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Article

On Stability of Iterative Sequences with Error

Salwa Salman Abed [1,*] and Noor Saddam Taresh [2]

1 Department of Mathematics, College of Education, Ibn Al-Haitham, University of Baghdad, Baghdad 10001, Iraq
2 Ministry of Higher Education and Scientific Research, Baghdad 10001, Iraq
* Correspondence: Salwa.s.a@ihcoedu.uobaghdad.edu.iq

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Abstract: Iterative methods were employed to obtain solutions of linear and non-linear systems of equations, solutions of differential equations, and roots of equations. In this paper, it was proved that s-iteration with error and Picard–Mann iteration with error converge strongly to the unique fixed point of Lipschitzian strongly pseudo-contractive mapping. This convergence was almost F-stable and F-stable. Applications of these results have been given to the operator equations $Fx = f$ and $x + Fx = f$, where F is a strongly accretive and accretive mappings of X into itself.

Keywords: Banach space; iterative sequences; stability; fixed points

1. Introduction and Preliminaries

Consider a normed space X, $F: X \to X$ is a mapping, M is an iteration procedure and $\lambda_n, \eta_n \in (0,1)$, we present the following iterative sequences.

$w_0 \in X$,

$$w_{n+1} = M(F,\ w_n),$$

is called s-iteration [1] if:

$$\begin{aligned} w_{n+1} &= \lambda_n F z_n + (1-\lambda_n) F w_n, \\ z_n &= \eta_n F w_n + (1-\eta_n) w_n,\ \forall\, n \geq 0. \end{aligned} \tag{1}$$

$x_0 \in X$,

$$x_{n+1} = M(F,\ x_n)$$

is called Picard–Mann iteration [2] if:

$$\begin{aligned} x_{n+1} &= F y_n \\ y_n &= \lambda_n F x_n + (1-\lambda_n) x_n, \forall\, n \geq 0. \end{aligned} \tag{2}$$

$w_0 \in X$,

$$w_{n+1} = M(F,\ w_n),$$

is called s-iteration with errors if

$$\begin{aligned} w_{n+1} &= \lambda_n F z_n + (1-\lambda_n) F w_n + a_n, \\ z_n &= \eta_n F w_n + (1-\eta_n) w_n + c_n,\ \forall\, n \geq 0. \end{aligned} \tag{3}$$

where $\sum_{n=0}^{\infty} \|a_n\| < \infty,\ \sum_{n=0}^{\infty} \|c_n\| < \infty$.

$x_0 \in X$,

$$x_{n+1} = M(F,\ x_n)$$

is called Picard–Mann iteration with errors if:

$$x_{n+1} = Fy_n + a_n \\ y_n = \lambda_n Fx_n + (1 - \lambda_n)x_n, \forall\, n \geq 0. \tag{4}$$

where $\sum_{n=0}^{\infty} \|a_n\| < \infty$.

Throughout this paper, we studied three cases: convergence, almost stability, and stability of schemes of sequences defined in Equations (3) and (4). In the following, we recall the needed definitions and lemmas.

Definition 1 ([3]). *Let $x_{n+1} = M(F, x_n)$ be an arbitrary iteration procedure such that $\{x_n\}$ converges to a fixed point p of F. For a sequence $\{q_n\}$ suppose that*

$$\delta_n = \|q_{n+1} - M(F, x_n)\|, n \geq 0.$$

Then the iteration procedure is said to be F–stable if $\lim_{n\to\infty} \delta_n = 0$, implies to $\lim_{n\to\infty} q_n = p$.

Definition 2 ([4]). *Let F, $\{x_{n+1}\}$, δ_n, q_n, and p be as shown in Definition 1. Then, the iteration procedure is said to be almost F-stable if $\sum_{n=0}^{\infty} \delta_n < \infty$ implies that $\lim_{n\to\infty} q_n = p$.*

Definition 3 ([5]). *Let X be a normed space and $F: X \to X$ be a mapping then for fixed m, $0 \leq m < \infty$, F* is said to be Lipschitzian if:

$$\|Fx - Fy\| \leq m\, \|x - y\| \;\; \forall x,\; y \in X. \tag{5}$$

Let X' be the dual of X, a set valued mapping $J: X \to 2^{X'}$ is said to be the normalized duality mapping [5] if:

$$J(x) = \{j \in X' :\; \langle x,\, j\rangle = \|j\|\|x\|,\; \|j\| = \|x\|\},\; \forall\, x \in X$$

where $\langle\, ,\rangle$ denotes the duality pairing, i.e., $\langle\, ,\rangle : X \times X' \to K$, $\langle x,\, j\rangle = j(x)$.

It is known that a Banach space X is smooth if and only if the duality mapping J is single [5].

Definition 4 ([6]). *Let X be a normed space, $F: X \to X$ be a mapping. Then, F is called strongly pseudo-contractive if for all x, $y \in X$, the following inequality holds:*

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Fx - Fy)\| \tag{6}$$

$\forall r > 0$ and some $t > 1$.

Or equivalently [7], if there exist $\mathfrak{r} = \frac{1}{l}$, where, $l > 1$ such that

$$\langle Fx - Fy, j(x - y)\rangle \leq \mathfrak{r}\|x - y\|^2, \forall x, y \in X.$$

If $t = 1$ in inequality (6), then F is called pseudo-contractive.

Definition 5 ([8]). *A mapping $F: X \to X$ is said to be*

i- *Strongly accretive, if there is $\mathfrak{r} > 0$ such that for each $x, y \in X$there exists $j(x - y) \in J(x - y)$*

$$\langle Fx - Fy, j(x - y)\rangle \geq \mathfrak{r}\|x - y\|^2. \tag{7}$$

ii- *Accretive, if $\mathfrak{r} = 0$ in Equation (7).*

Or equivalently [9]

$$\|x - y\| \leq \|x - y + \mathfrak{r}(Fx - Fy)\|, \text{ for some } \mathfrak{r} > 0 \tag{8}$$

Proposition 1 ([10]). *The relation between (strong) pseudo-contractive mapping and (strong) accretive mapping is that: F is (strong) pseudo-contractive if and only if $(I - F)$ is (strong) accretive.*

Lemma 1 ([11]). *Let $\{\rho_n\}$ be a non-negative sequence such that, $\rho_{n+1} \le (1-\gamma_n)\rho_n + \mu_n$, where $\gamma_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum \gamma_n = \infty$, and $\mu_n = o(\gamma_n)$. Then $\lim\limits_{n\to\infty} \rho_n = 0$.*

A general version of Lemma 1 is:

Lemma 2 ([12]). *Let $\{\xi_n\}$ be a non-negative sequence such that $\xi_{n+1} \le (1-\gamma_n)\xi_n + b_n + \mu_n$, $n \ge 0$, where $\gamma_n \in [0,1]$, $\forall n \in \mathbb{N}$, $\sum \gamma_n = \infty$, and $b_n = o(\gamma_n)$, $\sum_{n=0}^{\infty} \mu_n < \infty$. Then $\lim\limits_{n\to\infty} \xi_n = 0$.*

Lemma 3 ([13,14]). *Let X be a real Banach space, $F : X \to X$ be a mapping*

i- *If F is continuous and strongly pseudo-contractive, then F* has a unique fixed point.
ii- *If F is continuous and strongly accretive, then the equation $Fx = f$ has a unique solution for any $f \in X$.*
iii- *If F is continuous and accretive, then F is m-accretive and the equation $x + Fx = f$ has a unique solution for any $f \in X$.*

For more details about previous preliminaries and to determine the important aspects of the convergence of iterative sequences, we recommend the book by C. Chidume [5] and the paper by B.E. Rhoades and L. Saliga [15].

