mathematics

Article

A New Gronwall–Bellman Inequality in Frame of Generalized Proportional Fractional Derivative

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Received: 19 July 2019; Accepted: 12 August 2019; Published: 15 August 2019

Abstract: New versions of a Gronwall–Bellman inequality in the frame of the generalized (Riemann–Liouville and Caputo) proportional fractional derivative are provided. Before proceeding to the main results, we define the generalized Riemann–Liouville and Caputo proportional fractional derivatives and integrals and expose some of their features. We prove our main result in light of some efficient comparison analyses. The Gronwall–Bellman inequality in the case of weighted function is also obtained. By the help of the new proposed inequalities, examples of Riemann–Liouville and Caputo proportional fractional initial value problems are presented to emphasize the solution dependence on the initial data and on the right-hand side.

Keywords: Gronwall–Bellman inequality; proportional fractional derivative; Riemann–Liouville and Caputo proportional fractional initial value problem

1. Introduction

Integral inequalities have been used as fabulous instruments to explore the qualitative properties of differential equations [1]. Over the years, there have appeared many inequalities which have been established by many authors such as Ostrowski type inequality, Hardy type inequality, Olsen type inequality, Gagliardo–Nirenberg type inequality, Lyapunove type inequality, Opial type inequality and Hermite–Hadamard type inequality [2,3]. However, the most common and significant inequality is the Gronwall–Bellman inequality, which they introduced in [4,5]. The Gronwall–Bellman inequality allows one to provide an estimate for a function that is known to satisfy a certain integral inequality by the solution of the corresponding integral equation. In particular, it has been employed to provide a comparison that can be used to prove uniqueness of a solution to an initial value problem (see some recent relevant papers [6–9]).

Fractional differential equations (FDEs) is a rich area of research that has widespread applications in science and engineering. Indeed, it describes a large number of nonlinear phenomena in different fields such as physics, chemistry, biology, viscoelasticity, control hypothesis, speculation, fluid dynamics, hydrodynamics, aerodynamics, information processing system networking, notable and picture processing, control theory, etc. FDEs also provide marvellous tools for the depiction of memory and inherited properties of many materials and processes. In view of recent developments, one can consequently conclude that FDEs have emerged significant achievements in the last couple of decades [10–16]. The study of integral equations in the scope of non-integer-order equations has been in the spotlight in the recent years. Many mathematicians in the field of applied and pure mathematics have dedicated their efforts to extend, generalize and refine the integral inequalities carried over from integer order equations to the non-integer order equations. Meanwhile, different definitions of fractional derivatives have been recently introduced [17,18]. The Gronwall–Bellman inequality, which is our concern herein, has been under investigation and different versions of it have been established for different types of fractional operators [19–25].

In this paper, new versions for a Gronwall–Bellman inequality in the frame of the newly defined generalized (Riemann–Liouville and Caputo) proportional fractional derivative are provided. Before proceeding to the main results, we define the generalized Riemann–Liouville and Caputo proportional fractional derivatives and integrals and expose some of their features [26]. We prove our main result in light of some efficient comparison analysis. The Gronwall–Bellman inequality in the case of a weighted function is also obtained. By the help of the new proposed inequalities, examples of Riemann–Liouville and Caputo generalized proportional fractional initial value problems are presented to emphasize the solution dependence on the initial data and on the right-hand side. It worth mentioning that the new proposed derivative is well-behaved. Indeed, it has nonlocal character and satisfies the semigroup or the so-called index property. Besides, the resulting inequalities converge to the classical ones upon considering particular cases of the derivative. That is, our results not only extend the classical inequalities but also generalize the existing ones for non-integer-order equations.

2. The GPF Derivatives and Integrals

We assemble in this section some fundamental preliminaries that are used throughout the remaining part of the paper. For their justifications and proofs, the reader can consult the work in [26].

In control theory, a proportional derivative controller (PDC) for controller output u at time t with two tuning parameters has the algorithm

$$u(t) = \kappa_p E(t) + \kappa_d \frac{d}{dt} E(t),$$

where κ_p is the proportional gain, κ_d is the derivative gain, and E is the input deviation or the error between the state variable and the process variable. Recent investigations have shown that PDC has direct incorporation in the control of complex networks models (see [27] for more details).

For $\rho \in [0,1]$, let the functions $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that for all $t \in \mathbb{R}$ we have

$$\lim_{\rho\to 0^+} \kappa_1(\rho,t) = 1, \quad \lim_{\rho\to 0^+} \kappa_0(\rho,t) = 0, \ \lim_{\rho\to 1^-} \kappa_1(\rho,t) = 0, \quad \lim_{\rho\to 1^-} \kappa_0(\rho,t) = 1,$$

and $\kappa_1(\rho,t) \neq 0$, $\rho \in [0,1)$, $\kappa_0(\rho,t) \neq 0$, $\rho \in (0,1]$. Then, Anderson et al. [28] defined the proportional derivative of order ρ by

$$D^\rho \xi(t) = \kappa_1(\rho,t)\xi(t) + \kappa_0(\rho,t)\xi'(t) \tag{1}$$

provided that the right-hand side exists at $t \in \mathbb{R}$ and $\xi' := \frac{d}{dt}\xi$. For the operator given in Equation (1), κ_1 is a type of proportional gain κ_p, κ_0 is a type of derivative gain κ_d, ξ is the error and $u = D^\rho \xi$ is the controller output. The reader can consult the work in [29] for more details about the control theory of the proportional derivative and its component functions. We only consider here the case when $\kappa_1(\rho,t) = 1-\rho$ and $\kappa_0(\rho,t) = \rho$. Therefore, Equation (1) becomes

$$D^\rho \xi(t) = (1-\rho)\xi(t) + \rho\xi'(t). \tag{2}$$

It is easy to find that $\lim\limits_{\rho\to 0^+} D^\rho \xi(t) = \xi(t)$ and $\lim\limits_{\rho\to 1^-} D^\rho \xi(t) = \xi'(t)$. Thus, the derivative in Equation (2) is somehow more general than the conformable derivative, which certainly does not converge to the original functions as ρ tends to 0.

