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Makgeolli Structures and Its Application in BCK/BCI-Algebras

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Abstract: A fuzzy set is an extension of an existing set using fuzzy logic. Soft set theory is a generalization of fuzzy set theory. Fuzzy and soft set theory are good mathematical tools for dealing with uncertainty in a parametric manner. The aim of this article is to introduce the concept of makgeolli structures using fuzzy and soft set theory and to apply it to BCK/BCI-algebras. The notion of makgeolli algebra and makgeolli ideal in BCK/BCI-algebras is defined, and several properties are investigated. It deals with the relationship between makgeolli algebra and makgeolli ideal, and several examples are given. Characterization of makgeolli algebra and makgeolli ideal are discussed, and a new makgeolli algebra from old one is established. A condition for makgeolli algebra to be makgeolli ideal in BCK-soft universe is considered, and we give example to show that makgeolli ideal is not makgeolli algebra in BCI-soft universe. Conditions for makgeolli ideal to be makgeolli algebra in BCI-soft universe are provided.

Keywords: BCK/BCI-soft universe; makgeolli structure; makgeolli algebra; makgeolli ideal

1. Introduction

There are many things inherently uncertain, inaccurate, and ambiguous in the real world. Zadeh [1] pointed out: "Various problems in system identification involve characteristics which are essentially nonprobabilistic in nature," and he introduced fuzzy set theory as an alternative to probability theory (see the work by the authors of [2]). Zadeh [3] outlined the uncertainty, which is an attribute of information, by trying to address it more generally. It is difficult to deal with uncertainties by traditional mathematical tools. However, one can use a wider range of existing theories, such as theory of (intuitionistic) fuzzy sets, theory of interval mathematics, theory of vague sets, probability theory, and theory of rough sets for dealing with uncertainties. However, Molodtsov [4] pointed out all of these theories have their own difficulties. According to Maji et al. [5] and Molodtsov [4], these difficulties can be attributed to the inadequacy of the parametric tools of theory. Molodtsov [4] tried to overcome these difficulties. He introduced the concept of soft set as a new mathematical tool for dealing with uncertainties, and pointed out several directions for its applications. Globally, interest in soft set theory and its application has been growing rapidly in recent years. Soft set theory has been applied to decision making problem (see works by the authors of [5–12]), groups, rings, fields and modules (see works by the authors of [13–17]), BCK/BCI-algebras, etc. (see works by the authors of [18–27]).

In this paper, we introduce the notion of makgeolli structures using fuzzy and soft set theory and apply it to BCK/BCI-algebras. We define the concept of makgeolli algebra and makgeolli ideal in BCK/BCI-algebras, and investigate several properties. We deal with the relation between makgeolli algebra and makgeolli ideal, and consider several examples. We discuss characterization of makgeolli algebra and makgeolli ideal. We make a new makgeolli algebra from old one. We provide a condition for makgeolli algebra to be makgeolli ideal in BCK-soft universe. We give example to show that makgeolli ideal is not makgeolli algebra in BCI-soft universe, and provide conditions for makgeolli ideal to be makgeolli algebra in BCI-soft universe.

2. Preliminaries

In 1978 and 1980, K. Iséki [28,29] introduced a BCK/BCI-algebra, which is an important class of logical algebras.

By a BCI-algebra, we mean a set X with a a binary operation $*$ and special element 0 which satisfies the following conditions.

(I) $(\forall u, v, w \in X)\ (((u * v) * (u * w)) * (w * v) = 0)$,
(II) $(\forall u, v \in X)\ ((u * (u * v)) * v = 0)$,
(III) $(\forall u \in X)\ (u * u = 0)$,
(IV) $(\forall u, v \in X)\ (u * v = 0,\ v * u = 0\ \Rightarrow\ u = v)$.

If a BCI-algebra X satisfies the following identity,

(V) $(\forall u \in X)\ (0 * u = 0)$,

then X is called a BCK-algebra. Any BCK/BCI-algebra X satisfies the following conditions.

$$(\forall u \in X)\ (u * 0 = u)\,, \tag{1}$$

$$(\forall u, v, w \in X)\ (u \le v\ \Rightarrow\ u * w \le v * w,\ w * v \le w * u)\,, \tag{2}$$

$$(\forall u, v, w \in X)\ ((u * v) * w = (u * w) * v) \tag{3}$$

where $u \le v$ if, and only if, $u * v = 0$. A subset S of a BCK/BCI-algebra X is called a subalgebra of X if $u * v \in S$ for all $u, v \in S$. A subset I of a BCK/BCI-algebra X is called an ideal of X if it satisfies

$$0 \in I, \tag{4}$$

$$(\forall u \in X)\ (\forall v \in I)\ (u * v \in I\ \Rightarrow\ u \in I)\,. \tag{5}$$

We refer the reader to the books by the authors of [30,31] for further information regarding BCK/BCI-algebras.

Let U be a universal set and E a set of parameters, respectively. A pair (α, E) is called a soft set over a universe U (see [4]) where α is a mapping given by

$$\alpha : E \to \mathcal{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\alpha(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (α, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in the work by the authors of [4].

Given a nonempty subset A of E, denote by (α, A) a soft set (α, E) over U satisfying the following condition.

$$\alpha(x) = \emptyset \text{ for all } x \notin A. \tag{6}$$

3. Makgeolli Structures

In what follows, let E be a set of parameters and U a universal set unless otherwise specified. We say that the pair (U, E) is a soft universe.

Definition 1. *Let A and B be subsets of E. A makgeolli structure on U (related to A and B) is a structure of the form*

$$\mathcal{M}_{(A,B,U)} := \{\langle (a,b,x); M_A(a), G_B(b), \ell(x)\rangle \mid (a,b,x) \in A \times B \times U\} \tag{7}$$

where $M_A := (M, A)$ and $G_B := (G, B)$ are soft sets over U and ℓ is a fuzzy set in U.

For the sake of simplicity, the makgeolli structure in (7) will be denoted by $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$. The makgeolli structure $\mathcal{M}_{(A,A,U)} = (M_A, G_A, \ell)$ on U related to a subset A of E is simply denoted by $\mathcal{M}_{(A,U)} = (M_A, G_A, \ell)$.

Example 1. *Miss K (say) and Mr. J (say) are going to buy a house to live in after marriage. They are looking for the most reasonable house, considering its price, environment, and distance from the neighborhood (for example, hospital). There are six houses $U = \{h_i \mid i = 1,2,3,4,5,6\}$. They are considering two parameter sets $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and $B = \{\delta_1, \delta_2, \delta_3\}$ where each parameter ε_i and δ_i, $i = 1,2,3$, stands for*

ε_1: expensive, ε_2: intermediate price, ε_3: cheap,
δ_1: beautiful, δ_2: green surround, δ_3: pristine area,

and consider the distance from the neighborhood given by

$$\ell : U \to [0,1],\ x \mapsto \begin{cases} 0.4 & \text{if } x = h_1, \\ 0.7 & \text{if } x = h_2, \\ 0.6 & \text{if } x = h_3, \\ 0.2 & \text{if } x = h_4, \\ 0.5 & \text{if } x = h_5, \\ 0.1 & \text{if } x = h_6. \end{cases}$$

Here, for example, $\ell(h_1) = 0.4$ means that the distance from house to the neighborhood is 4 km. Suppose that $M_A(\varepsilon_1) = \{h_1, h_2\}$, $M_A(\varepsilon_2) = \{h_2, h_3, h_4\}$, $M_A(\varepsilon_3) = \{h_1, h_4, h_6\}$, $G_B(\delta_1) = \{h_2, h_4, h_6\}$, $G_B(\delta_2) = \{h_3, h_4, h_5\}$ and $G_B(\delta_3) = \{h_3, h_4, h_5, h_6\}$ Then the makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ on U is given by Table 1.

Table 1. Tabular representation of the makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$.

X	h_1	h_2	h_3	h_4	h_5	h_6
$(M_A(\varepsilon_1), G_B(\delta_1), \ell(x))$	$(1,0,0.4)$	$(1,1,0.7)$	$(0,0,0.6)$	$(0,1,0.2)$	$(0,0,0.5)$	$(0,1,0.1)$
$(M_A(\varepsilon_1), G_B(\delta_2), \ell(x))$	$(1,0,0.4)$	$(1,0,0.7)$	$(0,1,0.6)$	$(0,1,0.2)$	$(0,1,0.5)$	$(0,0,0.1)$
$(M_A(\varepsilon_1), G_B(\delta_3), \ell(x))$	$(1,0,0.4)$	$(1,0,0.7)$	$(0,1,0.6)$	$(0,1,0.2)$	$(0,1,0.5)$	$(0,1,0.1)$
$(M_A(\varepsilon_2), G_B(\delta_1), \ell(x))$	$(0,0,0.4)$	$(1,1,0.7)$	$(1,0,0.6)$	$(\mathbf{1},\mathbf{1},\mathbf{0.2})$	$(0,0,0.5)$	$(0,1,0.1)$
$(M_A(\varepsilon_2), G_B(\delta_2), \ell(x))$	$(0,0,0.4)$	$(1,0,0.7)$	$(1,1,0.6)$	$(1,1,0.2)$	$(0,1,0.5)$	$(0,0,0.1)$
$(M_A(\varepsilon_2), G_B(\delta_3), \ell(x))$	$(0,0,0.4)$	$(1,0,0.7)$	$(1,1,0.6)$	$(1,1,0.2)$	$(0,1,0.5)$	$(0,1,0.1)$
$(M_A(\varepsilon_3), G_B(\delta_1), \ell(x))$	$(1,0,0.4)$	$(0,1,0.7)$	$(0,0,0.6)$	$(1,1,0.2)$	$(0,0,0.5)$	$(1,1,0.1)$
$(M_A(\varepsilon_3), G_B(\delta_2), \ell(x))$	$(1,0,0.4)$	$(0,0,0.7)$	$(0,1,0.6)$	$(1,1,0.2)$	$(0,1,0.5)$	$(1,0,0.1)$
$(M_A(\varepsilon_3), G_B(\delta_3), \ell(x))$	$(1,0,0.4)$	$(0,0,0.7)$	$(0,1,0.6)$	$(1,1,0.2)$	$(0,1,0.5)$	$(1,1,0.1)$

The Gothic component $(\mathbf{1},\mathbf{1},\mathbf{0.2})$ in Table 1 means that the house h_4 is intermediate price, beautiful, and it is 2km away from the neighborhood (for example, hospital).

