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Article

Some Polynomial Sequence Relations

Chan-Liang Chung

College of Mathematics and Computer Science, Fuzhou University, No. 2, Xueyuan Road, Minhou County, Fuzhou 350108, China; andrechung@fzu.edu.cn

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Abstract: We give some polynomial sequence relations that are generalizations of the Sury-type identities. We provide two proofs, one based on an elementary identity and the other using the method of generating functions.

Keywords: polynomial sequence; Fibonacci sequence; Lucas sequence; generating function

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1. Introduction

Horadam [1] introduced two polynomial sequences $\{W_n^{(k)}(x)\}_{n\geq 0}$ and $\{w_n^{(k)}(x)\}_{n\geq 0}$ defined below. For a nonnegative integer k, the W-polynomial sequence of order k, $\{W_n^{(k)}(x)\}_{n\geq 0}$, is the sequence of polynomials defined by the recurrence relation

$$W_n^{(k)}(x) = p(x)W_{n-1}^{(k)}(x) + q(x)W_{n-2}^{(k)}(x),\ n \geq 2, \tag{1}$$

with the initial conditions

$$W_0^{(k)}(x) = \Delta^{k-1}(x)\left[1-(-1)^k\right] = \begin{cases} 0 & \text{if } k \text{ is even;} \\ 2\Delta^{k-1}(x) & \text{if } k \text{ is odd,} \end{cases}$$

$$W_1^{(k)}(x) = \Delta^{k-1}(x)\left[\alpha(x)-(-1)^k\beta(x)\right] = \begin{cases} \Delta^{k}(x) & \text{if } k \text{ is even;} \\ p(x)\Delta^{k-1}(x) & \text{if } k \text{ is odd,} \end{cases}$$

where $\alpha(x), \beta(x)$ are polynomials such that

$$\alpha(x) = \frac{p(x)+\Delta(x)}{2}, \qquad \text{and} \qquad \beta(x) = \frac{p(x)-\Delta(x)}{2},$$

with $\Delta(x) = \sqrt{p^2(x)+4q(x)}$. Note that $\alpha(x)+\beta(x) = p(x)$, $\alpha(x)\beta(x) = -q(x)$, and $\Delta(x) = \alpha(x)-\beta(x)$.

The companion sequence $\{w_n^{(k)}(x)\}_{n\geq 0}$, the w-polynomial sequence of order k, is defined with the same recurrence relation but with the different initial conditions

$$w_0^{(k)}(x) = \Delta^{k}(x)\left[1+(-1)^k\right] = \begin{cases} 2\Delta^{k}(x) & \text{if } k \text{ is even;} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

$$w_1^{(k)}(x) = \Delta^{k}(x)\left[\alpha(x)+(-1)^k\beta(x)\right] = \begin{cases} p(x)\Delta^{k}(x) & \text{if } k \text{ is even;} \\ \Delta^{k+1}(x) & \text{if } k \text{ is odd.} \end{cases}$$

Both polynomial sequences generalize the Lucas sequence to polynomials since we have Binet formulae

$$W_n^{(k)}(x) = \frac{\Delta^k(x)\left[\alpha^n(x)-(-1)^k\beta^n(x)\right]}{\alpha(x)-\beta(x)}, \tag{2}$$

and

$$w_n^{(k)}(x) = \Delta^k(x)\left[\alpha^n(x) + (-1)^k\beta^n(x)\right]. \tag{3}$$

The most common case $k = 0$ (zeroth order) gives the Fibonacci polynomial and the Lucas polynomial as well when $p(x) = x, q(x) = 1$.

In [2] Sury proved an interesting Fibonacci-Lucas relation for all positive integer n,

$$2^{n+1}F_{n+1} = \sum_{i=0}^{n} 2^i L_i,$$

where F_n, L_n are the n-th Fibonacci and the n-th Lucas number respectively. H. Kwong [3] provided another proof via the method of generating functions. D. Marques [4] proved the similar relation for $3^{n+1}F_{n+1}$ and T. Edgar [5] proved the following

$$r^{n+1}F_{n+1} = \sum_{i=0}^{n} r^i L_i + (r-2)\sum_{i=0}^{n+1} r^{i-1}F_i, \tag{4}$$

for any positive integer r. In fact, Edgar's proof was based on an elementary identity

$$L_n = F_{n-1} + F_{n+1}, \quad n \geq 1. \tag{5}$$

Theorem 6 of [6] stated that for integer $r \geq 2$ the following alternating Fibonacci-Lucas relation holds,

$$(-1)^n F_{n+1} = \sum_{i=0}^{n}(-1)^i r^{n-i}\left[L_{i+1} + (r-2)F_i\right]. \tag{6}$$