2. Main Results

The following condition is needed:
(Δ_1): If $\lambda_n, \eta_n \in (0,1)$, $\mathfrak{r} \in (0,1)$ and $m > 0$, then
$m\Big((m+1)(1+\eta_n) + \lambda_n m^2(2 + (m-1)\eta_n)\Big) - (2-\mathfrak{r})\lambda_n(2m + m(m-1)\eta_n) \;\le\; \mathfrak{r}m - e$, where $e \in (0, m)$.

Theorem 1. *Let X be a real Banach space and $F : X \to X$ be Lipschitzian strongly pseudo-contractive mapping with Lipschitz constant m. Suppose that $\{w_n\}$ be in (3), $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = 0$ and (Δ_1) is verified. Then:*

1- *$\{w_n\}$ converges strongly to the unique fixed point p.*
2- $||q_{n+1} - p|| \le \delta_n + ||a_n|| + (1 - \frac{\lambda_n e}{1+\lambda_n})||q_n - p|| + (3m + m^2)||c_n||$, $\forall n \ge 0$.

Proof. From Lemma 3, we obtain that F has a unique fixed point, and from Equations (3), (6), and Proposition 1 we have:

$$\begin{aligned} \mathrm{F}w_n &= w_{n+1} + \lambda_n \mathrm{F}w_n - \lambda_n \mathrm{F}z_n - a_n \\ &= w_{n+1} + \lambda_n \mathrm{F}w_n - \lambda_n \mathrm{F}z_n - a_n + 2\lambda_n w_{n+1} - 2\lambda_n w_{n+1} - \mathfrak{r}\lambda_n w_{n+1} \\ &\quad + \mathfrak{r}\lambda_n w_{n+1} - \lambda_n \mathrm{F}w_{n+1} + \lambda_n \mathrm{F}w_{n+1} \\ &= (1+\lambda_n)w_{n+1} + \lambda_n(\mathrm{I} - \mathrm{F} - \mathrm{rI})w_{n+1} - (1-\mathrm{r})\lambda_n \mathrm{F}w_n + (2-\mathrm{r})\lambda_n^2(\mathrm{F}w_n - \mathrm{F}z_n) + \lambda_n(\mathrm{F}w_{n+1} - \mathrm{F}z_n) \\ &\qquad -(1 + (2-\mathrm{r})\lambda_n)a_n \end{aligned} \tag{9}$$

Let p be a fixed point of F:

$$\mathrm{p} = (1+\lambda_n)\mathrm{p} + \lambda_n(\mathrm{I} - \mathrm{F} - \mathfrak{r}\mathrm{I})\mathrm{p} - (1-\mathfrak{r})\lambda_n \mathrm{p} \tag{10}$$

$$\begin{gathered} \mathrm{F}w_n - \mathrm{p} = (1+\lambda_n)(w_{n+1} - \mathrm{p}) + \lambda_n[(\mathrm{I} - \mathrm{F} - \mathfrak{r}\mathrm{I})w_{n+1} - (\mathrm{I} - \mathrm{F} - \mathfrak{r}\mathrm{I})\mathrm{p}] - \\ (1-\mathfrak{r})\lambda_n(\mathrm{F}w_n - \mathrm{p}) + (2-\mathfrak{r})\lambda_n^2(\mathrm{F}w_n - \mathrm{F}z_n) + \lambda_n(\mathrm{F}w_{n+1} - \mathrm{F}z_n) - (1 + (2-\mathfrak{r})\lambda_n) \end{gathered} \tag{11}$$

$$\|\mathrm{Fw}_n - \mathrm{p}\| \geq (1+\lambda_n)\|(\mathrm{w}_{n+1}-\mathrm{p}) + \frac{\lambda_n}{1+\lambda_n}[(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{w}_{n+1} - (\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{p}]\| \\ -(1-\mathrm{r})\lambda_n\|\mathrm{Fw}_n-\mathrm{p}\| - (2-\mathrm{r})\lambda_n^2\|\mathrm{Fw}_n - \mathrm{F}z_n\| - \lambda_n\|\mathrm{Fw}_{n+1}-\mathrm{F}z_n\| \\ -3\|a_n\|$$

Thus:

$$(1+\lambda_n)\|\mathrm{w}_{n+1}-\mathrm{p}\| \\ \leq (1+(1-\mathrm{r})\lambda_n)\|\mathrm{Fw}_n-\mathrm{p}\| + (2-\mathrm{r})\lambda_n^2\|\mathrm{Fw}_n-\mathrm{F}z_n\| + \lambda_n\|\mathrm{Fw}_{n+1}-\mathrm{F}z_n\| \\ +3\|a_n\| \\ \|\mathrm{w}_{n+1}-\mathrm{p}\| \leq \frac{1}{1+\lambda_n}[(1+(1-\mathrm{r})\lambda_n)\|\mathrm{Fw}_n-\mathrm{p}\| + (2-\mathrm{r})\lambda_n^2\|\mathrm{Fw}_n-\mathrm{F}z_n\| \\ +\lambda_n\|\mathrm{Fw}_{n+1}-\mathrm{F}z_n\| + 3\|a_n\|] \\ \|\mathrm{w}_{n+1}-\mathrm{p}\| \leq \frac{1}{1+\lambda_n}[(1+(1-\mathrm{r})\lambda_n)m\|\mathrm{w}_n-\mathrm{p}\| + (2-\mathrm{r})\lambda_n^2\|\mathrm{Fw}_n-\mathrm{F}z_n\| \\ +\lambda_n\|\mathrm{Fw}_{n+1}-\mathrm{F}z_n\| + 3\|a_n\|] \tag{12}$$