Definition 2. *Let (U, E) be a soft universe and let $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ and $\mathcal{N}_{(A,B,U)} = (N_A, H_B, \jmath)$ be makgeolli structures on U. The intersection of $\mathcal{M}_{(A,B,U)}$ and $\mathcal{N}_{(A,B,U)}$ is defined to be a makgeolli structure $(\mathcal{M} \Cap \mathcal{N})_{(A,B,U)} = (M_A \tilde{\cap} N_A, G_B \tilde{\cup} H_B, \ell \wedge \jmath)$ on U in which*

$$\begin{aligned} &M_A \tilde{\cap} N_A : A \to \mathcal{P}(U),\ a \mapsto M_A(a) \cap N_A(a), \\ &G_B \tilde{\cup} H_B : B \to \mathcal{P}(U),\ b \mapsto G_B(b) \cup H_B(b), \\ &\ell \wedge \jmath : U \to [0,1],\ x \mapsto \min\{\ell(x), \jmath(x)\}. \end{aligned}$$

4. Applications in BCK/BCI-Algebras

A BCK/BCI-soft universe is defined as a soft universe (U, E) in which U and E are BCK/BCI-algebras with binary operations "$*$" and "$\rightsquigarrow$", respectively.

Definition 3. *Let (U, E) be a BCK/BCI-soft universe and let A and B be subsets of E. A makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ on U is called a makgeolli algebra over U if it satisfies:*

$$\begin{aligned} &(\forall a_1, a_2 \in A)\,(a_1 \rightsquigarrow a_2 \in A \ \Rightarrow\ M_A(a_1 \rightsquigarrow a_2) \supseteq M_A(a_1) \cap M_A(a_2)), \\ &(\forall b_1, b_2 \in B)\,(b_1 \rightsquigarrow b_2 \in B \ \Rightarrow\ G_B(b_1 \rightsquigarrow b_2) \subseteq G_B(b_1) \cup G_B(b_2)), \\ &(\forall x, y \in U)(\forall t, r \in (0,1])\left(\tfrac{x}{t} \in \ell,\ \tfrac{y}{r} \in \ell \ \Rightarrow\ \tfrac{x*y}{\min\{t,r\}} \in \ell \right) \end{aligned} \tag{8}$$

where $\frac{x}{t} \in \ell$ means $\ell(x) \geq t$.

Example 2. *Assume that there are five houses in the universal set U, which is given by*

$$U = \{h_i \mid i = 0, 1, 2, 3, 4\}.$$

Then $(U, *, h_0)$ is a BCK-algebra in which the operation $*$ is given by Table 2.

Table 2. Cayley table for the binary operation "$*$".

$*$	h_0	h_1	h_2	h_3	h_4
h_0	h_0	h_0	h_0	h_0	h_0
h_1	h_1	h_0	h_0	h_1	h_0
h_2	h_2	h_1	h_0	h_2	h_0
h_3	h_3	h_3	h_3	h_0	h_3
h_4	h_4	h_4	h_4	h_4	h_0

Let $E = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ be a set of parameters in which each element ε_i, $i = 0, 1, 2, 3$, stands for

ε_0: beautiful, ε_1: in good location, ε_2: cheap, ε_3: pristine area.

If we give a binary operation $\rightsquigarrow$ to E by Table 3,

Table 3. Cayley table for the binary operation "$\rightsquigarrow$".

$\rightsquigarrow$	ε_0	ε_1	ε_2	ε_3
ε_0	ε_0	ε_0	ε_0	ε_0
ε_1	ε_1	ε_0	ε_1	ε_1
ε_2	ε_2	ε_2	ε_0	ε_2
ε_3	ε_3	ε_3	ε_3	ε_0

Then $(E, \rightsquigarrow, \varepsilon_0)$ is a BCK-algebra. If we take two sets, $A = \{\varepsilon_0, \varepsilon_2, \varepsilon_3\}$ and $B = \{\varepsilon_0, \varepsilon_3\}$ of E, then A and B are subalgebras of E. Let $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ be a makgeolli structure on U given as follows:

$$M_A : A \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{h_i \mid i = 0,1,2,3,4\} & \text{if } x = \varepsilon_0, \\ \{h_i \mid i = 0,2,3\} & \text{if } x = \varepsilon_2, \\ \{h_i \mid i = 1,2,4\} & \text{if } x = \varepsilon_3, \end{cases}$$

$$G_B : B \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{h_3\} & \text{if } x = \varepsilon_0, \\ \{h_3, h_4\} & \text{if } x = \varepsilon_3, \end{cases}$$

$$\ell : U \to [0,1],\ x \mapsto \begin{cases} 0.8 & \text{if } x = h_0, \\ 0.5 & \text{if } x = h_1, \\ 0.5 & \text{if } x = h_2, \\ 0.6 & \text{if } x = h_3, \\ 0.3 & \text{if } x = h_4. \end{cases}$$

It is routine to check that $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ is a makgeolli algebra over U.

Proposition 1. *Let (U, E) be a BCK/BCI-soft universe. For any subalgebras A and B of E, every makgeolli algebra $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ over U satisfies the following conditions.*

$$(\forall (a,b,x) \in A \times B \times U) \left(M_A(a) \subseteq M_A(0),\ G_B(b) \supseteq G_B(0),\ \tfrac{0}{\ell(x)} \in \ell \right). \tag{9}$$

Proof. If we take $a_1 = a_2 = a$ and $b_1 = b_2 = b$ in (8), then $a \rightsquigarrow a = 0 \in A$ and $b \rightsquigarrow b = 0 \in B$. Hence

$$M_A(0) = M_A(a \rightsquigarrow a) \supseteq M_A(a) \cap M_A(a) = M_A(a),$$
$$G_B(0) = G_B(b \rightsquigarrow b) \subseteq G_B(b) \cup G_B(b) = G_B(b).$$

Since $\frac{x}{\ell(x)} \in \ell$ for all $x \in U$, we have $\frac{0}{\ell(x)} = \frac{x*x}{\min\{\ell(x),\ell(x)\}} \in \ell$ for all $x \in U$. □

Theorem 1. *Let (U, E) be a BCK/BCI-soft universe and let A and B be subsets of E. Then a makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ on U is an makgeolli algebra over U if and only if the following assertions are valid.*

$$\begin{aligned} &(\forall a_1, a_2 \in A)\,(a_1 \rightsquigarrow a_2 \in A \ \Rightarrow\ M_A(a_1 \rightsquigarrow a_2) \supseteq M_A(a_1) \cap M_A(a_2))\,, \\ &(\forall b_1, b_2 \in B)\,(b_1 \rightsquigarrow b_2 \in B \ \Rightarrow\ G_B(b_1 \rightsquigarrow b_2) \subseteq G_B(b_1) \cup G_B(b_2))\,, \\ &(\forall x, y \in U)\,(\ell(x*y) \geq \min\{\ell(x), \ell(y)\})\,. \end{aligned} \tag{10}$$

Proof. Assume that

$$\tfrac{x}{t} \in \ell,\ \tfrac{y}{r} \in \ell \ \Rightarrow\ \tfrac{x*y}{\min\{t,r\}} \in \ell \tag{11}$$

for all $x, y \in U$ and $t, r \in (0,1]$. Since $\frac{x}{\ell(x)} \in \ell$ and $\frac{y}{\ell(y)} \in \ell$ for all $x, y \in U$, it follows from (11) that $\frac{x*y}{\min\{\ell(x),\ell(y)\}} \in \ell$. Thus $\ell(x*y) \geq \min\{\ell(x), \ell(y)\}$.

