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On Uniquely 3-Colorable Plane Graphs without Adjacent Faces of Prescribed Degrees

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Abstract: A graph G is *uniquely k-colorable* if the chromatic number of G is k and G has only one k-coloring up to the permutation of the colors. For a plane graph G, two faces f_1 and f_2 of G are *adjacent (i,j)-faces* if $d(f_1) = i$, $d(f_2) = j$, and f_1 and f_2 have a common edge, where $d(f)$ is the degree of a face f. In this paper, we prove that every uniquely three-colorable plane graph has adjacent $(3,k)$-faces, where $k \leq 5$. The bound of five for k is the best possible. Furthermore, we prove that there exists a class of uniquely three-colorable plane graphs having neither adjacent $(3,i)$-faces nor adjacent $(3,j)$-faces, where i, j are fixed in $\{3,4,5\}$ and $i \neq j$. One of our constructions implies that there exists an infinite family of edge-critical uniquely three-colorable plane graphs with n vertices and $\frac{7}{3}n - \frac{14}{3}$ edges, where $n(\geq 11)$ is odd and $n \equiv 2 \pmod 3$.

Keywords: plane graph; unique coloring; uniquely three-colorable plane graph; construction; adjacent (i,j)-faces

1. Introduction

Graph coloring is one of the most studied problems in graph theory, because it has many important applications [1–3]. The main aim of the problem is to assign colors to the elements of a graph, such as vertices, subject to certain constraints.

For a plane graph G, $V(G)$, $E(G)$, and $F(G)$ are the sets of vertices, edges, and faces of G, respectively. The *degree* of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of neighbors of v in G. The *degree* of a face $f \in F(G)$, denoted by $d_G(f)$, is the number of edges in its boundary, cut edges being counted twice. When no confusion can arise, $d_G(v)$ and $d_G(f)$ are simplified as $d(v)$ and $d(f)$, respectively. A face f is a *k-face* if $d(f) = k$ and a *k^+-face* if $d(f) \geq k$. Two faces f_1 and f_2 of G are *adjacent (i,j)-faces* if $d(f_1) = i$, $d(f_2) = j$, and f_1 and f_2 have at least one common edge. Two distinct paths of G are *internally disjoint* if they have no internal vertices in common. For other terminologies and notations in graph theory, we refer to [4].

A *k-coloring* of a graph G is an assignment of k colors to the vertices of G such that no two adjacent vertices are assigned the same color. A graph G is *k-colorable* if G admits a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G is k-colorable. A graph G is *uniquely k-colorable* if $\chi(G) = k$ and G has only one k-coloring up to the permutation of the colors, where the coloring is called a *unique k-coloring* of G. In other words, all k-colorings of G induce the same partition of $V(G)$ into k independent sets, in which an independent set is called a *color class* of G. In addition, uniquely colorable graphs may be defined in terms of their chromatic polynomials, which was initiated by Birkhoff [5] for planar graphs in 1912 and for general graphs by Whitney [6] in 1932.

A graph G is uniquely k-colorable if and only if its chromatic polynomial is $k!$. For a discussion of chromatic polynomials, see Read [7].

Uniquely colorable graphs were first studied by Harary and Cartwright [8] in 1968. They proved the following theorem.

Theorem 1 (Harary and Cartwright [8])**.** *Let G be a uniquely k-colorable graph. Then, for any unique k-coloring of G, the subgraph induced by the union of any two color classes is connected.*

As a corollary of Theorem 1, it can be seen that a uniquely k-colorable graph G has at least $(k-1)|V(G)| - \binom{k}{2}$ edges. There are many references on uniquely colorable graphs [9–13].

Dailey [14] proved that the problem of determining whether a graph G is uniquely colorable is NP-complete. However, it is still open for the case of planar graphs. Therefore, it is important to characterize the structure of uniquely colorable planar graphs.

Chartrand and Geller [10] in 1969 started to study uniquely colorable planar graphs. They proved that uniquely three-colorable planar graphs with at least four vertices contain at least two triangles, uniquely four-colorable planar graphs are maximal planar graphs, and uniquely five-colorable planar graphs do not exist. Aksionov [15] in 1977 improved the lower bound for the number of triangles in a uniquely three-colorable planar graph. He proved that a uniquely three-colorable planar graph with at least five vertices contains at least three triangles and gave a complete description of uniquely three-colorable planar graphs containing exactly three triangles. Li et al. [12] proved that if a uniquely three-colorable planar graph G has at most four triangles, then G has two adjacent triangles. Moreover, for any $k \geq 5$, they constructed a uniquely three-colorable planar graph with k triangles and without adjacent triangles.