Observe that

$$\|\mathrm{Fw}_n-\mathrm{F}z_n\| \leq \|\mathrm{Fw}_n-\mathrm{p}\| + \|\mathrm{p}-\mathrm{F}z_n\| \leq m\|\mathrm{w}_n-\mathrm{p}\| + m\|z_n-\mathrm{p}\| \\ \leq 2m + m(m-1)\eta_n\|\mathrm{w}_n-\mathrm{p}\| + m\|c_n\| \tag{13}$$

$$\|\mathrm{Fw}_{n+1}-\mathrm{F}z_n\| \leq m\|\mathrm{w}_{n+1}-z_n\| \leq [m(m+1)+\lambda_n m(2m+m(m-1)\eta_n\|\mathrm{w}_n- \\ \mathrm{p}\|) + \eta_n m(m+1)]\|\mathrm{w}_n-\mathrm{p}\| + m\|a_n\| + m\|c_n\| + \lambda_n m^2\|c_n\| \tag{14}$$

By substituting Equations (14) and (13) in (12), we get:

$$\|\mathrm{w}_{n+1}-\mathrm{p}\| \leq \frac{1}{1+\lambda_n}[(1+(1-\mathrm{r})\lambda_n)m\|\mathrm{w}_n-\mathrm{p}\| \\ +\lambda_n([m(m+1)+\lambda_n m(2m+m(m-1)\eta_n)+\eta_n m(m+1)]\|\mathrm{w}_n-\mathrm{p}\| \\ +m\|a_n\| + m\|c_n\| + \lambda_n m^2\|c_n\|) + \ (2-\mathrm{r})\lambda_n^2((2m+m(m \\ -1)\eta_n\|\mathrm{w}_n-\mathrm{p}\| + m\|c_n\|) + 3\|a_n\|] \\ = \frac{1}{1+\lambda_n}[(1+(1-\mathrm{r})\lambda_n)m + \lambda_n m(m+1)(1+\eta_n) + 2\lambda_n^2 m^2 + \lambda_n^2 m^2(m-1)\eta_n \\ +(2-\mathrm{r})\lambda_n^2((2m+m(m-1)\eta_n]\|\mathrm{w}_n-\mathrm{p}\| + \left[\frac{\lambda_n}{1+\lambda_n}m + \frac{3}{1+\lambda_n}\right]\|a_n\| \\ +\left[\frac{(2-\mathrm{r})\lambda_n^2}{1+\lambda_n}m + \frac{\lambda_n^2}{1+\lambda_n}\left(m^2+m\right)\right]\|c_n\| \\ \leq [1 - \frac{\lambda_n}{1+\lambda_n}[m\mathrm{r} - m\Big((m+1)(1+\eta_n) + \lambda_n m^2(2+(m-1)\eta_n)\Big) \\ + -(2-\mathrm{r})\lambda_n((2m+m(m-1)\eta_n]\|\mathrm{w}_n-\mathrm{p}\| + [m+3]\|a_n\| \\ +\left[3m+m^2\right]\|c_n\| \\ = [1 - \frac{\lambda_n e}{1+\lambda_n}\|\mathrm{w}_n-\mathrm{p}\| + [m+3]\|a_n\| + \left[3m+m^2\right]\|c_n\|$$

Lemma 1 yied to $\lim_{n\to\infty} \mathrm{w}_n = \mathrm{p}$.
For part(2):
Let $\{\mathrm{q}_n\}$ be a sequence in X, defined $\{\delta_n\}$ by $\delta_n = \|\mathrm{q}_{n+1} - \mathfrak{g}_n - a_n\|$, where

$$\mathfrak{g}_n = \lambda_n \mathrm{F}z_n + (1-\lambda_n)\mathrm{Fq}_n, z_n = \eta_n\mathrm{Fq}_n + (1-\eta_n)\mathrm{q}_n + c_n \ , n \geq 0.$$

$$\|\mathrm{q}_{n+1}-\mathrm{p}\| \leq \|\mathrm{q}_{n+1}-\mathfrak{g}_n-a_n\| + \|a_n\| + \|\mathfrak{g}_n-\mathrm{p}\| \leq \delta_n + \|a_n\| + \|\mathfrak{g}_n-\mathrm{p}\| \tag{15}$$

Since:

$$\mathrm{Fq}_n = \mathfrak{g}_n + \lambda_n\mathrm{Fq}_n - \lambda_n\mathrm{F}z_n \\ = (1+\lambda_n)\mathfrak{g}_n + \lambda_n(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathfrak{g}_n - (2-\mathrm{r})\lambda_n\mathfrak{g}_n + \lambda_n\mathrm{Fq}_n + \lambda_n(\mathrm{F}\mathfrak{g}_n - \mathrm{F}z_n) \\ = (1+\lambda_n)\mathfrak{g}_n + \lambda_n(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathfrak{g}_n - (1-\mathrm{r})\lambda_n\mathrm{Fq}_n + (2-\mathrm{r})\lambda_n^2\left(\mathrm{Fq}_n - \mathrm{F}z_n\right) \\ +\lambda_n(\mathrm{F}\mathfrak{g}_n - \mathrm{F}z_n) \tag{16}$$

Thus:

$$\mathrm{p} = (1+\lambda_n)\mathrm{p} + \lambda_n(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{p} - (1-\mathrm{r})\lambda_n\mathrm{p} \tag{17}$$

$$\mathrm{Fq}_n - \mathrm{p} = (1+\lambda_n)(\mathfrak{g}_n - \mathrm{p}) + \lambda_n[(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathfrak{g}_n - (\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{p}] - (1-\mathrm{r})\lambda_n\left(F\mathrm{q}_n - \mathrm{p}\right) \\ +(2-\mathrm{r})\lambda_n^2\left(F\mathrm{q}_n - \mathrm{F}z_n\right) + \lambda_n(\mathrm{F}\mathfrak{g}_n - \mathrm{F}z_n).$$

So that:

$$\begin{aligned} \|\mathrm{Fq}_n - \mathrm{p}\| &\geq (1+\lambda_n)\|(\mathfrak{g}_n - \mathrm{p}) + \tfrac{\lambda_n}{1+\lambda_n}[(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathfrak{g}_n - (\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{p}]\| \\ &-(1-\mathrm{r})\lambda_n \times \|\mathrm{Fq}_n - \mathrm{p}\| - (2-\mathrm{r})\lambda_n^2\|F\mathrm{q}_n - \mathrm{F}z_n\| - \lambda_n\|\mathrm{F}\mathfrak{g}_n - \mathrm{F}z_n\| \\ &\geq (1+\lambda_n)\|\mathfrak{g}_n - \mathrm{p}\| - (1-\mathrm{r})\lambda_n\|F\mathrm{q}_n - \mathrm{p}\| - (2-\mathrm{r})\lambda_n^2\|\mathrm{Fq}_n - \mathrm{F}z_n\| \\ &-\lambda_n\|\mathrm{F}\mathfrak{g}_n - \mathrm{F}z_n\| \end{aligned}$$