Conversely, let $x, y \in U$ and $t, r \in (0,1]$ be such that $\frac{x}{t} \in \ell$ and $\frac{y}{r} \in \ell$. Then $\ell(x) \geq t$ and $\ell(y) \geq r$. Hence $\ell(x*y) \geq \min\{\ell(x), \ell(y)\} \geq \min\{t, r\}$, and so $\frac{x*y}{\min\{t,r\}} \in \ell$. This completes the proof. □

Proposition 2. *Let (U, E) be a BCK/BCI-soft universe. For any makgeolli algebra $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ over U related to subalgebras A and B of E, the following are equivalent.*

(1) $$\begin{cases} (\forall a \in A)\,(M_A(a) = M_A(0))\,, \\ (\forall b \in B)\,(G_B(b) = G_B(0))\,, \\ (\forall x \in U)\,(\ell(x) = \ell(0))\,. \end{cases}$$

(2) $\begin{cases} (\forall a_1, a_2 \in A)\,(M_A(a_2) \subseteq M_A(a_1 \rightsquigarrow a_2))\,, \\ (\forall b_1, b_2 \in B)\,(G_B(b_2) \supseteq G_B(b_1 \rightsquigarrow b_2))\,, \\ (\forall x, y \in U)\,(\ell(x * y) \geq \ell(y))\,. \end{cases}$

Proof. Suppose that (1) is true. Using (10), we have

$$\begin{aligned} &(\forall a_1, a_2 \in A)\,(M_A(a_2) = M_A(0) \cap M_A(a_2) = M_A(a_1) \cap M_A(a_2) \subseteq M_A(a_1 \rightsquigarrow a_2))\,, \\ &(\forall b_1, b_2 \in B)\,(G_B(b_2) = G_B(0) \cup G_B(b_2) = G_B(b_1) \cup G_B(b_2) \supseteq G_B(b_1 \rightsquigarrow b_2))\,, \\ &(\forall x, y \in U)\,(\ell(y) = \min\{\ell(0), \ell(y)\} = \min\{\ell(x), \ell(y)\} \leq \ell(x * y))\,. \end{aligned}$$

Assume that (2) is valid. Since $a \rightsquigarrow 0 = a$ for all $a \in E$, we have $M_A(0) \subseteq M_A(a \rightsquigarrow 0) = M_A(a)$ for all $a \in A$ and $G_B(0) \supseteq G_B(b \rightsquigarrow 0) = G_B(b)$ for all $b \in B$. Since $x * 0 = x$ for all $x \in U$, we have $\ell(0) \leq \ell(x * 0) = \ell(x)$ for all $x \in U$. It follows from (9) that we have (1). □

Proposition 3. *Let (U, E) be a BCI-soft universe. Then every makgeolli algebra $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ over U related to subalgebras A and B of E satisfies the following conditions.*

$$\begin{aligned} &(\forall a_1, a_2 \in A)\,(M_A(a_1 \rightsquigarrow (0 \rightsquigarrow a_2)) \supseteq M_A(a_1) \cap M_A(a_2))\,, \\ &(\forall b_1, b_2 \in B)\,(G_B(b_1 \rightsquigarrow (0 \rightsquigarrow b_2)) \subseteq G_B(b_1) \cup G_B(b_2))\,, \\ &(\forall x, y \in U)\,(\ell(x * (0 * y)) \geq \min\{\ell(x), \ell(y)\})\,. \end{aligned} \tag{12}$$

Proof. Using Proposition 1, we have

$$\begin{aligned} M_A(a_1 \rightsquigarrow (0 \rightsquigarrow a_2)) &\supseteq M_A(a_1) \cap M_A(0 \rightsquigarrow a_2) \\ &\supseteq M_A(a_1) \cap M_A(0) \cap M_A(a_2) \\ &= M_A(a_1) \cap M_A(a_2), \end{aligned}$$

$$\begin{aligned} G_B(b_1 \rightsquigarrow (0 \rightsquigarrow b_2)) &\subseteq G_B(b_1) \cup G_B(0 \rightsquigarrow b_2) \\ &\subseteq G_B(b_1) \cup G_B(0) \cup G_B(b_2) \\ &= G_B(b_1) \cup G_B(b_2), \end{aligned}$$

$$\ell(x * (0 * y)) \geq \min\{\ell(x), \ell(0 * y)\} \geq \min\{\ell(x), \min\{\ell(0), \ell(y)\}\} = \min\{\ell(x), \ell(y)\}$$

for all $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $x, y \in U$. □

Theorem 2. *Let (U, E) be a BCK/BCI-soft universe and let $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ and $\mathcal{N}_{(A,B,U)} = (N_A, H_B, \jmath)$ be makgeolli algebras over U related to subalgebras A and B of E. Then the intersection of $\mathcal{M}_{(A,B,U)}$ and $\mathcal{N}_{(A,B,U)}$ is a makgeolli algebra over U.*

Proof. For any $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $x, y \in U$, we have

$$\begin{aligned} (M_A \tilde{\cap} N_A)(a_1 \rightsquigarrow a_2) &= M_A(a_1 \rightsquigarrow a_2) \cap N_A(a_1 \rightsquigarrow a_2) \\ &\supseteq (M_A(a_1) \cap M_A(a_2)) \cap (N_A(a_1) \cap N_A(a_2)) \\ &= (M_A(a_1) \cap N_A(a_1)) \cap (M_A(a_2) \cap N_A(a_2)) \\ &= (M_A \tilde{\cap} N_A)(a_1) \cap (M_A \tilde{\cap} N_A)(a_2), \end{aligned}$$

$$\begin{aligned}(G_B \tilde{\cup} H_B)(b_1 \rightsquigarrow b_2) &= G_B(b_1 \rightsquigarrow b_2) \cup H_B(b_1 \rightsquigarrow b_2)\\ &\subseteq (G_B(b_1) \cup G_B(b_2)) \cup (H_B(b_1) \cup H_B(b_2))\\ &= (G_B(b_1) \cup H_B(b_1)) \cup (G_B(b_2) \cup H_B(b_2))\\ &= (G_B \tilde{\cup} H_B)(b_1) \cup (G_B \tilde{\cup} H_B)(b_2),\end{aligned}$$

$$\begin{aligned}(\ell \wedge \jmath)(x * y) &= \min\{\ell(x * y), \jmath(x * y)\}\\ &\geq \min\{\min\{\ell(x), \ell(y)\}, \min\{\jmath(x), \jmath(y)\}\}\\ &\geq \min\{\min\{\ell(x), \jmath(x)\}, \min\{\ell(y), \jmath(y)\}\}\\ &\geq \min\{(\ell \wedge \jmath)(x), (\ell \wedge \jmath)(y)\}.\end{aligned}$$

Therefore $(\mathcal{M} \Cap \mathcal{N})_{(A,B,U)} = (M_A \tilde{\cap} N_A, G_B \tilde{\cup} H_B, \ell \wedge \jmath)$ is a makgeolli algebra over U. □

Let (U, E) be a BCK/BCI-soft universe. Gin a makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ on U related to A and B, consider the following sets.

$$\begin{aligned}&\mathcal{E}_A(M_A; \alpha) = \{a \in A \mid M_A(a) \supseteq \alpha\},\\ &\mathcal{E}_B(G_B; \beta) = \{b \in B \mid G_B(b) \subseteq \beta\},\\ &\mathcal{U}(\ell; t) = \{x \in U \mid \ell(x) \geq t\}\end{aligned}$$

where α and β are subsets of U and $t \in [0, 1]$.

Theorem 3. *Let (U, E) be a BCK/BCI-soft universe. Then a makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ on U related to subalgebras A and B of E is a makgeolli algebra over U if and only if the nonempty sets $\mathcal{E}_A(M_A; \alpha)$ and $\mathcal{E}_B(G_B; \beta)$ are subalgebras of E, and the nonempty set $\mathcal{U}(\ell; t)$ is a subalgebra of U for all $\alpha, \beta \in \mathcal{P}(U)$ and $t \in [0, 1]$.*

Proof. Suppose that $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ is a makgeolli algebra over U. Let $a_1, a_2 \in \mathcal{E}_A(M_A; \alpha)$, $b_1, b_2 \in \mathcal{E}_B(G_B; \beta)$ and $x, y \in \mathcal{U}(\ell; t)$ for all $\alpha, \beta \in \mathcal{P}(U)$ and $t \in [0, 1]$. Then $M_A(a_1) \supseteq \alpha$, $M_A(a_2) \supseteq \alpha$, $G_B(b_1) \subseteq \beta$, $G_B(b_2) \subseteq \beta$, $\ell(x) \geq t$ and $\ell(y) \geq t$. It follows from (10) that

$$\begin{aligned}&M_A(a_1 \rightsquigarrow a_2) \supseteq M_A(a_1) \cap M_A(a_2) \supseteq \alpha,\\ &G_B(b_1 \rightsquigarrow b_2) \subseteq G_B(b_1) \cup G_B(b_2) \subseteq \beta,\\ &\ell(x * y) \geq \min\{\ell(x), \ell(y)\} \geq t.\end{aligned}$$

Hence $a_1 \rightsquigarrow a_2 \in \mathcal{E}_A(M_A; \alpha)$, $b_1 \rightsquigarrow b_2 \in \mathcal{E}_B(G_B; \beta)$ and $x * y \in \mathcal{U}(\ell; t)$. Therefore, $\mathcal{E}_A(M_A; \alpha)$, $\mathcal{E}_B(G_B; \beta)$ and $\mathcal{U}(\ell; t)$ are subalgebras of U.

Conversely, let $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ be a makgeolli structure on U such that the nonempty sets $\mathcal{E}_A(M_A; \alpha)$ and $\mathcal{E}_B(G_B; \beta)$ are subalgebras of E, and the nonempty set $\mathcal{U}(\ell; t)$ is a subalgebra of U for all $\alpha, \beta \in \mathcal{P}(U)$ and $t \in [0, 1]$. Let $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $x, y \in U$ be such that $M_A(a_1) = \alpha_{a_1}$, $M_A(a_2) = \alpha_{a_2}$, $G_B(b_1) = \beta_{b_1}$, $G_B(b_2) = \beta_{b_2}$, $\ell(x) = t_x$ and $\ell(y) = t_y$. Taking $\alpha = \alpha_{a_1} \cap \alpha_{a_2}$, $\beta = \beta_{b_1} \cup \beta_{b_2}$ and $t = \min\{t_x, t_y\}$ imply that $a_1, a_2 \in \mathcal{E}_A(M_A; \alpha)$, $b_1, b_2 \in \mathcal{E}_B(G_B; \beta)$ and $x, y \in \mathcal{U}(\ell; t)$. Thus $a_1 \rightsquigarrow a_2 \in \mathcal{E}_A(M_A; \alpha)$, $b_1 \rightsquigarrow b_2 \in \mathcal{E}_B(G_B; \beta)$, and $x * y \in \mathcal{U}(\ell; t)$, which imply that

$$\begin{aligned}&M_A(a_1 \rightsquigarrow a_2) \supseteq \alpha = \alpha_{a_1} \cap \alpha_{a_2} = M_A(a_1) \cap M_A(a_2),\\ &G_B(b_1 \rightsquigarrow b_2) \subseteq \beta = \beta_{b_1} \cup \beta_{b_2} = G_B(b_1) \cup G_B(b_2),\\ &\ell(x * y) \geq t = \min\{t_x, t_y\} = \min\{\ell(x), \ell(y)\}.\end{aligned}$$