Let G be a uniquely k-colorable graph. G is *edge-critical* if $G - e$ is not uniquely k-colorable for any edge $e \in E(G)$. Obviously, if a uniquely k-colorable graph G has exactly $(k-1)|V(G)| - \binom{k}{2}$ edges, then G is edge-critical. Mel'nikov and Steinberg [16] in 1977 asked to find an exact upper bound for the number of edges in an edge-critical uniquely three-colorable planar graph with n vertices. In 2013, Matsumoto [17] proved that an edge-critical uniquely three-colorable planar graph has at most $\frac{8}{3}n - \frac{17}{3}$ edges and constructed an infinite family of edge-critical uniquely three-colorable planar graphs with n vertices and $\frac{9}{4}n - 6$ edges, where $n \equiv 0 \pmod 4$. This upper bound was improved by Li et al. [13] to $\frac{5}{2}n - 6$ when $n \geq 6$.

In this paper, we mainly prove Theorem 2.

Theorem 2. *If G is a uniquely three-colorable plane graph, then G has adjacent $(3,k)$-faces, where $k \leq 5$. The bound five for k is the best possible.*

Furthermore, by using constructions, we prove that there exist uniquely three-colorable plane graphs having neither adjacent $(3,i)$-faces nor adjacent $(3,j)$-faces, where i, j are fixed in $\{3,4,5\}$ and $i \neq j$. One of our constructions implies that there exists an infinite family of edge-critical uniquely three-colorable plane graphs with n vertices and $\frac{7}{3}n - \frac{14}{3}$ edges, where $n(\geq 11)$ is odd and $n \equiv 2 \pmod 3$. Our results further characterize the structure of the uniquely three-colorable plane graphs. The results can be used in optimal territorial distribution of mobile operators' transmitters.

2. Proof of Theorem 2

Now, we prove Theorem 2. First we give a useful Lemma 1.

Lemma 1. *Let G be a plane graph with three faces. If G has no adjacent $(3,k)$-faces, where $k \leq 5$, then $|E(G)| \geq 2|F(G)|$.*

Proof. We prove this by using a simple charging scheme. Since G has no adjacent $(3,k)$-faces when $k \leq 5$, for any edge e incident to a three-face f, e is incident to a face of degree at least six. Let

$ch(f) = d(f)$ for any face $f \in F(G)$, and we call $ch(f)$ the initial charge of the face f. Let initial charges in G be redistributed according to the following rule.

Rule: For each three-face f of G and each edge e incident with f, the 6^+-face incident with e sends $\frac{1}{3}$ charge to f through e.

Denote by $ch'(f)$ the charge of a face $f \in F(G)$ after applying the redistributed rule. Then:

$$\sum_{f \in F(G)} ch'(f) = \sum_{f \in F(G)} ch(f) = \sum_{f \in F(G)} d(f) = 2|E(G)| \quad (1)$$

On the other hand, for any three-face f of G, since the degree of each face adjacent to f is at least six, then by the redistributed rule, $ch'(f) = ch(f) + 3 \cdot \frac{1}{3} = d(f) + 1 = 4$. For any four-face or five-face f of G, $ch'(f) = ch(f) = d(f) \geq 4$. For any 6^+-face f of G, since f is incident to at most $d(f)$ edges, each of which is incident to a three-face, then $ch'(f) \geq ch(f) - \frac{1}{3}d(f) = \frac{2}{3}d(f) \geq 4$. Therefore, we have:

$$\sum_{f \in F(G)} ch'(f) \geq \sum_{f \in F(G)} 4 = 4|F(G)| \quad (2)$$

By Formulae (1) and (2), we have $|E(G)| \geq 2|F(G)|$. □

Proof of Theorem 2. Suppose that the theorem is not true, and let G be a counterexample to the theorem. Then, G has at least one three-face and no adjacent $(3,k)$-faces, where $k \leq 5$. By Lemma 1, $|E(G)| \geq 2|F(G)|$. Using Euler's Formula $|V(G)| - |E(G)| + |F(G)| = 2$, we can obtain:

$$|E(G)| \leq 2|V(G)| - 4.$$

Since G is uniquely three-colorable, then by Theorem 1, we have $|E(G)| \geq 2|V(G)| - 3$. This is a contradiction.

