sqrt{2}\right)}}$, $-1 < B < A \leq 1$ *and* $f \in \mathcal{A}$ *satisfy*
$$1 + \alpha z \left(\frac{f(z)}{z \partial_q f(z)}\right)^3 \partial_q \left(\frac{z \partial_q f(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz}.$$

Then $f \in \mathcal{S}^*_{\mathcal{L}_q}$.

4. Conclusions

In this article, we have studied some q-differential subordinations. We have determined conditions on α and
$$1 + \alpha \frac{z \partial_q h(z)}{(h(z))^n} \prec \frac{1+Az}{1+Bz} \ (n = 0,1,2,3), \tag{42}$$
then $h(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2$. Similar results are also investigated involving the function $1 + \sqrt{2}z + \frac{1}{2}z^2$. Further we have deduced sufficiency criterion for q-starlikeness related with cardioid and limacon from our main results. Moreover, by choosing particular functions instead of h, sufficient conditions for other analytic functions can be found.

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