Therefore $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ is a makgeolli algebra over U by Theorem 1. □

Let (U,E) be a soft universe. Given a makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ on U related to subsets A and B of E, let $\mathcal{M}^*_{(A,B,U)} = (M^*_A, G^*_B, \ell^*)$ be a makgeolli structure related to A and B where

$$M^*_A : A \to \mathcal{P}(U),\ x \mapsto \begin{cases} M_A(x) & \text{if } x \in \mathcal{E}_A(M_A;\alpha), \\ \eta & \text{otherwise,} \end{cases}$$

$$G^*_B : B \to \mathcal{P}(U),\ x \mapsto \begin{cases} G_B(x) & \text{if } x \in \mathcal{E}_B(G_B;\beta), \\ \rho & \text{otherwise} \end{cases}$$

$$\ell^* : U \to [0,1],\ x \mapsto \begin{cases} \ell(x) & \text{if } x \in \mathcal{U}(\ell;t), \\ k & \text{otherwise} \end{cases}$$

where $\alpha, \beta, \eta, \rho \in \mathcal{P}(U)$ and $t, k \in [0,1]$ with $\eta \subsetneq M_A(x)$, $\rho \supsetneq G_B(x)$ and $k < \ell(x)$.

Theorem 4. *Let (U,E) be a BCK/BCI-soft universe. If a makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ on U related to subalgebras A and B of E is a makgeolli algebra over U, then so is $\mathcal{M}^*_{(A,B,U)} = (M^*_A, G^*_B, \ell^*)$.*

Proof. Assume that $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ is a makgeolli algebra over U. Then the nonempty sets $\mathcal{E}_A(M_A;\alpha)$ and $\mathcal{E}_B(G_B;\beta)$ are subalgebras of E, and the nonempty set $\mathcal{U}(\ell;t)$ is a subalgebra of U for all $\alpha, \beta \in \mathcal{P}(U)$ and $t \in [0,1]$ by Theorem 3. Let $a_1, a_2 \in A$. If $a_1, a_2 \in \mathcal{E}_A(M_A;\alpha)$, then $a_1 \rightsquigarrow a_2 \in \mathcal{E}_A(M_A;\alpha)$, and so

$$M^*_A(a_1 \rightsquigarrow a_2) = M_A(a_1 \rightsquigarrow a_2) \supseteq M_A(a_1) \cap M_A(a_2) = M^*_A(a_1) \cap M^*_A(a_2).$$

If $a_1 \notin \mathcal{E}_A(M_A;\alpha)$ or $a_2 \notin \mathcal{E}_A(M_A;\alpha)$, then $M^*_A(a_1) = \eta$ or $M^*_A(a_2) = \eta$. Hence $M^*_A(a_1 \rightsquigarrow a_2) \supseteq \eta = M^*_A(a_1) \cap M^*_A(a_2)$. Let $b_1, b_2 \in B$. If $b_1, b_2 \in \mathcal{E}_B(G_B;\beta)$, then $b_1 \rightsquigarrow b_2 \in \mathcal{E}_B(G_B;\beta)$, which implies that

$$G^*_B(b_1 \rightsquigarrow b_2) = G_B(b_1 \rightsquigarrow b_2) \subseteq G_B(b_1) \cup G_B(b_2) = G^*_B(b_1) \cup G^*_B(b_2).$$

If $b_1 \notin \mathcal{E}_B(G_B;\beta)$ or $b_2 \notin \mathcal{E}_B(G_B;\beta)$, then $G^*_B(b_1) = \rho$ or $G^*_B(b_2) = \rho$. Hence $G^*_B(b_1 \rightsquigarrow b_2) \subseteq \rho = G^*_B(b_1) \cup G^*_B(b_2)$. Let $x, y \in U$. If $x, y \in \mathcal{U}(\ell;t)$, then $x * y \in \mathcal{U}(\ell;t)$, and so $\ell^*(x * y) = \ell(x * y) \geq \min\{\ell(x), \ell(y)\} = \min\{\ell^*(x), \ell^*(y)\}$. If $x \notin \mathcal{U}(\ell;t)$ of $y \notin \mathcal{U}(\ell;t)$, then $\ell^*(x) = k$ or $\ell^*(y) = k$. Hence $\ell^*(x * y) \geq k = \min\{\ell^*(x), \ell^*(y)\}$. Therefore $\mathcal{M}^*_{(A,B,U)} = (M^*_A, G^*_B, \ell^*)$ is a makgeolli algebra over U. □

The following example shows that the converse of Theorem 4 is not true in general.

Example 3. *Consider a soft universe (U,E) in which $U = \mathbb{Z}_{10} = \{\overline{a} \mid a = 0,1,2,\cdots,9\}$ and $E = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Define a binary operations "$*$" on U by*

$$\overline{a} * \overline{b} = \overline{a - b + 10} \tag{13}$$

for all $\overline{a}, \overline{b} \in U$. Then $(U, *, \overline{0})$ is a BCI-algebra. Let $\rightsquigarrow$ be a binary operation on E defined by Table 4.

Table 4. Cayley table for the binary operation "$\rightsquigarrow$".

$\rightsquigarrow$	ε_0	ε_1	ε_2	ε_3
ε_0	ε_0	ε_1	ε_2	ε_3
ε_1	ε_1	ε_0	ε_3	ε_2
ε_2	ε_2	ε_3	ε_0	ε_1
ε_3	ε_3	ε_2	ε_1	ε_0

Then $(E, *, \varepsilon_0)$ *is a BCI-algebra. Let* $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ *be a makgeolli structure on U defined by*

$$M_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} U & \text{if } x = \varepsilon_0, \\ \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\} & \text{if } x = \varepsilon_1, \\ \{\overline{2}, \overline{5}, \overline{6}, \overline{8}\} & \text{if } x = \varepsilon_2, \\ \{\overline{4}\} & \text{if } x = \varepsilon_3, \end{cases}$$

$$G_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{\overline{0}\} & \text{if } x = \varepsilon_0, \\ \{\overline{0}, \overline{5}\} & \text{if } x = \varepsilon_1, \\ \{\overline{0}, \overline{4}, \overline{6}, \overline{8}\} & \text{if } x = \varepsilon_2, \\ U & \text{if } x = \varepsilon_3, \end{cases}$$

$$\ell : U \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = \overline{0}, \\ 0.7 & \text{if } x \in \{\overline{2}, \overline{4}, \overline{6}, \overline{8}\}, \\ 0.6 & \text{if } x \in \{\overline{1}, \overline{3}\}, \\ 0.4 & \text{if } x \in \{\overline{5}, \overline{7}\}, \\ 0.3 & \text{if } x = \overline{9}. \end{cases}$$

Then $\mathcal{E}_E(M_E; \gamma) = \{\varepsilon_0, \varepsilon_1\} = \mathcal{E}_E(G_E; \eta)$ *and* $\mathcal{U}(\ell; r) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}$ *for* $\gamma = \{\overline{0}, \overline{6}, \overline{8}\}$, $\eta = \{\overline{0}, \overline{5}, \overline{6}\}$ *and* $r \in (0.6, 0.7]$. *Let* $\mathcal{M}^*_{(E,U)} = (M^*_E, G^*_E, \ell^*)$ *be a makgeolli structure on U given as follows.*

$$M^*_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} M_E(x) & \text{if } x \in \mathcal{E}_E(M_E; \gamma), \\ \emptyset & \text{otherwise}, \end{cases}$$

$$G^*_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} G_E(x) & \text{if } x \in \mathcal{E}_E(G_E; \eta), \\ U & \text{otherwise}, \end{cases}$$

$$\ell^* : U \to [0,1],\ x \mapsto \begin{cases} \ell(x) & \text{if } x \in \mathcal{U}(\ell; r), \\ 0 & \text{otherwise}, \end{cases}$$

that is,

$$M^*_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} U & \text{if } x = \varepsilon_0, \\ \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\} & \text{if } x = \varepsilon_1, \\ \emptyset & \text{if } x \in \{\varepsilon_2, \varepsilon_3\}, \end{cases}$$

$$G^*_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{\overline{0}\} & \text{if } x = \varepsilon_0, \\ \{\overline{0}, \overline{5}\} & \text{if } x = \varepsilon_1, \\ U & \text{if } x \in \{\varepsilon_2, \varepsilon_3\}, \end{cases}$$

$$\ell^* : U \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = \overline{0}, \\ 0.7 & \text{if } x \in \{\overline{2}, \overline{4}, \overline{6}, \overline{8}\}, \\ 0 & \text{otherwise}, \end{cases}$$

It is routine to verify that $\mathcal{M}^*_{(E,U)} = (M^*_E, G^*_E, \ell^*)$ *is a makgeolli algebra over U. But* $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ *is not a makgeolli algebra over U since*

$$M_E(\varepsilon_1) \cap M_E(\varepsilon_2) = \{\overline{2}, \overline{6}, \overline{8}\} \nsubseteq \{\overline{2}\} = M_E(\varepsilon_3) = M_E(\varepsilon_1 \rightsquigarrow \varepsilon_2),$$

$$G_E(\varepsilon_1) \cup G_E(\varepsilon_2) = \{\overline{0}, \overline{4}, \overline{5}, \overline{6}, \overline{8}\} \nsupseteq U = G_E(\varepsilon_3) = G_E(\varepsilon_1 \rightsquigarrow \varepsilon_2),$$

and/or

$$\ell(\overline{1} * \overline{2}) = \ell(\overline{9}) = 0.3 < 0.6 = \min\{\ell(\overline{1}), \ell(\overline{2})\}.$$