In what follows, we define the generalized proportional fractional (GPF) integral and derivative:

Definition 1 ([26]). *For $0 < \rho \leq 1$, $\alpha \in \mathbb{C}$ and $Re(\alpha) > 0$, the GPF integral of ξ of order α is*

$$({}_aI^{\alpha,\rho}\xi)(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-\tau)}(t-\tau)^{\alpha-1}\xi(\tau)d\tau = \rho^{-\alpha}e^{\frac{\rho-1}{\rho}t}\left({}_aI^{\alpha}\left(e^{\frac{1-\rho}{\rho}t}\xi(t)\right)\right). \tag{3}$$

Definition 2 ([26]). *For $0 < \rho \leq 1$, $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$ and $n = [Re(\alpha)] + 1$. Then, the Riemann–Liouville type GPF derivative of f of order α is*

$$({}_aD^{\alpha,\rho}\xi)(t) = D^{n,\rho}\,{}_aI^{n-\alpha,\rho}\xi(t) = \frac{D_t^{n,\rho}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-\tau)}(t-\tau)^{n-\alpha-1}\xi(\tau)d\tau. \tag{4}$$

Remark 1. *If we let $\rho = 1$ in Definition 2, then one can obtain the left Riemann–Liouville fractional derivative [12,14,15]. Moreover, it is obvious that*

$$\lim_{\alpha\to 0}(D^{\alpha,\rho}\xi)(t) = \xi(t) \;\; and \;\; \lim_{\alpha\to 1}(D^{\alpha,\rho}\xi)(t) = (D^{\rho}\xi)(t).$$

Proposition 1 ([26]). *Let $\alpha, \beta \in \mathbb{C}$ be such that $Re(\alpha) \geq 0$ and $Re(\beta) > 0$. Then, for any $0 < \rho \leq 1$, we have*

(1) $\left({}_aI^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\rho^{\alpha}}e^{\frac{\rho-1}{\rho}x}(x-a)^{\alpha+\beta-1}, \quad Re(\alpha) > 0.$

(2) $\left({}_aD^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}\right)(x) = \frac{\rho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{\frac{\rho-1}{\rho}x}(x-a)^{\beta-1-\alpha}, \quad Re(\alpha) \geq 0.$

In the following lemmas, we expose some features of Riemann–Liouville type GPF operator. The first result concerns with the index property of GPF which is of great significance.

Lemma 1 ([26]). *If $0 < \rho \leq 1$, $Re(\alpha) > 0$ and $Re(\beta) > 0$. For a continuous function ξ defined on $[a, \infty)$, we have*

$${}_aI^{\alpha,\rho}({}_aI^{\beta,\rho}\xi)(t) = {}_aI^{\beta,\rho}({}_aI^{\alpha,\rho}\xi)(t) = ({}_aI^{\alpha+\beta,\rho}\xi)(t). \tag{5}$$

The action of the operator ${}_aD^{\alpha,\rho}$ on the integral operator is demonstrated in the following results.

Lemma 2 ([26]). *Let $0 < \rho \leq 1$, $0 \leq m < [Re(\alpha)] + 1$ and ξ be integrable in each interval $[a, t]$, $t > a$. Then,*

$${}_aD^{m,\rho}({}_aI^{\alpha,\rho}\xi)(t) = ({}_aI^{\alpha-m,\rho}\xi)(t). \tag{6}$$

Corollary 1 ([26]). *Let $0 < \rho \leq 1$, $0 < Re(\beta) < Re(\alpha)$ and $m - 1 < Re(\beta) \leq m$. Then, we have*

$${}_aD^{\beta,\rho}({}_aI^{\alpha,\rho}\xi)(t) = ({}_aI^{\alpha-\beta,\rho}\xi)(t).$$

Lemma 3 ([26]). *Let f be integrable on $t \geq a$ and $Re[\alpha] > 0$, $0 < \rho \leq 1$, $n = [Re(\alpha)] + 1$. Then, we have*

$${}_aD^{\alpha,\rho}({}_aI^{\alpha,\rho}\xi)(t) = \xi(t).$$

Lemma 4 ([26]). *Let $0 < \rho \leq 1$, $Re(\alpha) > 0$, $n = [Re(\alpha)] + 1$, $\xi \in L_1(a,b)$ and $({}_aI^{\alpha,\rho}\xi)(t) \in AC^n[a,b]$. Then,*

$${}_aI^{\alpha,\rho}({}_aD^{\alpha,\rho}\xi)(t) = \xi(t) - e^{\frac{\rho-1}{\rho}(t-a)}\sum_{j=1}^{n}({}_aI^{j-\alpha,\rho}\xi)(a^+)\frac{(t-a)^{\alpha-j}}{\rho^{\alpha-j}\Gamma(\alpha+1-j)}. \tag{7}$$

The GPF derivative of Caputo type is defined as follows:

Definition 3 ([26]). *For $0 < \rho \le 1$, $\alpha \in \mathbb{C}$, $Re(\alpha) \ge 0$ and $n = [Re(\alpha)] + 1$. Then, the GPF derivative of Caputo type of ξ of order α is*

$$({}_a^C D^{\alpha,\rho}\xi)(t) = {}_aI^{n-\alpha,\rho}(D^{n,\rho}\xi)(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-\tau)}(t-\tau)^{n-\alpha-1}(D^{n,\rho}\xi)(\tau)d\tau. \tag{8}$$

Proposition 2 ([26]). *Let $\alpha, \beta \in \mathbb{C}$ be such that $Re(\alpha) > 0$ and $Re(\beta) > 0$. Then, for any $0 < \rho \le 1$ and $n = [Re(\alpha)] + 1$, we have*

$$\left({}_a^C D^{\alpha,\rho} e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}\right)(x) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\rho-1}{\rho}x}(x-a)^{\beta-1-\alpha}, \quad Re(\beta) > n.$$

For $k = 0, 1, \ldots, n-1$, we have $\left({}_a^C D^{\alpha,\rho} e^{\frac{\rho-1}{\rho}t}(t-a)^k\right)(x) = 0$.