Thus:

$$\begin{aligned} \|\mathfrak{g}_n - \mathrm{p}\| &\leq \tfrac{1}{1+\lambda_n}\Big[(1+(1-\mathfrak{r})\lambda_n)\|\mathrm{Fq}_n - \mathrm{p}\| + (2-\mathfrak{r})\lambda_n^2\|\mathrm{Fq}_n - \mathrm{F}z_n\| + \lambda_n\|\mathrm{F}\mathfrak{g}_n - \mathrm{F}z_n\|\Big] \\ \|\mathfrak{g}_n - \mathrm{p}\| &\leq \tfrac{1}{1+\lambda_n}\Big[(1+(1-\mathfrak{r})\lambda_n)m\|\mathrm{q}_n - \mathrm{p}\| + (2-\mathfrak{r})\lambda_n^2\|\mathrm{Fq}_n - \mathrm{F}z_n\| + \lambda_n\|\mathrm{F}\mathfrak{g}_n - \mathrm{F}z_n\|\Big] \end{aligned} \tag{18}$$

Observe that

$$\begin{aligned} \|\mathrm{Fq}_n - \mathrm{F}z_n\| &\leq \|\mathrm{Fq}_n - \mathrm{p}\| + \|\mathrm{p} - \mathrm{F}z_n\| \leq m\|\mathrm{q}_n - \mathrm{p}\| + m\|z_n - \mathrm{p}\| \\ &\leq 2m + m(m-1)\eta_n\|\mathrm{q}_n - \mathrm{p}\| + m\|c_n\| \end{aligned} \tag{19}$$

$$\begin{aligned} \|\mathrm{F}\mathfrak{g}_n - \mathrm{F}z_n\| &\leq m\|\mathfrak{g}_n - z_n\| \leq m\Big[\|\mathrm{Fq}_n - \mathrm{q}_n\| + \lambda_n\|\mathrm{Fq}_n - \mathrm{F}z_n\| + \eta_n\|\mathrm{q}_n - \mathrm{Fq}_n\| + \|c_n\|\Big] \\ &\leq [m(m+1) + \lambda_n m(2m + m(m-1)\eta_n) + \eta_n m(m+1)]\|\mathrm{q}_n - \mathrm{p}\| \\ &+m\|c_n\| \end{aligned} \tag{20}$$

By substituting Equations (20) and (19) in (18), we get:

$$\begin{aligned} \|\mathfrak{g}_n - \mathrm{p}\| &\leq \tfrac{1}{1+\lambda_n}[(1+(1-\mathfrak{r})\lambda_n)m\|\mathrm{q}_n - \mathrm{p}\| + \lambda_n([m(m+1) + \lambda_n m(2m + m(m-1)\eta_n) + \\ &\quad \eta_n m(m+1)]\|\mathrm{q}_n - \mathrm{p}\| + m\|c_n\| + \lambda_n m^2\|c_n\|) + (2-\mathfrak{r})\lambda_n^2 \\ &\quad \left(2m + m(m-1)\eta_n\|\mathrm{q}_n - \mathrm{p}\| + m\|c_n\|\right)] \\ &= \tfrac{1}{1+\lambda_n}[(1+(1-\mathfrak{r})\lambda_n)m + \lambda_n m(m+1)(1+\eta_n) + 2\lambda_n^2 m^2 + \lambda_n^2 m^2(m-1)\eta_n + \\ &(2-\mathfrak{r})\lambda_n^2((2m + m(m-1)\eta_n]\|\mathrm{q}_n - \mathrm{p}\| + \left[\tfrac{(2-\mathfrak{r})\lambda_n^2}{1+\lambda_n}m + \tfrac{\lambda_n^2}{1+\lambda_n}\left(m^2 + m\right)\right]\|c_n\| \\ &\leq [1 - \tfrac{\lambda_n}{1+\lambda_n}[m\mathfrak{r} - m\left((m+1)(1+\eta_n) + \lambda_n m^2(2 + (m-1)\eta_n)\right) \\ &\quad +(2-\mathfrak{r})\lambda_n((2m + m(m-1)\eta_n]\|\mathrm{q}_n - \mathrm{p}\| \\ &\quad +\left[(2-\mathfrak{r})\lambda_n^2 m + \left(m^2 + m\right)\lambda_n^2\right]\|c_n\| \\ &= [1 - \tfrac{\lambda_n e}{1+\lambda_n}\|\mathrm{q}_n - \mathrm{p}\| + \left[3m + m^2\right]\|c_n\|\|\|c_n\|. \end{aligned} \tag{21}$$

Substituting Equation (21) in (15) we obtain:

$$\begin{aligned} \|\mathrm{q}_{n+1} - \mathrm{p}\| &\leq \|\mathrm{q}_{n+1} - \mathfrak{g}_n - a_n\| + \|a_n\| + \|\mathfrak{g}_n - \mathrm{p}\| \leq \delta_n + \|a_n\| + \|\mathfrak{g}_n - \mathrm{p}\| \\ &\leq \delta_n + \|a_n\| + \left[1 - \tfrac{\lambda_n e}{1+\lambda_n}\right]\|\mathrm{q}_n - \mathrm{p}\| + \left[3m + m^2\right]\|c_n\|. \end{aligned} \tag{22}$$

□

Theorem 2. *Assume that X, F, p, m, $\{w_n\}$, $\{z_n\}$, $\{q_n\}$, $\{\lambda_n\}$, $\{\eta_n\}$, and $\{\delta_n\}$ be as in Theorem 1 and (Δ_1) is satisfied. Then the sequence (3) is almost F-stable.*

Proof. Assume that $\sum_{n=0}^{\infty} \delta_n < \infty$. Then, we prove that $\lim_{n\to\infty} \mathrm{q}_n = \mathrm{p}$.

Now, using Equation (22) such that $\xi_n = \|\mathrm{q}_n - \mathrm{p}\|, \gamma_n = \frac{\lambda_n e}{1+\lambda_n}$, $b_n = [3m + m^2]\|c_n\| + \|a_n\|$, and $\mu_n = \delta_n$, $\forall$ n ≥ 0.