Definition 4. *Let (U, E) be a BCK/BCI-soft universe. A makgeolli structure $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ on U is called a makgeolli ideal over U if it satisfies*

$$(\forall e \in E)(M_E(0) \supseteq M_E(e),\ G_E(0) \subseteq G_E(e)), \tag{14}$$

$$(\forall x \in U)\left(\frac{0}{\ell(x)} \in \ell\right), \tag{15}$$

$$(\forall a, b \in E)\left(\begin{array}{l} M_E(a) \supseteq M_E(a \rightsquigarrow b) \cap M_E(b) \\ G_E(a) \subseteq G_E(a \rightsquigarrow b) \cup G_E(b) \end{array}\right). \tag{16}$$

$$(\forall x, y \in U)(\forall t, r \in (0,1])\left(\frac{x*y}{t} \in \ell,\ \frac{y}{r} \in \ell \ \Rightarrow\ \frac{x}{\min\{t,r\}} \in \ell\right). \tag{17}$$

Example 4. *There are five woman patients in a hospital which is given by*

$$U = \{w_1, w_2, w_3, w_4, w_5\}.$$

Communication between two patients w_i and w_j for $i, j \in \{1,2,3,4,5\}$ in the hospital is expressed as $w_i * w_j$ and the result is w_k, i.e., $w_i * w_j = w_k$ for $k = 1,2,3,4,5$; this is what w_i informs w_j that the health condition of w_k is serious. In this case "$*$" is a binary operation given to U, where it is given as shown in Table 5.

Table 5. Cayley table for the binary operation "$*$".

$*$	w_1	w_2	w_3	w_4	w_5
w_1	w_1	w_1	w_3	w_4	w_5
w_2	w_2	w_1	w_3	w_4	w_5
w_3	w_3	w_3	w_1	w_5	w_4
w_4	w_4	w_4	w_5	w_1	w_3
w_5	w_5	w_5	w_4	w_3	w_1

Then $(U, *, w_1)$ is a BCI-algebra. Let a set of parameters $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$ be a set of status of patients in which each parameter means

ε_1: "chest pain"; ε_2: "headache"; ε_3: "toothache"; ε_4: "mental depression"; ε_5: "neurosis"

with the binary operation "$\rightsquigarrow$" in Table 6.

Table 6. Cayley table for the binary operation "$\rightsquigarrow$".

$\rightsquigarrow$	ε_1	ε_2	ε_3	ε_4	ε_5
ε_1	ε_1	ε_1	ε_1	ε_4	ε_4
ε_2	ε_2	ε_1	ε_2	ε_5	ε_4
ε_3	ε_3	ε_3	ε_1	ε_4	ε_4
ε_4	ε_4	ε_4	ε_4	ε_1	ε_1
ε_5	ε_5	ε_4	ε_5	ε_2	ε_1

Then $(E, \rightsquigarrow, \varepsilon_1)$ is a BCI-algebra. Hence (U, E) is a BCI-soft universe. Let $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ be a makgeolli structure on U defined by

$$M_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} U & \text{if } x = \varepsilon_1, \\ \{w_1, w_2, w_3, w_5\} & \text{if } x = \varepsilon_2, \\ \{w_1, w_3, w_5\} & \text{if } x = \varepsilon_3, \\ \{w_3, w_5\} & \text{if } x \in \{\varepsilon_4, \varepsilon_5\}, \end{cases}$$

$$G_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{w_1\} & \text{if } x = \varepsilon_1, \\ \{w_1, w_2, w_3\} & \text{if } x = \varepsilon_2, \\ \{w_1, w_2, w_5\} & \text{if } x = \varepsilon_3, \\ \{w_1, w_2, w_4, w_5\} & \text{if } x = \varepsilon_4, \\ U & \text{if } x = \varepsilon_5, \end{cases}$$

$$\ell : U \to [0,1],\ x \mapsto \begin{cases} 0.8 & \text{if } x = w_1, \\ 0.7 & \text{if } x = w_2, \\ 0.3 & \text{if } x = w_3, \\ 0.3 & \text{if } x = w_4, \\ 0.5 & \text{if } x = w_5, \end{cases}$$

It is routine to verify that $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ is a makgeolli ideal over U.

Assume that (17) is true. Since $\frac{x*y}{\ell(x*y)} \in \ell$ and $\frac{y}{\ell(y)} \in \ell$ for all $x, y \in U$, it follows from (17) that $\frac{x}{\min\{\ell(x*y), \ell(y)\}} \in \ell$, that is,

$$(\forall x, y \in U)\ (\ell(x) \geq \min\{\ell(x*y), \ell(y)\}). \tag{18}$$

Now, let $x, y \in U$ and $t, r \in (0,1]$ such that $\frac{x*y}{t} \in \ell$ and $\frac{y}{r} \in \ell$. Then $\ell(x*y) \geq t$ and $\ell(y) \geq r$. If (18) holds, then

$$\ell(x) \geq \min\{\ell(x*y), \ell(y)\} \geq \min\{t, r\},$$

and so $\frac{x}{\min\{t,r\}} \in \ell$. Therefore we have the following theorem.

Theorem 5. *Let (U, E) be a BCK/BCI-soft universe. A makgeolli structure $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ on U is an makgeolli ideal over U if, and only if, it satisfies (14), (16), (18), and*

$$(\forall x \in U)\ (\ell(0) \geq \ell(x)). \tag{19}$$

Proposition 4. *Let (U, E) be a BCK/BCI-soft universe. Every makgeolli ideal $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ over U satisfies the following assertions.*

(1) $(\forall a, b \in E)\ \left(a \leq b \Rightarrow M_E(a) \supseteq M_E(b),\ G_E(a) \subseteq G_E(b) \right)$.

(2) $(\forall x, y \in U)\ \left(x \leq y \Rightarrow \ell(x) \geq \ell(y) \right)$.

(3) $(\forall a, b, c \in E)\ \left(a \rightsquigarrow b \leq c \Rightarrow \begin{cases} M_E(a) \supseteq M_E(b) \cap M_E(c) \\ G_E(a) \subseteq G_E(b) \cup G_E(c) \end{cases} \right)$.

(4) $(\forall x, y, z \in U)\ \left(x*y \leq z \Rightarrow \ell(x) \geq \min\{\ell(y), \ell(z)\} \right)$.

Proof. Let $a, b \in E$ be such that $a \leq b$. Then $a \rightsquigarrow b = 0$, so the conditions (14) and (16) imply that

$$M_E(b) = M_E(0) \cap M_E(b) = M_E(a \rightsquigarrow b) \cap M_E(b) \subseteq M_E(a),$$

$$G_E(b) = G_E(0) \cup G_E(b) = G_E(a \rightsquigarrow b) \cup G_E(b) \supseteq G_E(a),$$

If $x \leq y$ for all $x, y \in U$, then $x * y = 0$. It follows from (18) and (19) that

$$\ell(y) = \min\{\ell(0), \ell(y)\} = \min\{\ell(x * y), \ell(y)\} \leq \ell(x).$$

Assume that $a \rightsquigarrow b \leq c$ for all $a, b, c \in E$. Then $(a \rightsquigarrow b) \rightsquigarrow c = 0$, and so

$$\begin{aligned} M_E(c) &= M_E(0) \cap M_E(c) = M_E((a \rightsquigarrow b) \rightsquigarrow c) \cap M_E(c) \subseteq M_E(a \rightsquigarrow b), \\ G_E(c) &= G_E(0) \cup G_E(c) = G_E((a \rightsquigarrow b) \rightsquigarrow c) \cup G_E(c) \supseteq G_E(a \rightsquigarrow b) \end{aligned} \tag{20}$$

by (14) and (16). If $x * y \leq z$ for all $x, y, z \in U$, then $(x * y) * z = 0$. Using (18) and (19), we have

$$\ell(z) = \min\{\ell(0), \ell(z)\} = \min\{\ell((x * y) * z), \ell(z)\} \leq \ell(x * y). \tag{21}$$

It follows from (16) and (18) that

$$\begin{aligned} &M_E(a) \supseteq M_E(a \rightsquigarrow b) \cap M_E(b) \supseteq M_E(b) \cap M_E(c), \\ &G_E(a) \subseteq G_E(a \rightsquigarrow b) \cup G_E(b) \subseteq G_E(b) \cup G_E(c), \\ &\ell(x) \geq \min\{\ell(x * y), \ell(y)\} \geq \min\{\ell(y), \ell(z)\}. \end{aligned}$$

This completes the proof. □

Proposition 5. *Let (U, E) be a BCK/BCI-soft universe. Every makgeolli ideal $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ over U satisfies the following assertions.*

$$(\forall a, b, c \in E) \left(\begin{array}{l} M_E(a \rightsquigarrow b) \supseteq M_E(a \rightsquigarrow c) \cap M_E(c \rightsquigarrow b) \\ G_E(a \rightsquigarrow b) \subseteq G_E(a \rightsquigarrow c) \cup G_E(c \rightsquigarrow b) \end{array} \right). \tag{22}$$

$$(\forall x, y, z \in U)(\forall t, r \in (0, 1]) \left(\tfrac{x*z}{t} \in \ell,\ \tfrac{z*y}{r} \in \ell \ \Rightarrow\ \tfrac{x*y}{\min\{t,r\}} \in \ell \right). \tag{23}$$

$$(\forall a, b \in E) \left(\begin{array}{l} M_E(a \rightsquigarrow b) = M_E(0) \ \Rightarrow\ M_E(a) \supseteq M_E(b) \\ G_E(a \rightsquigarrow b) = G_E(0) \ \Rightarrow\ G_E(a) \subseteq G_E(b) \end{array} \right). \tag{24}$$

$$(\forall x, y \in U) \left(\tfrac{x*y}{\ell(0)} \in \ell \ \Rightarrow\ \tfrac{x}{\ell(y)} \in \ell \right). \tag{25}$$