Lemma 5 ([26]). *For $\rho \in (0,1]$, $Re(\alpha) > 0$ and $n = [Re(\alpha)] + 1$. Then, we have*

$${}_aI^{\alpha,\rho}\left({}_a^C D^{\alpha,\rho}\xi\right)(t) = \xi(t) - e^{\frac{\rho-1}{\rho}(t-a)}\sum_{k=0}^{n-1}\frac{({}_aD^{k,\rho}\xi)(a)}{\rho^k k!}(t-a)^k. \tag{9}$$

3. Main Results

This section is devoted to provide our main results of this paper. We formulate new versions of the Gronwall–Bellman inequality within GPF operators in Riemann–Liouville and Caputo settings.

3.1. Gronwall–Bellman Inequality via the GPF Derivative of Riemann–Liouville Type

Consider the following generalized proportional Riemann–Liouville fractional initial value problem

$$\begin{cases} \left({}_aD^{\alpha,\rho}y\right)(t) = f(t, y(t)), \quad 0 < \alpha \le 1, \quad t \in [a,b], \\ \lim\limits_{t\to a^+}\left({}_aI^{1-\alpha,\rho}y\right)(t) = y(a) = y_a. \end{cases} \tag{10}$$

Applying the operator ${}_aI^{\alpha,\rho}$ to both sides of Equation (10), we obtain

$$y(t) = e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}y(a) + {}_aI^{\alpha,\rho}f(t, y(t)), \tag{11}$$

In the following, we present a comparison result for the GPF integral operator.

Theorem 1. *Let η and ζ be nonnegative continuous functions defined on $[a,b]$ and satisfying*

$$\eta(t) \ge e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\eta(a) + {}_aI^{\alpha,\rho}f(t,\eta(t)), \tag{12}$$

and

$$\zeta(t) \le e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\zeta(a) + {}_aI^{\alpha,\rho}f(t,\zeta(t)), \tag{13}$$

respectively. Suppose further that f satisfies a one-sided Lipschitz condition of the form

$$f(t,x) - f(t,y) \le \frac{L}{e^{\frac{\rho-1}{\rho}(a-t)}\left[e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + \frac{(t-a)^\alpha}{\alpha} + 1\right]}(x-y), \ \ for \ \ x \ge y, \ \ L > 0, \tag{14}$$

and $f(t,y)$ is nondecreasing in y. Then, $\eta(a) \ge \zeta(a)$ and $L < \left(1 + \frac{\alpha}{(t-a)^\alpha}\right)\Gamma(\alpha)\rho^\alpha e^{-\frac{\rho-1}{\rho}(t-a)}$ imply that $\eta(t) \ge \zeta(t)$ for all $t \in [a,b]$.

Proof. We start by setting

$$\eta_\varepsilon(t) = \eta(t) + \varepsilon\big[e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + \frac{(t-a)^\alpha}{\alpha} + 1\big], \text{ for small } \varepsilon > 0, \tag{15}$$

so that we have

$$\eta_\varepsilon(a) = \eta(a) + \varepsilon > \eta(a) \text{ and } \eta_\varepsilon(t) > \eta(t), \quad t \in [a,b]. \tag{16}$$

It follows that

$$\begin{aligned}\eta_\varepsilon(t) &\geq e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\eta(a) + \frac{1}{\Gamma(\alpha)\rho^\alpha}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}f(s,\eta(s))ds \\ &+ \varepsilon\Big[e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + \frac{(t-a)^\alpha}{\alpha} + 1\Big]\end{aligned}$$

or

$$\begin{aligned}\eta_\varepsilon(t) &\geq e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\eta(a) + \frac{1}{\Gamma(\alpha)\rho^\alpha}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}f(s,\eta(s))ds \\ &- \frac{1}{\Gamma(\alpha)\rho^\alpha}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}f(s,\eta_\varepsilon(s))ds \\ &+ \frac{1}{\Gamma(\alpha)\rho^\alpha}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}f(s,\eta_\varepsilon(s))ds \\ &+ \varepsilon\Big[e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + \frac{(t-a)^\alpha}{\alpha} + 1\Big].\end{aligned}$$

Using the Lipschitz condition in Equation (14) and the relations in Equations (15) and (16), we obtain

$$\begin{aligned}\eta_\varepsilon(t) &\geq e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\eta_\varepsilon(a) - \frac{\varepsilon L}{\Gamma(\alpha)\rho^\alpha}\int_a^t (t-s)^{\alpha-1}ds \\ &+ \frac{1}{\Gamma(\alpha)\rho^\alpha}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}f(s,\eta_\varepsilon(s))ds + \varepsilon\Big[\frac{(t-a)^\alpha}{\alpha} + 1\Big]\end{aligned}$$

Since $\int_a^t (t-s)^{\alpha-1}ds = \frac{(t-a)^\alpha}{\alpha}$ and $L < \big(1 + \frac{\alpha}{(t-a)^\alpha}\big)\Gamma(\alpha)\rho^\alpha e^{-\frac{\rho-1}{\rho}(t+a)}$, we arrive at

$$\eta_\varepsilon(t) > e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\eta_\varepsilon(a) + \frac{1}{\Gamma(\alpha)\rho^\alpha}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}f(s,\eta_\varepsilon(s))ds.$$

The remaining part of the proof can be completed by adopting the same steps followed in the proof of Theorem 2.1 in [30,31] to get $\eta_\varepsilon(t) \geq \zeta(t)$, $t \in [a,b]$. However, and since ε is arbitrary, we conclude that $\eta(t) \geq \zeta(t)$, $t \in [a,b]$ holds true.
□

Remark 2. *The Lipschitz condition in Equation (14) can be relaxed by relaxing the upper bound for the constant L.*

For our purpose, we replace $f(t,y(t))$ in Equation (11) by $x(t)y(t)$ where $|x(t)| < 1$, $t \in [a,b]$. Define the following operator

$$\Omega_x\phi = {}_aI^{\alpha,\rho}x(t)\phi(t). \tag{17}$$

The following results are important in the proof of the main theorem. We only state these lemmas as their proofs are straightforward.