Indeed, this identity holds for $r = 1$ as is easily checked. Relation (6) can be deduced from (5) or from the generating functions of the Fibonacci numbers and the Lucas numbers. Bhatnagar [7] proved relations (4) and (6) for the case r is an indeterminate by using Euler's telescoping lemma. Inspired by their work, we give some extensions of the Sury-type relation as follow.

Theorem 1. *Let r be an indeterminate. For any positive integer n, and for the both polynomial sequences given by (2) and (3), we have, for even k,*

$$r^{n+1}p(x)W_{n+1}^{(k)}(x) = \sum_{i=0}^{n} r^i w_i^{(k)}(x) + (rp(x)-2)\sum_{i=0}^{n+1} r^{i-1}W_i^{(k)}(x), \tag{7}$$

and, for odd k,

$$r^{n+1}p(x)w_{n+1}^{(k)}(x) = \left(p^2(x) + 4q(x)\right)\sum_{i=0}^{n} r^i W_i^{(k)}(x) + (rp(x)-2)\sum_{i=0}^{n+1} r^{i-1}w_i^{(k)}(x). \tag{8}$$

We note that when the order $k = 0$ and $p(x) = x, q(x) = 1$, (7) reduces to the Fibonacci-Lucas polynomial relation:

$$r^{n+1}xF_{n+1}(x) = \sum_{i=0}^{n} r^i L_i(x) + (rx-2)\sum_{i=0}^{n+1} r^{i-1}F_i(x).$$

By taking $x = 1$ into above, we recover (4).

Theorem 2. *Let r be an indeterminate. For any positive integer n, and for the both polynomial sequences given by (2) and (3), we have, for even k,*

$$(-1)^n p(x) W_{n+1}^{(k)}(x) = \sum_{i=0}^{n} (-1)^i r^{n-i} \left[w_{i+1}^{(k)}(x) + (rp(x) - 2q(x))\, W_i^{(k)}(x) \right], \tag{9}$$

and, for odd k,

$$(-1)^n p(x) w_{n+1}^{(k)}(x) = \sum_{i=0}^{n} (-1)^i r^{n-i} \left[\left(p^2(x) + 4q(x) \right) W_{i+1}^{(k)}(x) + (rp(x) - 2q(x))\, w_i^{(k)}(x) \right]. \tag{10}$$

We see immediately the identity (6) is just a special case of (9) by taking $p(x) = x, q(x) = 1$ and then $k = 0, x = 1$.

The rest of this paper is organized as follows. In Section 2, we will present a crucial relation between the polynomial sequences and then give an elementary proof of Theorems 1 and 2. We also give an alternative proof of our main results by using the method of generating functions in Section 3. We discuss some further examples in the final section.

2. A Crucial Relation

In this section, we present a crucial relation (11) which can be proved directly (see [1]). However, an induction proof of the following lemma could be given by dividing into two cases according to the order k is even or odd.

Lemma 1 ([1], Theorem D)**.** *For integers $n \geq 1$ and $k \geq 0$, we have*

$$w_n^{(k)}(x) = q(x) W_{n-1}^{(k)}(x) + W_{n+1}^{(k)}(x). \tag{11}$$

Proof. We will proceed by induction on n.