Note that the graph shown in Figure 1 is a uniquely three-colorable plane graph having neither adjacent $(3,3)$-faces nor adjacent $(3,4)$-faces. Therefore, the bound of five for k is the best possible. □

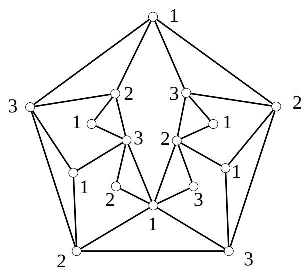

Figure 1. A uniquely three-colorable plane graph having neither adjacent $(3,3)$-faces nor adjacent $(3,4)$-faces.

Remark 1. *By piecing together more copies of the plane graph in Figure 1, one can construct an infinite class of uniquely three-colorable plane graphs having neither adjacent $(3,3)$-faces nor adjacent $(3,4)$-faces.*

3. Construction of Uniquely Three-Colorable Plane Graphs without Adjacent (3,3)-Faces or Adjacent (3,5)-Faces

There are many classes of uniquely three-colorable plane graphs having neither adjacent $(3,4)$-faces nor adjacent $(3,5)$-faces, such as even maximal plane graphs (maximal plane graphs in which each vertex has even degree) and maximal outerplanar graphs with at least six vertices. Now, we construct a class of uniquely three-colorable plane graphs having neither adjacent $(3,3)$-faces nor adjacent $(3,5)$-faces and prove that these graphs are edge-critical.

We construct a graph G_k as follows:

(1) $V(G_k) = \{u, w, v_0, v_1, \ldots, v_{3k-1}\}$;
(2) $E(G_k) = \{v_0v_1, v_1v_2, \ldots, v_{3k-2}v_{3k-1}, v_{3k-1}v_0\} \cup \{uv_i : i \equiv 1 \text{ or } 2 \pmod 3\} \cup \{wv_i : i \equiv 0 \text{ or } 1 \pmod 3\}$, where k is odd and $k \geq 3$ (see an example G_3 shown in Figure 2).

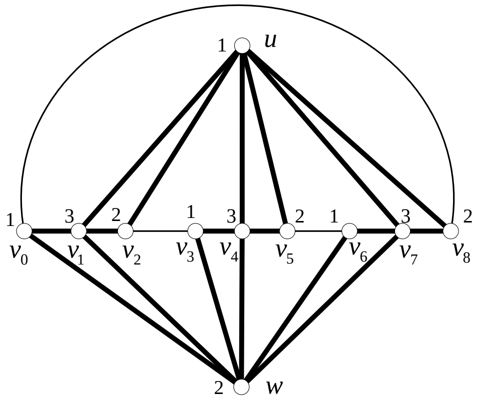

Figure 2. An example G_3.

Theorem 3. *For any odd k with $k \geq 3$, G_k is uniquely three-colorable.*

Proof. Let f be any three-coloring of G_k. Since $v_0v_1 \ldots v_{3k-1}v_0$ is a cycle of odd length and each v_i is adjacent to u or w, we have $f(u) \neq f(w)$. Without loss of generality, let $f(u) = 1$ and $f(w) = 2$. By the construction of G_k, we know that v_{3j+1} is adjacent to both u and w, where $j = 0, 1, \ldots, k-1$. Therefore, v_{3j+1} can only receive the color three, namely $f(v_{3j+1}) = 3$, $j = 0, 1, \ldots, k-1$. Since v_{3j} is adjacent to both w and v_{3j} in G_k, we have $f(v_{3j}) = 1$, $j = 0, 1, \ldots, k-1$. Similarly, we can obtain $f(v_{3j+2}) = 2$, $j = 0, 1, \ldots, k-1$. Therefore, the three-coloring f is uniquely decided as shown in Figure 2, and then, G_k is uniquely three-colorable. □

Theorem 4. *For any odd k with $k \geq 3$, G_k is edge-critical.*

Proof. To complete the proof, it suffices to show that $G_k - e$ is not uniquely three-colorable for any edge $e \in E(G_k)$. Let f be a uniquely three-coloring of G_k shown in Figure 2. Denote by E_{ij} the set of edges in G_k whose ends are colored by i and j, respectively, where $1 \leq i < j \leq 3$. Namely:

$$E_{ij} = \{xy : xy \in E(G_k), f(x) = i, f(y) = j\}, 1 \leq i < j \leq 3.$$

`Observation 1.` Both the subgraphs $G_k[E_{13}]$ and $G_k[E_{23}]$ of G_k induced by E_{13} and E_{23} are trees.
`Observation 2.` The subgraph $G_k[E_{12}]$ of G_k induced by E_{12} consists of k internally disjoint paths $uv_{3i-1}v_{3i}w$, where $i = 1, 2, \ldots, k$.