Lemma 6. *For any constant λ, one has*

$$\left|\Omega_{\lambda} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\right| \leq \Omega_{|\lambda|} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}. \tag{18}$$

Lemma 7. *For any constant λ, one has*

$$\left|\Omega_{\lambda}^{n} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\right| = \frac{|\lambda|^{n}(t-a)^{(n+1)\alpha-1}\Gamma(\alpha)}{\rho^{n\alpha}\Gamma((n+1)\alpha)} e^{\frac{\rho-1}{\rho}(t-a)}, \quad n = 0,1,2,\cdots. \tag{19}$$

Lemma 8. *Let $\lambda > 0$ be such that $|y(t)| \leq \lambda$ for $t \in [a,b]$. Then,*

$$\left|\Omega_{y}^{n} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\right| \leq \Omega_{\lambda}^{n} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}, \quad n = 0,1,2,\cdots. \tag{20}$$

Theorems 1 and 2 together give us the desired proportional Riemann–Liouville fractional Gronwall–Bellman-type inequality.

Theorem 2. *Let y be a nonnegative function on $[a,b]$. Then, the GPF integral equation*

$$y(t) = e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}y(a) + {}_{a}I^{\alpha,\rho}x(t)y(t), \quad t \in [a,b], \tag{21}$$

has a solution

$$y(t) = y(a)\sum_{k=0}^{\infty} \Omega_{x}^{k} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}. \tag{22}$$

Proof. The proof is accomplished by applying the successive approximation method. Set

$$y_0(t) = e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}y(a)$$

and

$$y_n(t) = e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}y(a) + {}_{a}I^{\alpha,\rho}x(t)y_{n-1}(t), \quad n \geq 1.$$

We observe that

$$\begin{aligned} y_1(t) &= e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}y(a) + {}_{a}I^{\alpha,\rho}x(t)y_0(t) \\ &= y(a)\Omega_{x}^{0} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + y(a)\Omega_{x}^{1} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}, \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}y(a) + {}_{a}I^{\alpha,\rho}x(t)y_1(t) \\ &= y(a)\Omega_{x}^{0} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + \Omega_{x}^{1}\left[y(a)\Omega_{x}^{0} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + y(a)\Omega_{x}^{1} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\right] \\ &= y(a)\Omega_{x}^{0} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + y(a)\Omega_{x}^{1} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + y(a)\Omega_{x}^{2} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}. \end{aligned}$$

It follows inductively that

$$y_n(t) = y(a)\sum_{k=0}^{n} \Omega_{x}^{k} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}, \quad n \geq 0. \tag{23}$$

Formally, taking the limit as $n \to \infty$ to obtain

$$y(t) = y(a) \sum_{k=0}^{\infty} \Omega_x^k e^{\frac{\rho-1}{\rho}(t-a)} (t-a)^{\alpha-1}. \tag{24}$$

We use Lemmas 6–8, the comparison test and the d'Alembert ratio test to show the absolute convergence of the series in Equation (24). Indeed, the infinite series

$$\sum_{n=0}^{\infty} \frac{\lambda^n (t-a)^{(n+1)\alpha-1}\Gamma(\alpha)}{\rho^{n\alpha}\Gamma((n+1)\alpha)} e^{\frac{\rho-1}{\rho}(t-a)},$$

is convergent for all $t \in [a,b]$ and for all $0 < \lambda, \rho \leq 1$. Let a_n be defined as

$$a_n = \frac{\lambda^n (t-a)^{(n+1)\alpha-1}\Gamma(\alpha)}{\rho^{n\alpha}\Gamma((n+1)\alpha)} e^{\frac{\rho-1}{\rho}(t-a)}. \tag{25}$$

Then, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{\lambda (t-a)^{\alpha}}{\rho^{\alpha}} \lim_{n\to\infty} \left| \frac{\Gamma((n+1)\alpha)}{\Gamma((n+2)\alpha)} \right|.$$

Next, we use Stirling approximation formula for the Gamma function $x\Gamma(x) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$, where x is large enough. It is a straightforward computation using this formula to show that

$$\lim_{x\to\infty} \frac{x\Gamma(x)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} = 1 \qquad \text{and} \qquad \lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = \frac{1}{e},$$

which are all we need. Hence, we have

$$\lim_{n\to\infty} \frac{(n+1)\alpha\Gamma((n+1)\alpha)}{\sqrt{2\pi(n+1)\alpha}\left(\frac{(n+1)\alpha}{e}\right)^{(n+1)\alpha}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{(n+2)\alpha\Gamma((n+2)\alpha)}{\sqrt{2\pi(n+2)\alpha}\left(\frac{(n+2)\alpha}{e}\right)^{(n+2)\alpha}} = 1.$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{\lambda (t-a)^{\alpha}}{\rho^{\alpha}} \lim_{n\to\infty} \left| \frac{\Gamma((n+1)\alpha)}{\Gamma((n+2)\alpha)} \right| \\
&= \frac{\lambda (t-a)^{\alpha}}{\rho^{\alpha}} \lim_{n\to\infty} \left[\sqrt{\frac{n+2}{n+1}} \left(\frac{n+1}{n+2}\right)^{\alpha} \left(\frac{e}{\alpha}\right)^{\alpha} \left(\frac{n+1}{n+2}\right)^{n\alpha} \left(\frac{1}{n+2}\right)^{\alpha} \right] \\
&= \frac{\lambda (t-a)^{\alpha}}{\rho^{\alpha}} \left[\left(\frac{e}{\alpha}\right)^{\alpha} \left(\frac{1}{e}\right)^{\alpha} 0 \right] \\
&= 0 < 1.
\end{aligned}$$

Hence, convergence is guaranteed. Besides, one can easily show that Equation (22) solves Equation (21). □

Remark 3. *Note that Equation* (22) *solves the inequality*

$$\zeta(t) \leq e^{\frac{\rho-1}{\rho}(t-a)} (t-a)^{\alpha-1} \zeta(a) + {}_aI^{\alpha,\rho}\zeta(t)y(t), \quad t \in [a,b], \tag{26}$$

where ζ *and* y *are nonnegative real valued functions such that* $0 \leq y(t) < \lambda < 1$.