Note that $\lim_{n\to\infty} \mathrm{b}_n = 0$, thus Lemma (1.8) holds, such that $\lim_{n\to\infty} \xi_n = 0$ yields $\lim_{n\to\infty} \mathrm{q}_n = \mathrm{p}$. □

Theorem 3. *Let X, F, p, m, $\{q_n\}$, $\{\lambda_n\}$, $\{\eta_n\}$, $\{a_n\}$, $\{c_n\}$, and $\{\delta_n\}$ be as in Theorem 1 and (Δ_1) is satisfied. Then $\{w_n\}$ is F-stable.*

Proof. Suppose that $\lim_{n\to\infty} \delta_n = 0$, then by applying Lemma 1 on (22) of Theorem 1, we obtain $\lim_{n\to\infty} q_n = p$. □

Example 1. *Let $X = (0,1])$, $F : X \to X$ by $Fx = \frac{x}{2}$, hence, the conditions in Equations (5) and (6) are satisfied as shown below.*

$$\begin{aligned} \|Fx - Fy\| = \|\tfrac{x}{2} - \tfrac{y}{2}\| \le \tfrac{1}{2}\|x - \ y\| \langle Fx - Fy, j(x-y)\rangle \le r\|x-y\|^2 \le (Fx - Fy)(x-y) \\ \le \left|\tfrac{x}{2} - \tfrac{y}{2}\right| |x-y| = \tfrac{1}{2}\|x-y\|^2 \end{aligned}$$

Now, put $\lambda_n = \frac{1}{2}$, $q_n = \frac{1}{n}$, $\forall\ n \ge 0$, since $\lim_{n\to\infty} q_n = 0$, to show that $\lim_{n\to\infty} \delta_n = p = 0$.

$$\begin{aligned} \delta_n &= \|q_{n+1} - x_{n+1}\| = \|q_{n+1} - Fq_n + a_n\| = \|\tfrac{1}{n+1} - \tfrac{q_n}{2}\| \\ &= \|\tfrac{1}{n+1} - \tfrac{(1-\lambda_n)}{2} q_n - \tfrac{\lambda_n}{2}\tfrac{q_n}{2}\| \\ &= \|\tfrac{1}{n+1} - \tfrac{1}{4n} - \tfrac{1}{8n}\| \Longrightarrow \lim_{n\to\infty} \delta_n = 0. \end{aligned}$$

Corollary 1. *Let X, F, p, m, $\{q_n\}$, $\{\lambda_n\}$, $\{\eta_n\}$, $\{a_n\}$, $\{c_n\}$, $\{\delta_n\}$ be as in Theorem 1, and $\{w_n\}$ defined by Equation (1), then $\{w_n\}$:*

1. *converges strongly to the unique fixed point p.*
2. *is almost F-stable*
3. *is F-stable.*

To prove the next results, we replace the inequality in the condition (Δ_1) by

$$(\Delta_2): \ m\Big(1 + m^2 + \lambda_n(1+m)\Big) \le rm^2 - e$$

Theorem 4. *Suppose that X is a real Banach space $F : X \to X$ is Lipschitzian strongly pseudo-contractive mapping with Lipschitz constant m. For $w_0 \in X$, let $\{x_n\}$ be in Equation (4), $\lim_{n\to\infty} a_n = 0$ (Δ_2) is satisfied. Then:*

1- $\{x_n\}$ *converges strongly to the unique fixed point p.*

2- $\|q_{n+1} - p\| \le \delta_n + [1 - \frac{\lambda_n e}{1+\lambda_n}]\|q_n - p\| + \|a_n\|, \ \forall n \ge 0.$

Proof. From Lemma 3, we obtained that F has a unique fixed point.

$$\begin{aligned} Fy_n &= x_{n+1} - a_n \\ &= x_{n+1} + 2\lambda_n x_{n+1} - 2\lambda_n x_{n+1} - r\lambda_n x_{n+1} + r\lambda_n Fx_{n+1} - \ \lambda_n Fx_{n+1} \\ &\quad + \lambda_n Fx_{n+1} - a_n \\ &= (1+\lambda_n)x_{n+1} + \lambda_n(I - F - rI)x_{n+1} + \lambda_n\Big(Fx_{n+1} - Fy_n\Big) - (1-r)\lambda_n Fy_n \\ &\quad -(1 + (2-r)\lambda_n)a_n \end{aligned} \tag{23}$$

$$= (1+\lambda_n)p + \lambda_n(I - F - rI)p - (1-r)\lambda_n p \tag{24}$$

So that:

$$\begin{aligned}\mathrm{F}y_n - \mathrm{p} = (1+\lambda_n)(x_{n+1}-\mathrm{p}) + \lambda_n[(\mathrm{I}-\mathrm{F}-\mathrm{rI})x_{n+1} - (\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{p}] - (1-\mathrm{r})\lambda_n\big(\mathrm{F}y_n - \mathrm{p}\big) \\ +\lambda_n\big(\mathrm{F}x_{n+1} - \mathrm{F}y_n\big) - (1+(2-\mathrm{r})\lambda_n)a_n \\ \|\mathrm{F}y_n - \mathrm{p}\| \geq (1+\lambda_n)\|(x_{n+1}-\mathrm{p}) + \tfrac{\lambda_n}{1+\lambda_n}[(\mathrm{I}-\mathrm{F}-\mathrm{rI})x_{n+1} - (\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{p}]\| \\ -(1-\mathrm{r})\lambda_n\|\mathrm{F}y_n - \mathrm{p}\| - \lambda_n\|\mathrm{F}x_{n+1} - \mathrm{F}y_n\| - 3\|a_n\|\end{aligned}$$

Thus:

$$\begin{aligned}(1+\lambda_n)\|x_{n+1}-\mathrm{p}\| \leq (1+(1-\mathrm{r})\lambda_n)\|\mathrm{F}y_n - \mathrm{p}\| + \lambda_n\|\mathrm{F}x_{n+1} - \mathrm{F}y_n\| + 3\|a_n\| \\ \|x_{n+1}-\mathrm{p}\| \leq \tfrac{1}{1+\lambda_n}\Big[(1+(1-\mathrm{r})\lambda_n)\|\mathrm{F}y_n - \mathrm{p}\| + \lambda_n\|\mathrm{F}x_{n+1} - \mathrm{F}y_n\| + 3\|a_n\|\Big]\end{aligned} \tag{25}$$

Observe that:

$$\begin{aligned}\|\mathrm{F}y_n - \mathrm{p}\| &\leq m[(1-\lambda_n)\|x_n - \mathrm{p}\| + \lambda_n\|\mathrm{F}x_n - p\|] = m(1-\lambda_n + m\lambda_n)\|x_n - \mathrm{p}\| \\ &\leq m^2\|x_n - \mathrm{p}\|\end{aligned} \tag{26}$$

Since $1 \leq m$ yields $(1-\lambda_n + m\lambda_n) \leq m$

$$\begin{aligned}\|\mathrm{F}x_{n+1} - \mathrm{F}y_n\| \leq m\|x_{n+1} - y_n\| \leq m\Big[\|\mathrm{F}y_n - x_n\| + \lambda_n\|x_n - \mathrm{F}x_n\| + \|a_n\|\Big] \\ = m\Big[\big(1+m^2+\lambda_n(1+m)\big)\|x_n - \mathrm{p}\| + \|a_n\|\Big]\end{aligned} \tag{27}$$