Proof. Since $(a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) \leq c \rightsquigarrow b$ for all $a, b, c \in E$, we have (22) by (3) in Proposition 4. Let $x, y, z \in U$ and $t, r \in (0, 1]$ be such that $\frac{x*z}{t} \in \ell$ and $\frac{z*y}{r} \in \ell$. Then $\ell(x * z) \geq t$ and $\ell(z * y) \geq r$. Since $(x * y) * (x * z) \leq z * y$ for all $x, y, z \in U$, it follows from (4) in Proposition 4 that

$$\ell(x * y) \geq \min\{\ell(x * z), \ell(z * y)\} \geq \min\{t, r\}.$$

Hence $\frac{x*y}{\min\{t,r\}} \in \ell$, and (23) is valid. Consider $a, b \in E$ satisfying $M_E(a \rightsquigarrow b) = M_E(0)$ and $G_E(a \rightsquigarrow b) = G_E(0)$. Then

$$M_E(a) \supseteq M_E(a \rightsquigarrow b) \cap M_E(b) = M_E(0) \cap M_E(b) = M_E(b)$$

and

$$G_E(a) \subseteq G_E(a \rightsquigarrow b) \cup G_E(b) = G_E(0) \cup M_E(b) = M_E(b).$$

Suppose that $\frac{x*y}{\ell(0)} \in \ell$ for all $x, y \in U$. Then $\ell(x * y) = \ell(0)$, and so

$$\ell(x) \geq \min\{\ell(x * y), \ell(y)\} = \min\{\ell(0), \ell(y)\} = \ell(y),$$

that is, $\frac{x}{\ell(y)} \in \ell$. This completes the proof. □

Proposition 6. *Let (U, E) be a BCK/BCI-soft universe. For every makgeolli ideal $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ over U, the following are equivalent.*

(1) $\begin{cases} (\forall a, b \in E) \begin{pmatrix} M_E(a \rightsquigarrow b) \supseteq M_E((a \rightsquigarrow b) \rightsquigarrow b) \\ G_E(a \rightsquigarrow b) \subseteq G_E((a \rightsquigarrow b) \rightsquigarrow b) \end{pmatrix}. \\ (\forall x, y \in U)\, (\ell(x * y) \geq \ell((x * y) * y)). \end{cases}$

(2) $\begin{cases} (\forall a, b, c \in E) \begin{pmatrix} M_E((a \rightsquigarrow c) \rightsquigarrow (b \rightsquigarrow c)) \supseteq M_E((a \rightsquigarrow b) \rightsquigarrow c) \\ G_E((a \rightsquigarrow c) \rightsquigarrow (b \rightsquigarrow c)) \subseteq G_E((a \rightsquigarrow b) \rightsquigarrow c) \end{pmatrix}. \\ (\forall x, y, z \in U)\, (\ell((x * z) * (y * z)) \geq \ell((x * y) * z)). \end{cases}$

Proof. Let $a, b, c \in E$ and assume that (1) is valid. Since

$$((a \rightsquigarrow (b \rightsquigarrow c)) \rightsquigarrow c) \rightsquigarrow c = ((a \rightsquigarrow c) \rightsquigarrow (b \rightsquigarrow c)) \rightsquigarrow c \leq (a \rightsquigarrow b) \rightsquigarrow c,$$

it follows from Proposition 4 that

$$\begin{aligned} M_E((a \rightsquigarrow c) \rightsquigarrow (b \rightsquigarrow c)) &= M_E((a \rightsquigarrow (b \rightsquigarrow c)) \rightsquigarrow c) \\ &\supseteq M_E(((a \rightsquigarrow (b \rightsquigarrow c)) \rightsquigarrow c) \rightsquigarrow c) \\ &\supseteq M_E((a \rightsquigarrow b) \rightsquigarrow c) \end{aligned}$$

and

$$\begin{aligned} G_E((a \rightsquigarrow c) \rightsquigarrow (b \rightsquigarrow c)) &= G_E((a \rightsquigarrow (b \rightsquigarrow c)) \rightsquigarrow c) \\ &\subseteq G_E(((a \rightsquigarrow (b \rightsquigarrow c)) \rightsquigarrow c) \rightsquigarrow c) \\ &\subseteq G_E((a \rightsquigarrow b) \rightsquigarrow c). \end{aligned}$$

Using (1), (2), and (3), we get $((x * (y * z)) * z) * z \leq (x * y) * z$ for all $x, y, z \in X$. Hence

$$\ell((x * z) * (y * z)) = \ell((x * (y * z)) * z) \geq \ell(((x * (y * z)) * z) * z) \geq \ell((x * y) * z)$$

for all $x, y, z \in X$.

Conversely, suppose that (2) is true. If we take $b = c$ and $y = z$ in (2), then

$$\begin{aligned} M_E(a \rightsquigarrow c) &= M_E((a \rightsquigarrow c) \rightsquigarrow (c \rightsquigarrow c)) \supseteq M_E((a \rightsquigarrow c) \rightsquigarrow c) \\ G_E(a \rightsquigarrow c) &= G_E((a \rightsquigarrow c) \rightsquigarrow (c \rightsquigarrow c)) \subseteq G_E((a \rightsquigarrow c) \rightsquigarrow c) \end{aligned}$$

and

$$\ell(x * z) = \ell((x * z) * (z * z)) \geq \ell((x * z) * z)$$

by (III) and (1). This proves (1). □

Theorem 6. *In a BCK-soft universe (U, E), every makgeolli ideal is a makgeolli algebra.*

Proof. Let $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ be a makgeolli ideal over U. For any $a, b \in E$ and $x, y \in U$, we have

$$\begin{aligned} M_E(a \rightsquigarrow b) &\supseteq M_E((a \rightsquigarrow b) \rightsquigarrow a) \cap M_E(a) = M_E((a \rightsquigarrow a) \rightsquigarrow b) \cap M_E(a) \\ &= M_E(0 \rightsquigarrow b) \cap M_E(a) = M_E(0) \cap M_E(a) \supseteq M_E(a) \cap M_E(b), \end{aligned}$$

$$\begin{aligned} G_E(a \rightsquigarrow b) &\subseteq G_E((a \rightsquigarrow b) \rightsquigarrow a) \cup G_E(a) = G_E((a \rightsquigarrow a) \rightsquigarrow b) \cup G_E(a) \\ &= G_E(0 \rightsquigarrow b) \cup G_E(a) = G_E(0) \cup G_E(a) \subseteq G_E(a) \cup G_E(b), \end{aligned}$$

and

$$\ell(x * y) \geq \min\{\ell((x * y) * x), \ell(x)\} = \min\{\ell((x * x) * y), \ell(x)\}$$
$$= \min\{\ell(0 * y), \ell(x)\} = \min\{\ell(0), \ell(x)\} \geq \min\{\ell(x), \ell(y)\}.$$

Therefore $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ is a makgeolli algebra over U by Theorem 1. □

The following example shows that the converse of Theorem 6 is not true in general.

Example 5. *Let $U = \mathcal{P}(\mathbb{N})$. Define a binary operation $*$ on U by*

$$(\forall A, B \in U) \left(A * B = \begin{cases} \emptyset & \text{if } A \subseteq B \\ A \setminus B & \text{otherwise} \end{cases} \right). \tag{26}$$

Then $(U, *, \emptyset)$ is a BCK-algebra (see the work by the authors of [31]). Consider a BCK-algebra $E = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ in Table 7.

Table 7. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	3	1	0

Then (U, E) is a BCK-soft universe. Let $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ be a makgeolli structure on U defined by

$$M_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} \mathbb{N} & \text{if } x = 0, \\ 4\mathbb{N} & \text{if } x = 1, \\ 2\mathbb{N} & \text{if } x = 2, \\ 3\mathbb{N} & \text{if } x = 3, \\ 8\mathbb{N} & \text{if } x = 4, \end{cases}$$

$$G_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} 12\mathbb{N} & \text{if } x = 0, \\ 3\mathbb{N} & \text{if } x = 1, \\ 6\mathbb{N} & \text{if } x = 2, \\ 5\mathbb{N} & \text{if } x = 3, \\ \mathbb{N} & \text{if } x = 4, \end{cases}$$

$$\ell : U \to [0,1],\ x \mapsto \begin{cases} 0.8 & \text{if } x \in S, \\ 0.3 & \text{if } x \notin S \end{cases}$$

where S is a subalgebra of U. It is routine to verify that $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ is a makgeolli algebra over U. But it is not a makgeolli ideal over U since $M_E(4 * 2) \cap M_E(2) = M_E(3) \cap M_E(2) = 3\mathbb{N} \cap 2\mathbb{N} = 6\mathbb{N} \nsubseteq 8\mathbb{N} = M_E(4)$ and/or $G_E(4 * 2) \cap G_E(2) = G_E(3) \cup G_E(2) = 5\mathbb{N} \cup 6\mathbb{N} \nsupseteq \mathbb{N} = G_E(4)$.

We provide a condition for a makgeolli algebra to be a makgeolli ideal in BCK-soft universe.