Now, we are in a position to state the main theorem, which is a new version of the Gronwall–Bellman inequality within the generalized proportional fractional Riemann–Liouville settings.

Corollary 2. *Let ζ and y be nonnegative real valued functions such that $0 \le y(t) < \lambda < 1$ and*

$$\zeta(t) \le e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\zeta(a) + {}_aI^{\alpha,\rho}\zeta(t)y(t), \quad t \in [a,b]. \tag{27}$$

Then,

$$\zeta(t) \le \zeta(a)\sum_{k=0}^{\infty}\Omega_y^k e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}. \tag{28}$$

The proof of the corollary is a straightforward implementation of Theorems 1 and 2. Indeed, it is immediately obtained by setting $\eta(t) = \zeta(a)\sum_{k=0}^{\infty}\Omega_y^k e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}$.

3.2. Gronwall–Bellman Inequality via the GPF Derivative of Caputo Type

Consider the following generalized proportional Caputo fractional initial value problem

$$\begin{cases} ({}_a^CD^{\alpha,\rho}y)(t) = f(t,y(t)), \quad 0 < \alpha \le 1, \quad t \in [a,b], \\ y(a) = y_a. \end{cases} \tag{29}$$

Applying the operator ${}_aI^{\alpha,\rho}$ to both sides of Equation (29), we obtain

$$y(t) = e^{\frac{\rho-1}{\rho}(t-a)}y(a) + {}_aI^{\alpha,\rho}f(t,y(t)), \tag{30}$$

The results of this subsection resemble the ones proved in Section 3.1. To avoid redundancy, therefore, we skip some steps of the proofs. We start by the following comparison result for the generalized proportional Caputo fractional integral operator.

Theorem 3. *Let η and ζ be nonnegative continuous functions defined on $[a,b]$ and satisfy*

$$\eta(t) \ge e^{\frac{\rho-1}{\rho}(t-a)}\eta(a) + {}_aI^{\alpha,\rho}f(t,\eta(t)), \tag{31}$$

and

$$\zeta(t) \le e^{\frac{\rho-1}{\rho}(t-a)}\zeta(a) + {}_aI^{\alpha,\rho}f(t,\zeta(t)), \tag{32}$$

respectively. Suppose further that f satisfies one-sided Lipschitz condition of the form

$$f(t,x) - f(t,y) \le \frac{L}{e^{\frac{\rho-1}{\rho}(a-t)}\left[e^{\frac{\rho-1}{\rho}(t-a)} + \frac{(t-a)^{\alpha}}{\alpha}\right]}(x-y), \ \ for \ \ x \ge y, \quad L > 0, \tag{33}$$

and $f(t,y)$ is nondecreasing in y. Then, $\eta(a) \ge \zeta(a)$ and $L < \Gamma(\alpha)\rho^{\alpha}e^{-\frac{\rho-1}{\rho}(t-a)}$ imply that $\eta(t) \ge \zeta(t)$ for all $t \in [a,b]$.

The proof of the above theorem can be completed by setting $\eta_\varepsilon(t) = \eta(t) + \varepsilon\left[e^{\frac{\rho-1}{\rho}(t-a)} + \frac{(t-a)^{\alpha}}{\alpha}\right]$, for small $\varepsilon > 0$, and following similar steps as the proof of Theorem 1.

In the sequel, we replace $f(t,y(t))$ in Equation (30) by $x(t)y(t)$, where $|x(t)| < 1, \ t \in [a,b]$. Define the following operator

$$\Phi_x\phi = {}_aI^{\alpha,\rho}x(t)\phi(t). \tag{34}$$

In similar manner, the following lemmas are formulated for Caputo type operator.

Lemma 9. *For any constant λ, one has*

$$\left|\Phi_{\lambda} e^{\frac{\rho-1}{\rho}(t-a)}\right| \leq \Phi_{|\lambda|} e^{\frac{\rho-1}{\rho}(t-a)}. \tag{35}$$

Lemma 10. *For any constant λ, one has*

$$\left|\Phi_{\lambda}^{n} e^{\frac{\rho-1}{\rho}(t-a)}\right| = \frac{|\lambda|^{n}(t-a)^{n\alpha}}{\rho^{n\alpha}\Gamma(n\alpha+1)} e^{\frac{\rho-1}{\rho}(t-a)}, \quad n = 0, 1, 2, \cdots. \tag{36}$$

Lemma 11. *Let $\lambda > 0$ be such that $|y(t)| \leq \lambda$ for $t \in [a, b]$. Then,*

$$\left|\Phi_{y}^{n} e^{\frac{\rho-1}{\rho}(t-a)}\right| = \Phi_{\lambda}^{n} e^{\frac{\rho-1}{\rho}(t-a)}, \qquad n = 0, 1, 2, \cdots. \tag{37}$$

Theorem 4. *Let y be a nonnegative function on $[a, b]$. Then, the generalized proportional fractional integral equation*

$$y(t) = e^{\frac{\rho-1}{\rho}(t-a)} y(a) + {}_{a}I^{\alpha,\rho} x(t) y(t), \quad t \in [a, b], \tag{38}$$

has a solution

$$y(t) = y(a) \sum_{k=0}^{\infty} \Phi_{x}^{k} e^{\frac{\rho-1}{\rho}(t-a)}. \tag{39}$$

