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Statistical Solitons and Inequalities for Statistical Warped Product Submanifolds

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Abstract: Warped products play crucial roles in differential geometry, as well as in mathematical physics, especially in general relativity. In this article, first we define and study statistical solitons on Ricci-symmetric statistical warped products $\mathbb{R} \times_f N_2$ and $N_1 \times_f \mathbb{R}$. Second, we study statistical warped products as submanifolds of statistical manifolds. For statistical warped products statistically immersed in a statistical manifold of constant curvature, we prove Chen's inequality involving scalar curvature, the squared mean curvature, and the Laplacian of warping function (with respect to the Levi–Civita connection). At the end, we establish a relationship between the scalar curvature and the Casorati curvatures in terms of the Laplacian of the warping function for statistical warped product submanifolds in the same ambient space.

Keywords: statistical warped product submanifold; statistical manifold; B.Y.Chen inequality; Casorati curvatures; statistical soliton

MSC: 53B30; 53C15; 53C25

1. Introduction

Statistical manifolds were introduced in 1985 by S. Amari [1] in terms of information geometry, and they were applied by Lauritzen in [2]. Such manifolds have an important role in statistics as the statistical model often forms a geometrical manifold.

Let $\tilde{\nabla}$ be an affine connection on a (pseudo-)Riemannian manifold $(\tilde{N}, \tilde{g})$. The affine connection $\tilde{\nabla}^*$ on $\tilde{N}$ satisfying:

$$E\tilde{g}(F,G) = \tilde{g}(\tilde{\nabla}_E F, G) + \tilde{g}(F, \tilde{\nabla}^*_E G), \quad \forall E, F, G \in \Gamma(T\tilde{N}),$$

is called a *dual connection* of $\tilde{\nabla}$ with respect to $\tilde{g}$.

The triplet $(\tilde{N}, \tilde{\nabla}, \tilde{g})$ is called a *statistical manifold* if:

(a) the Codazzi equation $(\tilde{\nabla}_E \tilde{g})(F,G) = (\tilde{\nabla}_F \tilde{g})(E,G)$ holds, for any $E, F, G \in \Gamma(T\tilde{N})$;
(b) the torsion tensor field of $\tilde{\nabla}$ vanishes.

If $(\tilde{\nabla}, \tilde{g})$ is a statistical structure on $\tilde{N}$, then $(\tilde{\nabla}^*, \tilde{g})$ is also a statistical structure. The connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfy $(\tilde{\nabla}^*)^* = \tilde{\nabla}$. On the other hand, we have $\tilde{\nabla}^0 = \frac{1}{2}(\tilde{\nabla} + \tilde{\nabla}^*)$, where $\tilde{\nabla}^0$ is the Levi–Civita connection of $\tilde{N}$.

One of the most fruitful generalizations of Riemannian products is the warped product defined in [3]. The notion of warped products plays very important roles in differential geometry and in mathematical physics, especially in general relativity. For instance, space-time models in general relativity are usually expressed in terms of warped products (cf., e.g., [4,5]).

In 2006, L. Todjihounde [6] defined a suitable dualistic structure on warped product manifolds. Furthermore, Furuhata et al. [7] defined Kenmotsu statistical manifolds and studied how to construct such structures on the warped product of a holomorphic statistical manifold [8] and a line. In [9], H. Aytimur and C. Ozgur studied Einstein statistical warped product manifolds. Further, C. Murathan and B. Sahin [10] studied and obtained the Wintgen-like inequality for statistical submanifolds of statistical warped product manifolds.

The Ricci solitons are special solutions of the Ricci flow of the Hamilton. In Section 4, we define statistical solitons and study the problem under what conditions the base manifold or fiber manifold of a statistical warped product manifold is a statistical soliton.

Curvature invariants play the most fundamental and natural roles in Riemannian geometry. A fundamental problem in the theory of Riemannian submanifolds is (cf. [11]):

Problem A. *"Establish simple optimal relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold."*

The first solutions of this problem for warped product submanifolds were given in [11,12]. In Section 5, we study this fundamental problem for statistical warped product submanifolds in any statistical manifolds of constant curvature. Our solution to this problem given in this section is derived via the fundamental equations of statistical submanifolds.

An extrinsic curvature of a Riemannian submanifold was defined by Casorati in [13], as the normalized square of the length of the second fundamental form. Casorati curvature has nice applications in computer vision. It was preferred by Casorati over the traditional curvature since it corresponds better to the common intuition of curvature.

Several sharp inequalities between extrinsic and intrinsic curvatures for different submanifolds in real, complex, and quaternionic space forms endowed with various connections have been obtained (e.g., [14–21]). Such inequalities with a pair of conjugate affine connections involving the normalized scalar curvature of statistical submanifolds in different ambient spaces were obtained in [22–26].