By substituting Equations (27) and (26) in (25), we yielded:

$$\begin{aligned}\|x_{n+1}-\mathrm{p}\| \leq \tfrac{1}{1+\lambda_n}\Big[[(1+(1-\mathrm{r})\lambda_n)m^2 + \lambda_n m\Big[\big(1+m^2+\lambda_n(1+m)\big)\Big]\|x_n - \mathrm{p}\| + \|a_n\|\Big] \\ +3\|a_n\|] \\ \|x_{n+1}-\mathrm{p}\| = \tfrac{1}{1+\lambda_n}[(1+(1-\mathrm{r})\lambda_n)m^2 + \lambda_n m\Big(\big(1+m^2+\lambda_n(1+m)\big)\Big]\|x_n - \mathrm{p}\| \\ +\Big[\tfrac{\lambda_n}{1+\lambda_n}m + \tfrac{3}{1+\lambda_n}\Big]\|a_n\| \\ \leq [1 - \tfrac{\lambda_n}{1+\lambda_n}[m^2\mathrm{r} - m\big(1+m^2+\lambda_n(1+m)\big)]\|x_n - \mathrm{p}\| + [m+3]\|a_n\| \\ = [1 - \tfrac{\lambda_n e}{1+\lambda_n}\|x_n - \mathrm{p}\| + [m+3]\|a_n\|\end{aligned}$$

By applying Lemma 1, we get $\lim\limits_{n\to\infty} x_n = \mathrm{p}$.

For prove part (2):

Let $\{\mathrm{q}_n\} \subset \mathrm{X}$, defined $\{\delta_n\}$ by $\delta_n = \|\mathrm{q}_{n+1} - \mathfrak{g}_n - a_n\|$, where

$$\begin{aligned}\mathfrak{g}_n = \mathrm{F}y_n,\ y_n = \lambda_n\mathrm{F}\mathrm{q}_n + (1-\lambda_n)\mathrm{q}_n + c_n\ , n \geq 0. \\ \|\mathrm{q}_{n+1} - \mathrm{p}\| \leq \|\mathrm{q}_{n+1} - \mathfrak{g}_n - a_n\| + \|a_n\| + \|\mathfrak{g}_n - \mathrm{p}\| \leq \delta_n + \|a_n\| + \|\mathfrak{g}_n - \mathrm{p}\|\end{aligned} \tag{28}$$

Since:

$$\begin{aligned}\mathrm{F}y_n = \mathfrak{g}_n &= \mathfrak{g}_n + 2\lambda_n\mathfrak{g}_n - 2\lambda_n\mathfrak{g}_n - \mathrm{r}\lambda_n\mathfrak{g}_n + \mathrm{r}\lambda_n\mathrm{F}\mathfrak{g}_n - \lambda_n\mathrm{F}\mathfrak{g}_n + \lambda_n\mathrm{F}\mathfrak{g}_n \\ &= (1+\lambda_n)\mathfrak{g}_n + \lambda_n(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathfrak{g}_n - (2-\mathrm{r})\lambda_n\mathrm{F}y_n + \lambda_n\mathrm{F}\mathfrak{g}_n \\ &= (1+\lambda_n)\mathfrak{g}_n + \lambda_n(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathfrak{g}_n + \lambda_n\big(\mathrm{F}\mathfrak{g}_n - \mathrm{F}y_n\big) - (1-\mathrm{r})\lambda_n\mathrm{F}y_n\end{aligned} \tag{29}$$

$$= (1+\lambda_n)\mathrm{p} + \lambda_n(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{p} - (1-\mathrm{r})\lambda_n\mathrm{p} \tag{30}$$

So that:

$$\begin{aligned}\mathrm{F}y_n - \mathrm{p} = (1+\lambda_n)(\mathfrak{g}_n - \mathrm{p}) + \lambda_n[(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathfrak{g}_n - (\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{p}] - (1-\mathrm{r})\lambda_n\big(\mathrm{F}y_n - \mathrm{p}\big) \\ +\lambda_n\big(\mathrm{F}\mathfrak{g}_n - \mathrm{F}y_n\big) \\ \|\mathrm{F}y_n - \mathrm{p}\| \geq (1+\lambda_n)\|(\mathfrak{g}_n - \mathrm{p}) + \tfrac{\lambda_n}{1+\lambda_n}[(\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathfrak{g}_n - (\mathrm{I}-\mathrm{F}-\mathrm{rI})\mathrm{p}]\| \\ -(1-\mathrm{r})\lambda_n\|\mathrm{F}y_n - \mathrm{p}\| - \lambda_n\|\mathrm{F}\mathfrak{g}_n - \mathrm{F}y_n\| \\ \geq (1+\lambda_n)\|\mathfrak{g}_n - \mathrm{p}\| - (1-\mathrm{r})\lambda_n\|\mathrm{F}y_n - \mathrm{p}\| - \lambda_n\|\mathrm{F}\mathfrak{g}_n - \mathrm{F}y_n\|\end{aligned}$$

This implies that:

$$\|g_n - p\| \leq \frac{1}{1+\lambda_n}\Big[(1+(1-r)\lambda_n)\|Fy_n - p\| + \lambda_n\|Fg_n - Fy_n\|\Big] \tag{31}$$

Hence:

$$\begin{aligned} \|Fy_n - p\| &\leq m\Big[(1-\lambda_n)\|q_n - p\| + \lambda_n\|Fq_n - p\|\Big] = m(1-\lambda_n + m\lambda_n)\|q_n - p\| \\ &\leq m^2\|q_n - p\| \end{aligned} \tag{32}$$

Since $1 \leq m$ yields $(1-\lambda_n + m\lambda_n) \leq m$

$$\begin{aligned} \|Fg_n - Fy_n\| \leq m\|g_n - y_n\| &\leq m\Big[\|Fy_n - q_n\| + \lambda_n\|q_n - Fq_n\|\Big] \\ &= m\Big[\Big(1+m^2+\lambda_n(1+m)\Big)\Big]\|q_n - p\| \end{aligned} \tag{33}$$

Substituting Equations (33) and (32) in (31) yielded that:

$$\begin{aligned} \|g_n - p\| &\leq \tfrac{1}{1+\lambda_n}\Big[(1+(1-r)\lambda_n)m^2 + \lambda_n m\Big(1+m^2+\lambda_n(1+m)\Big)\Big]\|q_n - p\| \\ &\leq [1 - \tfrac{\lambda_n}{1+\lambda_n}[m^2 r - m\Big(1+m^2+\lambda_n(1+m)\Big)]]\|q_n - p\| \\ &= [1 - \tfrac{\lambda_n e}{1+\lambda_n}]\|x_n - p\| \end{aligned} \tag{34}$$

Substitute Equation (34) in (28), to obtain:

$$\|q_{n+1} - p\| \leq \delta_n + [1 - \frac{\lambda_n e}{1+\lambda_n}]\|q_n - p\| + \|a_n\|. \tag{35}$$

□

Theorem 5. *Assume that X, F, p, m, $\{x_n\}$, $\{q_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ be as in Theorem 4 and the hypothesis that the condition (Δ_2) is satisfied. Then $\{x_n\}$ in Equation (4) is almost F-stable.*

Proof. Let $\sum_{n=0}^{\infty} \delta_n < \infty$, to prove that $\lim_{n\to\infty} q_n = p$.