Proof. We employ the successive approximation method to complete the proof. Set

$$\begin{aligned} y_0(t) &= e^{\frac{\rho-1}{\rho}(t-a)} y(a) \\ y_n(t) &= e^{\frac{\rho-1}{\rho}(t-a)} y(a) + {}_{a}I^{\alpha,\rho} x(t) y_{n-1}(t), \quad n \geq 1. \end{aligned}$$

We observe that

$$y_1(t) = y(a)\Phi_{x}^{0} e^{\frac{\rho-1}{\rho}(t-a)} + y(a)\Phi_{x}^{1} e^{\frac{\rho-1}{\rho}(t-a)}$$

and

$$\begin{aligned} y_2(t) &= e^{\frac{\rho-1}{\rho}(t-a)} y(a) + {}_{a}I^{\alpha,\rho} x(t) y_1(t) \\ &= y(a)\Phi_{x}^{0} e^{\frac{\rho-1}{\rho}(t-a)} + y(a)\Phi_{x}^{1} e^{\frac{\rho-1}{\rho}(t-a)} + y(a)\Phi_{x}^{2} e^{\frac{\rho-1}{\rho}(t-a)}. \end{aligned}$$

It follows inductively that $y_n(t) = y(a) \sum_{k=0}^{n} \Phi_{x}^{k} e^{\frac{\rho-1}{\rho}(t-a)}$. Taking the limit as $n \to \infty$ to obtain

$$y(t) = y(a) \sum_{k=0}^{\infty} \Phi_{x}^{k} e^{\frac{\rho-1}{\rho}(t-a)}. \tag{40}$$

Following the same arguments as in the proof of Theorem 2, we use Lemmas 9–11, the comparison test and the d'Alembert ratio test to show the absolute convergence of the series in Equation (40). Moreover, it is clear to verify that Equation (39) solves Equation (38). The proof is finished. □

Remark 4. *Note that Equation (39) solves the inequality*

$$\zeta(t) \leq e^{\frac{\rho-1}{\rho}(t-a)} \zeta(a) + {}_{a}I^{\alpha,\rho} \zeta(t) y(t), \quad t \in [a, b], \tag{41}$$

where ζ and y are nonnegative functions on $[a, b]$ such that $0 \leq y(t) < \lambda < 1$.

The Gronwall–Bellman inequality in generalized proportional Caputo fractional is stated as follows.

Corollary 3. *Let ζ and y be nonnegative real valued functions such that $0 \le y(t) < \lambda < 1$ and*

$$\zeta(t) \le e^{\frac{\rho-1}{\rho}(t-a)}\zeta(a) + {}_aI^{\alpha,\rho}\zeta(t)y(t), \qquad t \in [a,b]. \tag{42}$$

Then,

$$\zeta(t) \le \zeta(a)\sum_{k=0}^{\infty}\Phi_y^k e^{\frac{\rho-1}{\rho}(t-a)}. \tag{43}$$

To prove Equation (43), we set $\eta(t) = \zeta(a)\sum_{k=0}^{\infty}\Phi_y^k e^{\frac{\rho-1}{\rho}(t-a)}$ and the rest follows as a direct application of Theorems 3 and 4.

4. Gronwall–Bellman Inequality via Weighted Function

In this section, we extend the results obtained in Section 3 to the case of weighted function. The analysis can be carried out for the Riemann–Liouville and Caputo operators. However, we only present the results for the case of Riemann–Liouville proportional fractional operator. Unlike previous relevant results in the literature [32], the weighted function w in the following first two theorems requires no monotonic restriction.

Theorem 5. *Let η, ζ, w be nonnegative continuous functions on $[a,b]$ where η and ζ satisfy*

$$\eta(t) \ge e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\eta(a) + w(t)_aI^{\alpha,\rho}f(t,\eta(t)), \tag{44}$$

and

$$\zeta(t) \le e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\zeta(a) + w(t)_aI^{\alpha,\rho}f(t,\zeta(t)), \tag{45}$$

respectively. Suppose further that f satisfies one-sided Lipschitz condition of the form

$$f(t,x) - f(t,y) \le \frac{L}{e^{\frac{\rho-1}{\rho}(a-t)}\left[e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + w(t)\frac{(t-a)^\alpha}{\alpha} + 1\right]}(x-y), \quad for \ x \ge y, \quad L > 0, \tag{46}$$

and $f(t,y)$ is nondecreasing in y. Then, $\eta(a) \ge \zeta(a)$ and $L < \left(1 + \frac{\alpha}{w(t)(t-a)^\alpha}\right)\Gamma(\alpha)\rho^\alpha e^{-\frac{\rho-1}{\rho}(t-a)}$ imply that $\eta(t) \ge \zeta(t)$ for all $t \in [a,b]$.

To prove the above theorem, we set $\eta_\varepsilon(t) = \eta(t) + \varepsilon\left[e^{\frac{\rho-1}{\rho}(t-a)} + w(t)\frac{(t-a)^\alpha}{\alpha} + 1\right]$, for small $\varepsilon > 0$, and follow similar steps as the proof of Theorem 1.

Remark 5. *The Lipschitz condition in Equation (46) can be relaxed by relaxing the upper bound for the constant L.*

Theorem 6. *Let x, y be nonnegative functions on $[a,b]$ and w be a nonnegative continuous function defined on $[a,b]$. Further, assume that $|x(t)| < 1$ for $t \in [a,b]$ and $\max_{t\in[a,b]} w(t) = M$. Then, the generalized proportional fractional integral equation*

$$y(t) = e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}y(a) + w(t)_aI^{\alpha,\rho}x(t)y(t), \quad t \in [a,b], \tag{47}$$

has a solution

$$y(t) = y(a)\Omega_x^0 e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + y(a)w(t)\sum_{k=1}^{\infty}M^{k-1}\Omega_x^k e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}. \tag{48}$$

Remark 6. *Note that Equation (48) solves the inequality*

$$\zeta(t) \le e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\zeta(a) + w(t)_a I^{\alpha,\rho}\zeta(t)y(t), \quad t \in [a,b], \tag{49}$$

where ζ, y are nonnegative functions on $[a,b]$ and w is a nonnegative continuous function defined on $[a,b]$ and $0 \le y(t) < \lambda < 1$ and $\max_{t\in[a,b]} w(t) = M$.