If $e \in E_{13} \cup E_{23}$, then $G_k - e$ is not uniquely three-colorable by Observation 1. Suppose that $e \in E_{12}$. By Observation 2, there exists a number $t \in \{1, 2, \ldots, k\}$ such that $e \in \{uv_{3t-1}, v_{3t-1}v_{3t}, v_{3t}w\}$. Moreover, $G_k - e$ contains at least one vertex of degree two. By repeatedly deleting vertices of degree two in $G_k - e$, we can obtain a subgraph $G_k - \{v_{3t-1}, v_{3t}\}$ of G_k. Now, we prove that $G_k - \{v_{3t-1}, v_{3t}\}$ is not uniquely three-colorable.

It can be seen that the restriction f_0 of f to the vertices of $G_k - \{v_{3t-1}, v_{3t}\}$ is a three-coloring of $G_k - \{v_{3t-1}, v_{3t}\}$. On the other hand, $G_k - \{v_{3t-1}, v_{3t}, u, w\}$ is a path, denoted by P. Let $f'(u) = f'(w) = 1$, and alternately, color the vertices of P by the other two colors. We can obtain a three-coloring f' of $G_k - \{v_{3t-1}, v_{3t}\}$, which is distinct from f_0. Since each three-coloring of $G_k - \{v_{3t-1}, v_{3t}\}$ can be extended to a three-coloring of $G_k - e$, we know that $G_k - e$ is not uniquely three-colorable when $e \in E_{12}$.

Since $E(G_k) = E_{12} \cup E_{13} \cup E_{23}$, $G_k - e$ is not uniquely three-colorable for any edge $e \in E(G_k)$. □

Note that G_k has $3k+2$ vertices and $7k$ edges by the construction. From Theorem 4, we can obtain the following result.

Corollary 1. *There exists an infinite family of edge-critical uniquely three-colorable plane graphs with n vertices and* $\frac{7}{3}n - \frac{14}{3}$ *edges, where* $n(\geq 11)$ *is odd and* $n \equiv 2 \pmod 3$.

Denote by $size(n)$ the upper bound of the number of edges of edge-critical uniquely three-colorable planar graphs with n vertices. Then, by Corollary 1 and the result due to Li et al. [13], we can obtain the following result.

Corollary 2. *For any odd integer n such that* $n \equiv 2 \pmod 3$ *and* $n \geq 11$, *we have* $\frac{7}{3}n - \frac{14}{3} \leq size(n) \leq \frac{5}{2}n - 6$.

Proof. First, in [13], Li et al. proved that $size(n) \leq \frac{5}{2}n - 6$ for any edge-critical uniquely three-colorable planar graph G with $n(n \geq 6)$ vertices. Then, by Corollary 1, we can conclude that Corollary 2 is true. □

Corollary 2 improves the lower bound $\frac{9}{4}n - 6$ of $size(n)$ given by Matsumoto [17] and gives a negative answer to a problem proposed by Mel'nikov and Steinberg [16], who asked that $size(n) = \frac{9}{4}n - 6$ for any $n \geq 12$.

4. Conclusions and Conjectures

In this paper, we obtained a structural property of uniquely three-colorable plane graphs. We proved that every uniquely three-colorable plane graph has adjacent $(3,k)$-faces, where $k \leq 5$, and the bound of five for k is the best possible. The graph in Figure 1 shows a uniquely three-colorable plane graph having neither adjacent $(3,3)$-faces nor adjacent $(3,4)$-faces. However this plane graph is two-connected. This prompts us to propose the following conjecture.

Conjecture 1. *Let G be a three-connected uniquely three-colorable plane graph. Then, G has adjacent* $(3,k)$*-faces, where* $k \leq 4$.

It can be seen that the uniquely three-colorable plane graph G_k constructed in Section 3 is three-connected. So Therefore, Conjecture 1 is true, then the bound of four for k is the best possible. Moreover, because the family of graphs G_k is the edge-critical uniquely three-colorable planar graphs with the largest number of edges found at present, we recall the follow conjecture proposed by Li et al [13].

Conjecture 2 ([13])**.** *Let G be an edge-critical uniquely three-colorable planar graph with n vertices. Then,* $size(n) \leq \frac{7}{3}n - \frac{14}{3}$.

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