Theorem 7. *In a BCK-soft universe (U, E), let $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ be a makgeolli algebra over U satisfying the conditions (3) and (4) in Proposition 4. Then $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ is a makgeolli ideal over U.*

Proof. By Proposition 1, we know that $M_E(a) \subseteq M_E(0)$, $G_E(b) \supseteq G_E(0)$ and $\frac{0}{\ell(x)} \in \ell$ for all $a \in E$ and $x \in U$. Sine $a \rightsquigarrow (a \rightsquigarrow b) \leq b$ and $x * (x * y) \leq y$ for all $a, b \in E$ and $x, y \in U$, it follows from the conditions (3) and (4) in Proposition 4 that $M_E(a) \supseteq M_E(a \rightsquigarrow b) \cap M_E(b)$, $G_E(a) \subseteq G_E(a \rightsquigarrow b) \cup G_E(b)$ and $\ell(x) \geq \min\{\ell(x * y), \ell(y)\}$. Therefore, $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ is a makgeolli ideal over U. □

The following example shows that Theorem 6 is not true in a BCI-soft universe (U, E).

Example 6. *Consider the two BCI-algebras* $U = \{0, 1, a, b, c\}$ *and* $E = \{0, a, b, c\}$ *with binary operation* $*$ *and* $\rightsquigarrow$ *given by Tables 8 and 9, respectively.*

Table 8. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Table 9. Cayley table for the binary operation "$\rightsquigarrow$".

$\rightsquigarrow$	**0**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then (U, E) *is a BCI-soft universe. Let* $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ *be a makgeolli structure on* U *defined by*

$$M_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} U & \text{if } x = 0, \\ \{0, 1, a\} & \text{if } x = a, \\ \{0, 1\} & \text{if } x = b, \\ \{0\} & \text{if } x = c, \end{cases}$$

$$G_E : E \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{0, 1\} & \text{if } x = 0, \\ \{0, 1, b\} & \text{if } x = a, \\ \{0, 1, c\} & \text{if } x = b, \\ U & \text{if } x = c, \end{cases}$$

$$\ell : U \to [0, 1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.8 & \text{if } x = 1, \\ 0.3 & \text{if } x \in \{a, b\}, \\ 0.6 & \text{if } x = c. \end{cases}$$

It is routine to verify that $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ *is a makgeolli ideal over* U*, but it is not a makgeolli algebra over* U *since*

$$M_E(a) \cap M_E(b) = \{0, 1, a\} \cap \{0, 1\} = \{0, 1\} \nsubseteq \{0\} = M_E(c) = M_E(a \rightsquigarrow b).$$

We provide a condition for Theorem 6 to be true in a BCI-soft universe (U, E).

Theorem 8. *In a BCI-soft universe (U, E), let $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ be a makgeolli ideal over U satisfying the following condition.*

$$(\forall a \in E, \forall x \in U)(M_E(0 \rightsquigarrow a) \supseteq M_E(a), G_E(0 \rightsquigarrow a) \subseteq G_E(a), \tfrac{0*x}{\ell(x)} \in \ell). \tag{27}$$

Then $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ is a makgeolli algebra over U.

Proof. Let $a, b \in E$ and $x, y \in U$. Then

$$M_E(a \rightsquigarrow b) \supseteq M_E((a \rightsquigarrow b) \rightsquigarrow a) \cap M_E(a) = M_E(0 \rightsquigarrow b) \cap M_E(a) \supseteq M_E(a) \cap M_E(b),$$

$$G_E(a \rightsquigarrow b) \subseteq G_E((a \rightsquigarrow b) \rightsquigarrow a) \cup G_E(a) = G_E(0 \rightsquigarrow b) \cup G_E(a) \subseteq G_E(a) \cup G_E(b),$$

and $\ell(x * y) \geq \min\{\ell((x * y) * x), \ell(x)\} = \min\{\ell(0 * y), \ell(x)\} \geq \min\{\ell(y), \ell(x)\}$. It follows from Theorem 1 that $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ is a makgeolli algebra over U. □

Let $(X, *, 0)$ be a BCI-algebra and $B(X) := \{x \in X \mid 0 \leq x\}$. For any $x \in X$ and $n \in \mathbb{N}$, we define x^n by

$$x^1 = x, \; x^{n+1} = x * (0 * x^n).$$

The element x of X is said to be of finite periodic (see the work by the authors of [32]) if there exists $n \in \mathbb{N}$ such that $x^n \in B(X)$. The period of x is denoted by $|x|$ and it is given as follows.

$$|x| = \min\{n \in \mathbb{N} \mid x^n \in B(X)\}.$$

Theorem 9. *Let (U, E) be a BCI-soft universe in which every element of U (resp., E) is of finite period. Then every makgeolli ideal over U is a makgeolli algebra over U.*

Proof. Let $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ be a makgeolli ideal over U. For any $a \in E$ and $x \in U$, assume that $|a| = m$ and $|x| = n$. Then $a^m \in B(E)$ and $x^n \in B(U)$. Note that

$$\begin{aligned}(0 \rightsquigarrow a^{m-1}) \rightsquigarrow a &= (0 \rightsquigarrow (0 \rightsquigarrow (0 \rightsquigarrow a^{m-1}))) \rightsquigarrow a = (0 \rightsquigarrow a) \rightsquigarrow (0 \rightsquigarrow (0 \rightsquigarrow a^{m-1})) \\ &= 0 \rightsquigarrow (a \rightsquigarrow (0 \rightsquigarrow a^{m-1})) = 0 \rightsquigarrow a^m = 0\end{aligned}$$

and

$$\begin{aligned}(0 * x^{n-1}) * x &= (0 * (0 * (0 * x^{n-1}))) * x = (0 * x) * (0 * (0 * x^{n-1})) \\ &= 0 * (x * (0 * x^{n-1})) = 0 * x^n = 0.\end{aligned}$$

Hence $M_E((0 \rightsquigarrow a^{m-1}) \rightsquigarrow a) = M_E(0) \supseteq M_E(a)$, $G_E((0 \rightsquigarrow a^{m-1}) \rightsquigarrow a) = G_E(0) \subseteq G_E(a)$ and $\ell((0 * x^{n-1}) * x) = \ell(0) \geq \ell(x)$ by (14) and (19). It follows from (16) and (18) that

$$\begin{aligned}&M_E(0 \rightsquigarrow a^{m-1}) \supseteq M_E((0 \rightsquigarrow a^{m-1}) \rightsquigarrow a) \cap M_E(a) \supseteq M_E(a), \\ &G_E(0 \rightsquigarrow a^{m-1}) \subseteq G_E((0 \rightsquigarrow a^{m-1}) \rightsquigarrow a) \cup G_E(a) \subseteq G_E(a), \\ &\ell(0 * x^{n-1}) \geq \min\{\ell((0 * x^{n-1}) * x), \ell(x)\} \geq \ell(x).\end{aligned} \tag{28}$$

Also, note that

$$\begin{aligned}(0 \rightsquigarrow a^{m-2}) \rightsquigarrow a &= (0 \rightsquigarrow (0 \rightsquigarrow (0 \rightsquigarrow a^{m-2}))) \rightsquigarrow a = (0 \rightsquigarrow a) \rightsquigarrow (0 \rightsquigarrow (0 \rightsquigarrow a^{m-2})) \\ &= 0 \rightsquigarrow (a \rightsquigarrow (0 \rightsquigarrow a^{m-2})) = 0 \rightsquigarrow a^{m-1}\end{aligned}$$

and

$$(0 * x^{n-2}) * x = (0 * (0 * (0 * x^{n-2}))) * x = (0 * x) * (0 * (0 * x^{n-2}))$$
$$= 0 * (x * (0 * x^{n-2})) = 0 * x^{n-1}.$$

Using (28), we have

$$M_E((0 \rightsquigarrow a^{m-2}) \rightsquigarrow a) = M_E(0 \rightsquigarrow a^{m-1}) \supseteq M_E(a),$$
$$G_E((0 \rightsquigarrow a^{m-2}) \rightsquigarrow a) = G_E(0 \rightsquigarrow a^{m-1}) \subseteq G_E(a),$$
$$\ell((0 * x^{n-2}) * x) = \ell(0 * x^{n-1}) \geq \ell(x).$$

It follows from (16) and (18) that

$$M_E(0 \rightsquigarrow a^{m-2}) \supseteq M_E((0 \rightsquigarrow a^{m-2}) \rightsquigarrow a) \cap M_E(a) \supseteq M_E(a),$$
$$G_E(0 \rightsquigarrow a^{m-2}) \subseteq G_E((0 \rightsquigarrow a^{m-2}) \rightsquigarrow a) \cup G_E(a) \subseteq G_E(a),$$
$$\ell(0 * x^{n-2}) \geq \min\{\ell((0 * x^{n-2}) * x), \ell(x)\} \geq \ell(x).$$

Continuing this prosess, we get $M_E(0 \rightsquigarrow a) \supseteq M_E(a)$, $G_E(0 \rightsquigarrow a) \subseteq G_E(a)$ and $\ell(0 * x) \geq \ell(x)$, i.e., $\frac{0*x}{\ell(x)} \in \ell$. Hence $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ satisfies the condition (27), and therefore $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ is a makgeolli algebra over U by Theorem 8. □

Theorem 10. *Let (U, E) be a BCK/BCI-soft universe. Then a makgeolli structure $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ on U is a makgeolli ideal over U if and only if the sets $\mathcal{E}_E(M_E;\alpha)$, $\mathcal{E}_E(G_E;\beta)$, and $\mathcal{U}(\ell;t)$ are ideals of E and U, respectively, for all $\alpha, \beta \in \mathcal{P}(U)$ and $t \in [0,1]$.*