The Gronwall–Bellman inequality in case of weighted function w is stated as follows.

Theorem 7. *Let ζ, y be nonnegative functions on $[a,b]$ and w be a nonnegative continuous function defined on $[a,b]$. Further, assume that $0 \le y(t) < \lambda < 1$ for $t \in [a,b]$ and $\max_{t\in[a,b]} w(t) = M$ and*

$$\zeta(t) \le e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\zeta(a) + w(t)_a I^{\alpha,\rho}\zeta(t)y(t), \quad t \in [a,b]. \tag{50}$$

Then,

$$\zeta(t) \le \zeta(a)\Omega_y^0 e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + \zeta(a)w(t)\sum_{k=1}^{\infty} M^{k-1}\Omega_y^k e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}. \tag{51}$$

If the weighted function w possesses a monotonic behavior, then Theorem 6 and Theorem 7 can be reformulated, respectively, in the following forms.

Theorem 8. *Let y, x be nonnegative functions on $[a,b]$ and w be a nonnegative continuous function defined on $[a,b]$. Further, assume that $|x(t)| < 1$ for $t \in [a,b]$ and w is a nondecreasing function. Then, the generalized proportional fractional integral equation*

$$y(t) = e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}y(a) + w(t)_a I^{\alpha,\rho}x(t)y(t), \quad t \in [a,b], \tag{52}$$

has a solution

$$y(t) = y(a)\sum_{k=0}^{\infty} w^k(t)\Omega_x^k e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}. \tag{53}$$

Theorem 9. *Let ζ, y be nonnegative functions on $[a,b]$ and w be a nonnegative continuous function defined on $[a,b]$. Assume that $0 \le y(t) < \lambda < 1$ for $t \in [a,b]$ and w is a nondecreasing function and*

$$\zeta(t) \le e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\zeta(a) + w(t)_a I^{\alpha,\rho}\zeta(t)y(t), \quad t \in [a,b]. \tag{54}$$

Then,

$$\zeta(t) \le \zeta(a)\sum_{k=0}^{\infty} w^k(t)\Omega_y^k e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}. \tag{55}$$

5. Applications

In this section, two examples of Riemann–Liouville and Caputo generalized proportional fractional initial value problems are presented. By the help of the new proposed Gronwall–Bellman inequalities in Theorems 2 and 3, we show that the solution of the initial value problems depend on the initial data and on the right-hand side.

Consider the proportional Riemann–Liouville fractional initial value problem in Equation (10). In the remaining part of this section, we assume that the nonlinearity function $f(t,y)$ satisfies a Lipschitz condition with a constant $L \in [0,1)$ for all (t,y).

Example 1. *Consider the following Riemann–Liouville proportional fractional initial value problems of the form*

$$({}_aD^{\alpha,\rho}\beta)(t) = f(t,\beta(t)), \quad \lim_{t\to a^+}\left({}_aI^{1-\alpha,\rho}\beta\right)(t) = \beta(a) = \beta_0, \quad 0<\alpha\le 1,\, t\in[a,b], \tag{56}$$

and

$$({}_aD^{\alpha,\rho}\gamma)(t) = f(t,\gamma(t)), \quad \lim_{t\to a^+}\left({}_aI^{1-\alpha,\rho}\gamma\right)(t) = \gamma(a) = \gamma_0, \quad 0<\alpha\le 1,\, t\in[a,b]. \tag{57}$$

We claim that a small change in the initial condition implies a small change in the solution.

Proof. Applying the generalized proportional fractional integral operator in Equations (56) and (57), we have

$$\beta(t) \;=\; e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\beta_0 + {}_aI^{\alpha,\rho}f(t,\beta(t)),$$

and

$$\gamma(t) \;=\; e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}\gamma_0 + {}_aI^{\alpha,\rho}f(t,\gamma(t)).$$

It follows that

$$\beta(t)-\gamma(t) = e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}(\beta_0-\gamma_0) + {}_aI^{\alpha,\rho}[f(t,\beta(t)) - f(t,\gamma(t))].$$

Taking the absolute value, we obtain

$$\begin{aligned}
|\beta(t)-\gamma(t)| &\le e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}|\beta_0-\gamma_0| + {}_aI^{\alpha,\rho}|f(t,\beta(t)) - f(t,\gamma(t))| \\
&\le e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}|\beta_0-\gamma_0| + L_aI^{\alpha,\rho}|\beta(t)-\gamma(t)|.
\end{aligned}$$

By employing Theorem 2, we get

$$\begin{aligned}
|\beta(t)-\gamma(t)| &\le |\beta_0-\gamma_0|\sum_{k=0}^{\infty}\Omega_L^k e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} \\
&= |\beta_0-\gamma_0|\sum_{k=0}^{\infty}\frac{L^k(t-a)^{(k+1)\alpha-1}\Gamma(\alpha)}{\rho^{k\alpha}\Gamma((k+1)\alpha)}e^{\frac{\rho-1}{\rho}(t-a)}.
\end{aligned}$$

Consider the initial value problem

$$\begin{cases}
({}_aD^{\alpha,\rho}\nu)(t) = f(t,\nu(t)), \quad 0<\alpha\le 1, \quad t\in[a,b] \\
\lim_{t\to a^+}\left({}_aI^{1-\alpha,\rho}\nu\right)(t) = \nu(a) = \beta_n,
\end{cases} \tag{58}$$

where $\beta_n \to \beta_0$. If the solution of Equation (58) is denoted by ν_n, then, for all $t\in[a,b]$, we have

$$|\beta(t)-\nu_n(t)| \le |\beta_0-\beta_n|\sum_{k=0}^{\infty}\frac{L^k(t-a)^{(k+1)\alpha-1}\Gamma(\alpha)}{\rho^{k\alpha}\Gamma((k+1)\alpha)}e^{\frac{\rho-1}{\rho}(t-a)}.$$