Assume that k is even. When $n = 1$, notice that

$$w_1^{(k)}(x) = p(x)\Delta^k(x),$$

and $q(x)W_0^{(k)}(x) + W_2^{(k)}(x) = q(x) \times 0 + p(x)\Delta^k(x)$. Thus, (11) holds for $n = 1$. Suppose the relation (11) holds for some $m > 1$ and then by definition we arrive at

$$\begin{aligned} w_{m+1}^{(k)}(x) &= p(x) w_m^{(k)}(x) + q(x) w_{m-1}^{(k)}(x) \\ &= p(x)q(x)W_{m-1}^{(k)}(x) + p(x)W_{m+1}^{(k)}(x) + q^2(x)W_{m-2}^{(k)}(x) + q(x)W_m^{(k)}(x) \\ &= q(x)W_m^{(k)}(x) + W_{m+2}^{(k)}(x). \end{aligned}$$

For the odd k, we just note that

$$\begin{aligned} &w_1^{(k)}(x) = \Delta^{k+1}(x),\ w_2^{(k)}(x) = p(x)\Delta^{k+1}(x); \\ &W_0^{(k)}(x) = 2\Delta^{k-1}(x),\ W_1^{(k)}(x) = p(x)\Delta^{k-1}(x),\ W_2^{(k)}(x) = \left[p^2(x) + 2q(x) \right] \Delta^{k-1}(x), \end{aligned}$$

and $W_3^{(k)}(x) = \left[p^3(x) + 3p(x)q(x) \right] \Delta^{k-1}(x)$. The induction procedure is straightforward, so we leave it to the reader. □

Put $k = 0$, $p(x) = x$, and $q(x) = 1$ in Lemma 1. Then the identity (5) is obtained by taking $x = 1$. So Lemma 1 is essentially an extension of (5). In a similar way(use induction or prove directly), one can prove that

$$\Delta^2(x) W_n^{(k)}(x) = q(x) w_{n-1}^{(k)}(x) + w_{n+1}^{(k)}(x).$$

In particular, it gives $(x^2+4)F_n(x) = L_{n+1}(x) + L_{n-1}(x)$, and hence $5F_n = L_{n+1} + L_{n-1}$.

We now show the details for proving (7) and (10), and leave (8) and (9) to the readers.

For (7), we have

$$\begin{aligned}
&\sum_{i=0}^{n} r^i w_i^{(k)}(x) + (rp(x)-2) \sum_{i=0}^{n+1} r^{i-1} W_i^{(k)}(x) \\
&= \sum_{i=0}^{n} r^i \left(w_i^{(k)}(x) + p(x) W_i^{(k)}(x) \right) + r^{n+1} p(x) W_{n+1}^{(k)}(x) - 2 \sum_{i=0}^{n+1} r^{i-1} W_i^{(k)}(x) \\
&= \sum_{i=0}^{n} r^i \left(w_i^{(k)}(x) + p(x) W_i^{(k)}(x) \right) + r^{n+1} p(x) W_{n+1}^{(k)}(x) - 2 \sum_{i=0}^{n} r^i W_{i+1}^{(k)}(x) \\
&= \sum_{i=0}^{n} r^i \left(w_i^{(k)}(x) + p(x) W_i^{(k)}(x) - 2W_{i+1}^{(k)}(x) \right) + r^{n+1} p(x) W_{n+1}^{(k)}(x),
\end{aligned}$$

since k is even. By Lemma 1 the first inner term

$$\begin{aligned}
&w_i^{(k)}(x) + p(x) W_i^{(k)}(x) - 2W_{i+1}^{(k)}(x) \\
&= q(x) W_{i-1}^{(k)}(x) + W_{i+1}^{(k)}(x) + p(x) W_i^{(k)}(x) - 2W_{i+1}^{(k)}(x) \\
&= 0.
\end{aligned}$$

This proves the relation in (7).

Note that

$$\begin{aligned}
&2q(x) w_m^{(k)}(x) + p(x) w_{m+1}^{(k)}(x) \\
&= 2q^2(x) W_{m-1}^{(k)}(x) + 2q(x) W_{m+1}^{(k)}(x) + p(x) q(x) W_m^{(k)}(x) + p(x) W_{m+2}^{(k)}(x) \\
&= \left(p^2(x) + 4q(x) \right) W_{m+1}^{(k)}(x).
\end{aligned}$$

So we see the right hand side of (10) is equal to

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i r^{n-i} p(x) \left[w_{i+1}^{(k)}(x) + r w_i^{(k)}(x) \right] \\
&= p(x) \left[(-1)^0 r^{n+1} w_0^{(k)}(x) + (-1)^n r^0 w_{n+1}^{(k)}(x) \right] \\
&= (-1)^n p(x) w_{n+1}^{(k)}(x).
\end{aligned}$$

Thus the proof of (10) is done.