By using the conclusion of Equation (35) of Theorem 4 and an application of Lemma 1, we get $\lim_{n\to\infty} q_n = p$. □

Theorem 6. *Let X, F, p, m, $\{q_n\}$, $\{\lambda_n\}$, $\{a_n\}$, and $\{\delta_n\}$ be as in Theorem 4 and (Δ_2) is satisfied. Then $\{x_n\}$ in (2) is F-stable.*

Proof. Suppose that $\lim_{n\to\infty} \delta_n = 0$. □

By expressing Equation (35) in the form $\rho_{n+1} \leq (1-\gamma_n)\rho_n + \mu_n$, of Lemma 1,where $\gamma_n = \frac{\lambda_n e}{1+\lambda_n}$, $\rho_n = \|q_n - p\|$ and $\mu_n = \delta_n + \|a_n\|$, this implies to $\lim_{n\to\infty} q_n = p$.

Corollary 2. *Let X, F, p, m, $\{q_n\}$, $\{\lambda_n\}$, $\{a_n\}$, and $\{\delta_n\}$ be as in Theorem 4 and $\{x_n\}$ be in Equation (2). Then $\{x_n\}$:*

1. *converges strongly to the unique fixed point p.*
2. *is almost F-stable.*
3. *is F -stable.*

3. Applications

Theorem 7. *Let X be a real Banach space and* $F : X \to X$ *be Lipschitzian strongly accretive mapping with Lipschitz constant m. Define* $\mathcal{S} : X \to X$ *by* $\mathcal{S}x = f + x - Fx$*. Let* $\{\lambda_n\}$*,* $\{\eta_n\}$*,* $\{a_n\}$*, and* $\{c_n\}$ *as are in Theorem 1. For* $w_0,\ f \in X$*,*

$$\begin{aligned} w_{n+1} &= \lambda_n \mathcal{S} z_n + (1-\lambda_n)\mathcal{S} w_n + a_n, \\ z_n &= \eta_n \mathcal{S} w_n + (1-\eta_n) w_n + c_n,\ \forall\ n \geq 0. \end{aligned}$$

Then $\{w_n\}$*:*

1. *converges strongly the unique solution* p^* *of the equation* $Fx = f$*.*
2. *is almost* $\mathcal{S}$*-stable.*
3. *is* $\mathcal{S}$*-stable.*

Proof. The mapping $\mathcal{S}$ is Lipschitzian with a constant $m_* = 1 + m$, and from Lemma 3 the equation $Fx = f$ has a unique solution p^*, this implies that $\mathcal{S}$ has a unique fixed point p^*.

From Equation (7) and Proposition (6), hence $\langle (I-\mathcal{S})x - (I-\mathcal{S})y, j(x-y)\rangle = \langle Fx - Fy, j(x-y)\rangle \geq \mathfrak{r}\|x-y\|^2$, this implies $\mathcal{S}$ is strongly pseudo-contractive, therefore, the proof follows from Theorems 1–3. □

Corollary 3. *Let* $X,\ F,\ \mathcal{S},\ p^*,\ m,\ \{q_n\},\ \{\lambda_n\},\ \{\eta_n\}$*, and*$\{\delta_n\}$ *be as in Theorem 7 and* $\{w_n\}$ *defined by*

$$\begin{aligned} w_{n+1} &= \lambda_n \mathcal{S} z_n + (1-\lambda_n)\mathcal{S} w_n, \\ z_n &= \eta_n \mathcal{S} w_n + (1-\eta_n) w_n,\ \forall\ n \geq 0. \end{aligned}$$

Then $\{w_n\}$ *:*

1. *converges strongly to the unique solution*p^**of the equation* $Fx = f$*.*
2. *is almost* $\mathcal{S}$*-stable.*
3. *is* $\mathcal{S}$*-stable.*

Theorem 8. *Let X be a real Banach space and* $F : X \to X$ *be Lipschitzian accretive mapping with Lipschitz constant. Define* $\mathcal{S} : X \to X$ *by* $\mathcal{S}x = f - Fx$*. Let* $\{\lambda_n\}$*,* $\{\eta_n\}$*,* $\{a_n\}$*, and* $\{c_n\}$ *as are in Theorem 1. For* $w_0,\ f \in X$*,*

$$\begin{aligned} w_{n+1} &= \lambda_n \mathcal{S} z_n + (1-\lambda_n)\mathcal{S} w_n + a_n, \\ z_n &= \eta_n \mathcal{S} w_n + (1-\eta_n) w_n + c_n,\ \forall\ n \geq 0. \end{aligned}$$

Then $\{w_n\}$*:*

1. *converges strongly to the unique solution* p^* *of the equation* $x + Fx = f$*.*
2. *is almost* $\mathcal{S}$*-stable.*
3. *is* $\mathcal{S}$*-stable.*

Proof. From Lemma 3, hence, the equation $x + Fx = f$ has a unique fixed point p^*, (i.e., $\mathcal{S}$ has a unique fixed point p^*). By using Equation (8), we obtained:

$$\|x-y\| \leq \|x-y+\mathfrak{r}(Fx-Fy)\| = \|x-y+\mathfrak{r}(\mathcal{S}x-\mathcal{S}y)\| \quad (36)$$

Since:

$$\begin{aligned}\mathcal{S}w_n &= w_{n+1} + \lambda_n \mathcal{S}w_n - \lambda_n \mathcal{S}z_n - a_n\\ &= (1+\lambda_n)w_{n+1} - \lambda_n \mathcal{S}w_{n+1} + \lambda_n(\mathcal{S}w_{n+1} - \mathcal{S}z_n)\mathcal{S}w_n + \lambda_n{}^2(\mathcal{S}w_n - \mathcal{S}z_n)\\ &\quad - (1+\lambda_n)a_n\\ p^* &= (1+\lambda_n)p^* - \lambda_n \mathcal{S}p^*\end{aligned}$$