Hence, $|\beta(t)-\nu_n(t)| \to 0$ when $\beta_n\to\beta_0$ as $n\to\infty$. We conclude that a small change in the initial condition implies a small change in the solution. □

Example 2. *Consider the following Caputo generalized proportional fractional initial value problems of the form*

$$({}_a^CD^{\alpha,\rho}\beta)(t) = f(t,\beta(t)), \quad \beta(a)=\beta_0, \quad \alpha\in(0,1], \quad t\in[a,b]. \tag{59}$$

and

$$({}_{a}^{C}D^{\alpha,\rho}\sigma)(t) = f(t,\sigma(t)) + g(t,\sigma(t)), \quad \sigma(a) = \sigma_0, \quad \alpha \in (0,1], \quad t \in [a,b]. \tag{60}$$

We claim that the solution of Equation (60) depends continuously on the right-hand side of Equation (60) if $|g(t,\sigma)| \le Ke^{\frac{\rho-1}{\rho}(t-a)}$ *for all* $t \in [a,b]$ *and for a positive number* K.

Proof. If the solution of Equation (60) is denoted by σ, then, for all $t \in [a,b]$, we have

$$\begin{aligned}|\beta(t)-\sigma(t)| &\le e^{\frac{\rho-1}{\rho}(t-a)}|\beta_0-\sigma_0| + {}_aI^{\alpha,\rho}|f(t,\beta(t))-f(t,\sigma(t))| + {}_aI^{\alpha,\rho}|g(t,\sigma(t))| \\ &\le e^{\frac{\rho-1}{\rho}(t-a)}|\beta_0-\sigma_0| + L\,{}_aI^{\alpha,\rho}|\beta(t)-\sigma(t)| + {}_aI^{\alpha,\rho}|g(t,\sigma(t))|.\end{aligned}$$

By the assumption, we have

$$|\beta(t)-\sigma(t)| \le e^{\frac{\rho-1}{\rho}(t-a)}|\beta_0-\sigma_0| + L\,{}_aI^{\alpha,\rho}|\beta(t)-\sigma(t)| + {}_aI^{\alpha,\rho}Ke^{\frac{\rho-1}{\rho}(t-a)}$$

or

$$|\beta(t)-\sigma(t)| + \frac{K}{L}e^{\frac{\rho-1}{\rho}(t-a)} \le e^{\frac{\rho-1}{\rho}(t-a)}\left(|\beta_0-\sigma_0| + \frac{K}{L}\right) + L\,{}_aI^{\alpha,\rho}\left(|\beta(t)-\sigma(t)| + \frac{K}{L}e^{\frac{\rho-1}{\rho}(t-a)}\right).$$

Let $r(t) = |\beta(t)-\sigma(t)| + \frac{K}{L}e^{\frac{\rho-1}{\rho}(t-a)}$. Then, if we apply Theorem 3, we obtain

$$r(t) \le \left(|\beta_0-\sigma_0| + \frac{K}{L}\right)\sum_{k=0}^{\infty}\Phi_L^k e^{\frac{\rho-1}{\rho}(t-a)},$$

or

$$|\beta(t)-\sigma(t)| \le \left(|\beta_0-\sigma_0| + \frac{K}{L}\right)\sum_{k=0}^{\infty}\frac{L^k(t-a)^{(k+1)\alpha-1}}{\rho^{k\alpha}\Gamma((k+1)\alpha)}e^{\frac{\rho-1}{\rho}(t-a)} - \frac{K}{L}e^{\frac{\rho-1}{\rho}(t-a)}.$$

For $a \le t \le b$, letting $Ke^{\frac{\rho-1}{\rho}(t-a)} < \delta$ implies that

$$\begin{aligned}|\beta(t)-\sigma(t)| &\le |\beta_0-\sigma_0|\sum_{k=0}^{\infty}\frac{L^k(t-a)^{(k+1)\alpha-1}}{K\rho^{k\alpha}\Gamma((k+1)\alpha)}\delta + \frac{\delta}{L}\left[\sum_{k=0}^{\infty}\frac{L^k(t-a)^{(k+1)\alpha-1}}{\rho^{k\alpha}\Gamma((k+1)\alpha)} - 1\right] \\ &\le \delta\left\{|\beta_0-\sigma_0|\sum_{k=0}^{\infty}\frac{L^kb^{(k+1)\alpha-1}}{K\rho^{k\alpha}\Gamma((k+1)\alpha)} + \frac{1}{L}\left[\sum_{k=0}^{\infty}\frac{L^kb^{(k+1)\alpha-1}}{\rho^{k\alpha}\Gamma((k+1)\alpha)} - 1\right]\right\} = \epsilon,\end{aligned}$$

which implies that a small change on the right-hand side of Equation (59) implies a small change in its solution. □

6. Conclusions

One of the most crucial issues in the theory of differential equations is to study qualitative properties for solutions of these equations. Integral inequalities are significant instruments that facilitate exploring such properties. In this paper, we accommodate a newly defined generalized proportional fractional (GPF) derivative to establish new versions for the well–known Gronwall–Bellman inequality. We prove our results in the frame of GPF operators within the Riemann–Liouville and Caputo settings. The main results are also extended to the weighted function case. One can easily figure out that the current results generalize the ones previously obtained in the literature. Indeed, the case $\rho = 1$ covers the results of classical Riemann–Liouville and Caputo fractional derivatives. As an application, we provide two efficient examples that demonstrate the solution dependence on the initial data and on the right-hand side of the initial value problems.

The results of this paper have strong potential to be used for establishing new substantial investigations in the future for equations involving the GPF operators.

Author Contributions: All authors contributed equally and significantly to this paper. All authors have read and approved the final version of the manuscript.

Funding: The first author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Acknowledgments: The authors would like to express their sincere thanks to the handling editor and the referees for their constructive comments and suggestions. We believe that their efforts have significantly helped in improving the contents of this paper.

Conflicts of Interest: The authors declare that they have no competing interests.

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