3. Generating Functions

Proofs of Theorems 1 and 2 based on (11) are not intuitive. One may wonder how to discover the extension of Sury-type relation such as (8). Well, a good way is to look at the generating function. In this section, we obtain the generating functions for W- and w-polynomial sequences of order k and reprove our main results via the method of generating functions. Let

$$W^{(k)}(x;y) = \sum_{n=0}^{\infty} W_n^{(k)}(x) y^n,$$

and

$$w^{(k)}(x;y) = \sum_{n=0}^{\infty} w_n^{(k)}(x)y^n.$$

Proposition 1. *The generating functions for the W- and w-polynomial sequence of order k are given respectively by*

$$W^{(k)}(x;y) = \begin{cases} \dfrac{\Delta^k(x)y}{1-p(x)y-q(x)y^2} & \text{if } k \text{ is even;} \\ \dfrac{\Delta^{k-1}(x)\,(2-p(x)y)}{1-p(x)y-q(x)y^2} & \text{if } k \text{ is odd,} \end{cases} \tag{12}$$

and

$$w^{(k)}(x;y) = \begin{cases} \dfrac{\Delta^k(x)\,(2-p(x)y)}{1-p(x)y-q(x)y^2} & \text{if } k \text{ is even;} \\ \dfrac{\Delta^{k+1}(x)y}{1-p(x)y-q(x)y^2} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. From the recurrence relation, we deduce that

$$\sum_{n=2}^{\infty} W_n^{(k)}(x)y^n = p(x)y\sum_{n=2}^{\infty} W_{n-1}^{(k)}(x)y^{n-1} + q(x)y^2\sum_{n=2}^{\infty} W_{n-2}^{(k)}(x)y^{n-2}.$$

This can be rewritten as

$$W^{(k)}(x;y) - W_0^{(k)}(x) - W_1^{(k)}(x)y = p(x)y\left[W^{(k)}(x;y) - W_0^{(k)}(x)\right] + q(x)y^2W^{(k)}(x;y).$$

Hence,

$$W^{(k)}(x;y) = \frac{W_0^{(k)}(x) + \left[W_1^{(k)}(x) - p(x)W_0^{(k)}(x)\right]y}{1-p(x)y-q(x)y^2}.$$

The Equation (12) follows by substituting in the explicit values of $W_0^{(k)}(x)$ and $W_1^{(k)}(x)$ stated at the beginning of Section 1. The proof of the second equation of Proposition 1 is similar. □

Alternative proof of (8). Assume that k is an odd integer. Consider the generating function

$$\sum_{n=0}^{\infty}\left[\sum_{i=0}^{n} r^i\Delta^2(x)W_i^{(k)}(x) + (rp(x)-2)\sum_{i=0}^{n+1} r^{i-1}w_i^{(k)}(x)\right]y^n. \tag{13}$$

We have

$$\sum_{n=0}^{\infty}\left[\sum_{i=0}^{n} r^i\Delta^2(x)W_i^{(k)}(x)\right]y^n = \frac{\Delta^2(x)}{1-y}W^{(k)}(x;ry).$$

Likewise,

$$\sum_{n=0}^{\infty}\left[\sum_{i=0}^{n+1} r^i p(x)w_i^{(k)}(x)\right]y^n = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} r^{i+1}p(x)w_{i+1}^{(k)}(x)y^{i+j} = \frac{p(x)w^{(k)}(x;ry)}{y(1-y)},$$

since $w_0^{(k)}(x) = 0$ when k is odd. And,

$$\sum_{n=0}^{\infty}\left[\sum_{i=0}^{n+1}(-2)r^{i-1}w_i^{(k)}(x)\right]y^n = -2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} r^i w_{i+1}^{(k)}(x)y^{i+j} = \frac{-2w^{(k)}(x;ry)}{ry(1-y)}.$$