Proof. Assume that $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ on U is a makgeolli ideal over U. It is clear that 0 is contained in $\mathcal{E}_E(M_E;\alpha)$, $\mathcal{E}_E(G_E;\beta)$ and $\mathcal{U}(\ell;t)$ for all $\alpha, \beta \in \mathcal{P}(U)$ and $t \in [0,1]$. Let $a, b \in E$ be such that $a \rightsquigarrow b \in \mathcal{E}_E(M_E;\alpha)$ and $b \in \mathcal{E}_E(M_E;\alpha)$ (resp., $a \rightsquigarrow b \in \mathcal{E}_E(G_E;\beta)$ and $b \in \mathcal{E}_E(G_E;\beta)$). Then

$$M_E(a) \supseteq M_E(a \rightsquigarrow b) \cap M_E(b) \supseteq \alpha$$

(respectively, $G_E(a) \subseteq zg_E(a \rightsquigarrow b) \cup G_E(b) \subseteq \beta$), and thus $a \in \mathcal{E}_E(M_E;\alpha)$ (resp., $a \in \mathcal{E}_E(G_E;\beta)$). For any $x, y \in U$, let $x * y \in \mathcal{U}(\ell;t)$ and $y \in \mathcal{U}(\ell;t)$. Then $\ell(x * y) \geq t$ and $\ell(y) \geq t$. It follows from Theorem 1 that $\ell(x) \geq \min\{\ell(x * y), \ell(y)\} \geq t$. Hence $x \in \mathcal{U}(\ell;t)$. Therefore $\mathcal{E}_E(M_E;\alpha)$, $\mathcal{E}_E(G_E;\beta)$ and $\mathcal{U}(\ell;t)$ are ideals of E and U, respectively.

Conversely, suppose that the sets $\mathcal{E}_E(M_E;\alpha)$, $\mathcal{E}_E(G_E;\beta)$ and $\mathcal{U}(\ell;t)$ are ideals of E and U, respectively, for all $\alpha, \beta \in \mathcal{P}(U)$ and $t \in [0,1]$. Let $a, b \in E$ and $x \in U$ be such that $M_E(a) = \alpha$, $G_E(b) = \beta$ and $\ell(x) = t$. Then $M_E(a) = \alpha \subseteq M_E(0)$, $G_E(b) = \beta \subseteq G_E(0)$ and $\ell(x) = t \leq \ell(0)$. Let $a, b \in E$ and $x, y \in U$ be such that $M_E(a \rightsquigarrow b) = \alpha_1$, $M_E(b) = \alpha_2$ (resp., $G_E(a \rightsquigarrow b) = \beta_1$, $G_E(b) = \beta_2$) and $\ell(x * y) = t_1$, $\ell(y) = t_2$. If we take $\alpha = \alpha_1 \cap \alpha_2$ (resp., $\beta = \beta_1 \cup \beta_2$) and $t = \min\{t_1, t_2\}$, then $a \rightsquigarrow b \in \mathcal{E}_E(M_E;\alpha)$, $b \in \mathcal{E}_E(M_E;\alpha)$ (resp., $a \rightsquigarrow b \in \mathcal{E}_E(G_E;\alpha)$, $b \in \mathcal{E}_E(G_E;\alpha)$) and $x * y \in \mathcal{U}(\ell;t)$, $y \in \mathcal{U}(\ell;t)$. It follows that $a \in \mathcal{E}_E(M_E;\alpha)$ (resp., $a \in \mathcal{E}_E(G_E;\alpha)$) and $x \in \mathcal{U}(\ell;t)$. Hence

$$M_E(a) \supseteq \alpha = \alpha_1 \cap \alpha_2 = M_E(a \rightsquigarrow b) \cap M_E(b)$$

(resp., $G_E(a) \subseteq \beta = \beta_1 \cup \beta_2 = G_E(a \rightsquigarrow b) \cup G_E(b)$) and

$$\ell(x) \geq t = \min\{t_1, t_2\} = \min\{\ell(x * y), \ell(y)\}.$$

Therefore $\mathcal{M}_{(E,U)} = (M_E, G_E, \ell)$ on U is a makgeolli ideal over U by Theorem 1. □

5. Applications in Medical Sciences

Miss J (say) has cancer and needs surgery. She tries to find a hospital with excellent medical skills, low treatment costs, and friendly nurses. There are six hospitals, $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and there are two parameter sets, $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, and $B = \{\delta_1, \delta_2\}$, where each parameter ε_i for $i = 1, 2, 3$ and δ_j for $j = 1, 2$, stands for

ε_1: Medical expenses are low; ε_2: Medical expenses are intermediate

ε_3: Medical expenses are expensive

δ_1: Nurses are kind; δ_2: Nurses are unkind

The medical skills of the hospital are indicated by the following functions.

$$\ell : U \to [0,1],\ x \mapsto \begin{cases} 0.1 & \text{if } x = h_1, \\ 0.7 & \text{if } x = h_2, \\ 0.3 & \text{if } x = h_3, \\ 0.9 & \text{if } x = h_4, \\ 0.8 & \text{if } x = h_5, \\ 0.5 & \text{if } x = h_6, \end{cases}$$

where, the higher the number, the better the medical skill. Assume that $M_A(\varepsilon_1) = \{h_1, h_2\}$, $M_A(\varepsilon_2) = \{h_3, h_6\}$, $M_A(\varepsilon_3) = \{h_4, h_5\}$, $G_B(\delta_1) = \{h_2, h_4, h_6\}$ and $G_B(\delta_2) = \{h_1, h_3, h_5\}$. Then the makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$ on U is given by Table 10.

Table 10. Tabular representation of the makgeolli structure $\mathcal{M}_{(A,B,U)} = (M_A, G_B, \ell)$.

X	h_1	h_2	h_3	h_4	h_5	h_6
$(M_A(\varepsilon_1), G_B(\delta_1), \ell(x))$	$(1,0,0.1)$	$(1,1,0.7)$	$(0,0,0.3)$	$(0,1,0.9)$	$(0,0,0.8)$	$(0,1,0.5)$
$(M_A(\varepsilon_1), G_B(\delta_2), \ell(x))$	$(1,1,0.1)$	$(1,0,0.7)$	$(0,1,0.3)$	$(0,0,0.9)$	$(0,1,0.8)$	$(0,0,0.5)$
$(M_A(\varepsilon_2), G_B(\delta_1), \ell(x))$	$(0,0,0.1)$	$(0,1,0.7)$	$(1,0,0.3)$	$(0,1,0.9)$	$(0,0,0.8)$	$(1,1,0.5)$
$(M_A(\varepsilon_2), G_B(\delta_2), \ell(x))$	$(0,1,0.1)$	$(0,0,0.7)$	$(1,1,0.3)$	$(0,0,0.9)$	$(0,1,0.8)$	$(1,0,0.5)$
$(M_A(\varepsilon_3), G_B(\delta_1), \ell(x))$	$(0,0,0.1)$	$(0,1,0.7)$	$(0,0,0.3)$	$(1,1,0.9)$	$(1,0,0.8)$	$(0,1,0.5)$
$(M_A(\varepsilon_3), G_B(\delta_2), \ell(x))$	$(0,1,0.1)$	$(0,0,0.7)$	$(0,1,0.3)$	$(1,0,0.9)$	$(1,1,0.8)$	$(0,0,0.5)$

You know that, in the first row of Table 10, if you find a hospital that responds to the element $(1, 1, 0.9)$, the hospital has excellent medical skills, friendly nurses, and medical costs are also low, but you cannot see it. However, you can see the element $(1, 1, 0.7)$ in the first row of Table 10, and the corresponding hospital is h_2. Therefore, although the medical skill of h_2 is slightly lower than that of h_4 and h_5, it can be found that the nurse is kind and also the treatment cost is cheap. Therefore Miss J will choose hospital h_2 for surgery. Even if the cost of treatment is high, if Miss J find the hospital which the medical skills are excellent and the nurses are kind, she can select the hospital h_4 that corresponds to $(M_A(\varepsilon_3), G_B(\delta_1), \ell(x)) = (1, 1, 0.9)$. We can see that the cost of treatment in the hospital (h_4) with the best medical skills is the most expensive. If a mild cold patient tries to visit a hospital, he or she does not need high-level medical skills. Regardless of the nurse's kindness, he/she will try to find a hospital where treatment costs are low. In this case, he or she can select the hospital h_1.

6. Conclusions

Soft set theory, which was proposed by Molodtsov in 1999, is a generalization of fuzzy set theory. It is a good mathematical tool for dealing with uncertainty in a parametric manner. Soft set has many applications in medical diagnosis and decision making etc. As an extension of the classical set, Zadeh introduced the fuzzy set in 1965, which has been applied in so many areas. In this paper, we have introduced the concept of makgeolli structures (see Definition 1) using fuzzy and soft set theory and have applied it to BCK/BCI-algebras. We have defined the notion of makgeolli algebra (see Definition 3) and makgeolli ideal (see Definition 4) in BCK/BCI-algebras, and have investigated several properties. We have shown that every makgeolli ideal is a makgeolli algebra in BCK-soft universes (see Theorem 6). We have considered an example to show that any makgeolli algebra may not be a makgeolli ideal in BCK-soft universes (see Example 5). We have provided a condition for a makgeolli algebra to be a makgeolli ideal in BCK-soft universes (see Theorem 7). We have considered an example to show that any makgeolli ideal may not be a makgeolli algebra in BCI-soft universe (see Example 6), and have provided a condition for a makgeolli ideal to be a makgeolli algebra in BCI-soft universes (see Theorem 8). We have discussed characterization of makgeolli algebra and makgeolli ideal (see Theorems 1, 3, 5, and 10). We have made a new makgeolli algebra from old one (see Theorem 4). In the final section, we have considered an application in medical sciences. In the forthcoming research and papers, we will continue these ideas and will define new notions in several algebraic structures.

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