By using Equation (36), we obtained:

$$\begin{aligned}\|\mathcal{S}w_n - p^*\| &\geq (1+\lambda_n)\|(w_{n+1} - p^*) + \tfrac{\lambda_n}{1+\lambda_n}(\mathcal{S}w_{n+1} - \mathcal{S}p^*)\| - \lambda_n\|\mathcal{S}w_{n+1} - \mathcal{S}z_n\|\\ &\quad -\lambda_n^2\|\mathcal{S}w_n - \mathcal{S}z_n\| - (1+\lambda_n)\|a_n\|\\ &\geq (1+\lambda_n)\|w_{n+1} - p^*\| - \lambda_n^2\|\mathcal{S}w_n - \mathcal{S}z_n\| - \lambda_n\|\mathcal{S}w_{n+1} - \mathcal{S}z_n\|\\ &\quad -(1+\lambda_n)\|a_n\|\end{aligned}$$

This implies:

$$\|w_{n+1} - p^{*}\| \leq \frac{1}{1+\lambda_n}\|\mathcal{S}w_n - p^*\| + \frac{\lambda_n}{1+\lambda_n}\|\mathcal{S}w_{n+1} - \mathcal{S}z_n\| + \frac{\lambda_n^2}{1+\lambda_n}\|\mathcal{S}w_n - \mathcal{S}z_n\| + \|a_n\|$$

The proof completes by the same way as Theorems 1–3. □

Corollary 4. *Let X, F, $\mathcal{S}$, p^*, m, $\{q_n\}$, $\{\lambda_n\}$, $\{\eta_n\}$, $\{\delta_n\}$ be as in Theorem 8 and $\{w_n\}$ defined by*

$$\begin{aligned}w_{n+1} &= \lambda_n \mathcal{S}z_n + (1-\lambda_n)\mathcal{S}w_n,\\ z_n &= \eta_n \mathcal{S}w_n + (1-\eta_n)w_n,\ \forall\, n \geq 0.\end{aligned}$$

Then $\{w_n\}$:

1. *converges strongly to the unique solution $p^*$of the equation $x + Fx = f$.*
2. *is almost $\mathcal{S}$-stable.*
3. *is $\mathcal{S}$-stable.*

Theorem 9. *Suppose that X is a real Banach space and $F : X \to X$ is Lipschitzian strongly accretive mapping. Define $\mathcal{S} : X \to X$ by $\mathcal{S}x = f + x - Fx$. Let $\{\lambda_n\}$ and $\{a_n\}$, as are in Theorem 4. For x_0, $f \in X$,*

$$\begin{aligned}x_{n+1} &= \mathcal{S}y_n + a_n,\\ y_n &= \lambda_n \mathcal{S}x_n + (1-\lambda_n)x_n, \forall\, n \geq 0.\end{aligned}$$

Then $\{x_n\}$

1. *converges strongly to the unique solution p^* of the equation $Fx = f$.*
2. *is almost $\mathcal{S}$-stable.*
3. *is $\mathcal{S}$-stable.*

Proof. We can prove this the same way for Theorem 7. □

Corollary 5. *Let X, F, $\mathcal{S}$, p^*, m, $\{q_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ be as in Theorem 8 and $\{x_n\}$ defined by*

$$\begin{aligned}x_{n+1} &= \mathcal{S}y_n,\\ y_n &= \lambda_n \mathcal{S}x_n + (1-\lambda_n)x_n, \forall\, n \geq 0.\end{aligned}$$

Then$\{x_n\}$:

1. *converges strongly to the fixed point p^* the unique solution of the equation $Fx = f$.*

2. *is almost $\mathcal{S}$-stable.*
3. *is $\mathcal{S}$-stable.*

Theorem 10. *Let X be a real Banach space, $F : X \to X$ is Lipschitzian accretive mapping with Lipschitz constant m. Define $\mathcal{S} : X \to X$ by $\mathcal{S}x = f - Fx$. Let $\{\lambda_n\}$ and $\{a_n\}$, be as in Theorem 4. For $x_0,\ f \in X$,*

$$x_{n+1} = \mathcal{S}y_n + a_n,$$
$$y_n = \lambda_n \mathcal{S}x_n + (1 - \lambda_n)x_n, \forall\ n \geq 0.$$

Then $\{x_n\}$:

1. *converges strongly to the unique solution p^* of the equation $x + Fx = f$.*
2. *is almost $\mathcal{S}$-stable.*
3. *is $\mathcal{S}$-stable.*

Proof. The proof follows the same way as Theorem 8. □

Corollary 6. *Let X, F, $\mathcal{S}$, p^*, m, $\{q_n\}$, $\{\lambda_n\}$, and$\{\delta_n\}$ be as in Theorem 10 and$\{x_n\}$ defined by*

$$x_{n+1} = \mathcal{S}y_n,$$
$$y_n = \lambda_n \mathcal{S}x_n + (1 - \lambda_n)x_n, \forall\ n \geq 0.$$

Then $\{x_n\}$:

1. *converge strongly to the unique solution p^* of the equation $x + Fx = f$.*
2. *is almost $\mathcal{S}$-stable.*
3. *is $\mathcal{S}$ -stable.*

4. Conclusions

For real Banach spaces, very interesting results were proved which say that for a Lipschitzian strongly pseudo-contractive operator, the s-iteration with error and Picard–Mann iteration with error processes converge strongly to the unique fixed point of the operator (Theorems 1 and 4). Some applications were also given (Theorem 7).

Open Problem

Let B be a non-empty closed convex subset of a Banach space X and $\{T_i, S_i,\ \forall i = 1,\ 2, \ldots,\ k\}$ be two families of total asymptotically quasi-nonexpansive self-mappings. Abed and Hasan [16] studied the convergence of the iterative sequence $\{w_n\}$, defined as:

$$\begin{aligned}
&w_1 \in B\\
&w_{n+1} = (1 - \alpha_{in})S_i^n w_n + w_{in} T_i^n b_{in}\\
&b_{in} = (1 - w_{in})S_i^n a_n + w_{in} T_i^n b_{(i-1)n}\\
&b_{(i-1)n} = \left(1 - \alpha_{(i-1)n}\right) S_{i-1}^n w_n + \alpha_{(i-1)n} T_{i-1}^n b_{(i-2)n}\\
&b_{2n} = (1 - w_{2n})S_2^n a_n + \alpha_{2n} T_2^n b_{1n}\\
&b_{1n} = (1 - \alpha_{1n})S_1^n w_n + \alpha_{1n} T_1^n b_{0n},
\end{aligned}$$

where $b_{0n} = w_n$ *and* $\{\alpha_n\}_{n=1}^{\infty}$ are sequences in (0, 1).

We suggest studying the stability of this iterative sequence.

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