According to Proposition 1 we evaluate (13) as

$$\frac{(2-rp(x)y)\,\Delta^{k+1}(x)+rp(x)\Delta^{k+1}(x)-2\Delta^{k+1}(x)}{(1-y)(1-rp(x)y-r^2q(x)y^2)},$$

or

$$\frac{rp(x)\Delta^{k+1}(x)}{1-rp(x)y-r^2q(x)y^2}=\frac{p(x)w^{(k)}(x;ry)}{y}=\sum_{m=0}^{\infty}r^m p(x)w_m^{(k)}(x)y^{m-1}. \tag{14}$$

Relation (8) follows by comparing the coefficient of y^n in (13) and (14). □

Alternative proof of (9). Notice that

$$\sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}(-1)^i r^{n-i}w_{i+1}^{(k)}(x)\right]y^n=\frac{1}{1-ry}\cdot\frac{w^{(k)}(x;-y)-2\Delta^k(x)}{-y}.$$

Likewise,

$$\sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}(-1)^i r^{n-i}\left(rp(x)-2q(x)\right)W_i^{(k)}(x)\right]y^n=\frac{rp(x)-2q(x)}{1-ry}W^{(k)}(x;-y).$$

Thus by Proposition 1,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}(-1)^i r^{n-i}\left[w_{i+1}^{(k)}(x)+\left(rp(x)-2q(x)\right)W_i^{(k)}(x)\right]\right\}y^n\\
&=\frac{1}{1-ry}\cdot\frac{w^{(k)}(x;-y)-2\Delta^k(x)}{-y}+\frac{rp(x)-2q(x)}{1-ry}W^{(k)}(x;-y)\\
&=\frac{\Delta^k(x)(p(x)-2q(x)y)}{(1-ry)(1+p(x)y-q(x)y^2)}+\frac{\Delta^k(x)(-rp(x)y+2q(x)y)}{(1-ry)(1+p(x)y-q(x)y^2)}\\
&=\frac{p(x)\Delta^k(x)}{1+p(x)y-q(x)y^2}\\
&=\frac{p(x)W^{(k)}(x;-y)}{-y}\\
&=\sum_{m=0}^{\infty}(-1)^{m-1}p(x)W_m^{(k)}(x)y^{m-1}.
\end{aligned}$$

Relation (9) follows from the comparing coefficient of y^n. □

4. Conclusions

There are a lots of polynomials satisfying the recurrence relation (1) when the order is zero, such as Pell polynomial and its companion Pell-Lucas polynomial, Jacobsthal and Jacobsthal-Lucas polynomial, Chebyshev polynomial and so on. As we see in (7), the result only relates to the polynomial factor $p(x)$ but not to $q(x)$. So if we consider the Jacobsthal numbers $\{J_n\}_{n\geq 0}$ and the Jacobsthal-Lucas numbers $\{j_n\}_{n\geq 0}$, we quickly get the relation of same fashion with (4):

$$r^{n+1}J_{n+1}=\sum_{i=0}^{n}r^i j_i+(r-2)\sum_{i=0}^{n+1}r^{i-1}J_i,\ \text{for } n,r\geq 1,$$

since the recurrence relation they share is $u_{n+1}=u_n+2u_{n-1}$.

Here is another example, which involves the m-Fibonacci $\{F_{m,n}\}_{n\geq 0}$ and m-Lucas sequence $\{L_{m,n}\}_{n\geq 0}$. Both sequences are defined recursively by $u_{n+1} = mu_n + u_{n-1}$ for $n \geq 1$, with respective initial conditions $F_{m,0} = 0, F_{m,1} = 1$, and $L_{m,0} = 2, L_{m,1} = m$. By (7), we get a Sury-type identity [8],

$$mr^{n+1}F_{m,n+1} = \sum_{i=0}^{n} r^i L_{m,i} + (mr-2)\sum_{i=0}^{n+1} r^{i-1}F_{m,i}.$$

We remark that our Theorem 1 is actually equivalent to the Theorem 2. To see this, we substitute r in (7) for $-1/r$ and then use Lemma 1 to obtain (9).

In addition, one can obtain immediately, by (7), the divisibility relation between polynomials

$$p(x) \;\Big|\; \sum_{i=0}^{n} r^i \left(w_i^{(k)}(x) - 2W_{i+1}^{(k)}(x)\right), \text{ for } k \text{ is even and any integers } n, r \geq 1.$$

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