Inspired by historical development on the classifications of Casorati curvatures and Ricci curvatures, we establish in Section 6 an inequality for statistical warped product submanifolds in a statistical manifold of constant curvature. In the last section, we provide two examples of statistical warped product submanifolds in the same environment.

2. Preliminaries

Let $(\tilde{N}, \tilde{\nabla}, \tilde{g})$ be a statistical manifold and N be a submanifold of $\tilde{N}$. Then, (N, ∇, g) is also a statistical manifold with the statistical structure (∇, g) on N induced from $(\tilde{\nabla}, \tilde{g})$, and we call (N, ∇, g) a statistical submanifold.

The fundamental equations in the geometry of Riemannian submanifolds are the Gauss and Weingarten formulae and the equations of Gauss, Codazzi, and Ricci (cf. [4,5,27]). In the statistical setting, the Gauss and Weingarten formulae are defined respectively by [28]:

$$\left.\begin{array}{ll} \tilde{\nabla}_E F = \nabla_E F + h(E,F), & \tilde{\nabla}^*_E F = \nabla^*_E F + h^*(E,F), \\ \tilde{\nabla}_E \xi = -A_\xi(E) + \nabla^\perp_E \xi, & \tilde{\nabla}^*_E \xi = -A^*_\xi(E) + \nabla^{\perp *}_E \xi, \end{array}\right\} \quad (1)$$

for any $E, F \in \Gamma(TN)$ and $\xi \in \Gamma(T^\perp N)$, where $\tilde{\nabla}$ and $\tilde{\nabla}^*$ (resp., ∇ and ∇^*) are the dual connections on $\tilde{N}$ (resp., on N).

The symmetric and bilinear imbedding curvature tensor of N in $\tilde{N}$ with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$ is denoted as h and h^*, respectively. The relation between h (resp. h^*) and A_ξ (resp. A^*_ξ) is defined by [28]:

$$\left.\begin{aligned} \tilde{g}(h(E,F),\xi) = g(A^*_\xi E,F), \\ \tilde{g}(h^*(E,F),\xi) = g(A_\xi E,F), \end{aligned}\right\} \tag{2}$$

for any $E,F \in \Gamma(TN)$ and $\xi \in \Gamma(T^\perp N)$.

Let $\tilde{R}$ and R be the curvature tensor fields of $\tilde{\nabla}$ and ∇, respectively. The corresponding Gauss, Codazzi, and Ricci equations are given by [28]:

$$\begin{aligned} \tilde{g}(\tilde{R}(E,F)G,H) &= g(R(E,F)G,H) + \tilde{g}(h(E,G),h^*(F,H)) \\ &\quad - \tilde{g}(h^*(E,H),h(F,G)), \end{aligned} \tag{3}$$

$$\begin{aligned} (\tilde{R}(E,F)G)^\perp &= \nabla^\perp_E h(F,G) - h(\nabla_E F,G) - h(F,\nabla_E G) \\ &\quad - \{\nabla^\perp_F h(E,G) - h(\nabla_F E,G) - h(E,\nabla_F G)\}, \end{aligned} \tag{4}$$

$$\tilde{g}(\tilde{R}^\perp(E,F)\xi,\eta) = \tilde{g}(R(E,F)\xi,\eta) + g([A^*_\xi, A_\eta]E,F), \tag{5}$$

for any $E,F,G,H \in \Gamma(TN)$ and $\xi,\eta \in \Gamma(T^\perp N)$, where $R^\perp$ is the Riemannian curvature tensor on $T^\perp N$.

Similarly, $\tilde{R}^*$ and R^* are respectively the curvature tensor fields with respect to $\tilde{\nabla}^*$ and ∇^*. We can obtain the duals of all Equations (3)–(5) with respect to $\tilde{\nabla}^*$ and ∇^*. Furthermore,

$$\tilde{S} = \frac{1}{2}(\tilde{R} + \tilde{R}^*) \text{ and } S = \frac{1}{2}(R + R^*) \tag{6}$$

are respectively the curvature tensor fields of $\tilde{N}$ and N given by [7]. Thus, the sectional curvature $\mathbb{K}^{\nabla,\nabla^*}$ on N of $\tilde{N}$ is defined by [29,30]:

$$\begin{aligned} \mathbb{K}^{\nabla,\nabla^*}(E \wedge F) &= g(S(E,F)F,E) \\ &= \frac{1}{2}(g(R(E,F)F,E) + g(R^*(E,F)F,E)), \end{aligned} \tag{7}$$

for any orthonormal vectors $E,F \in T_pN$, $p \in N$.

Suppose that $\dim(N) = m$ and $\dim(\tilde{N}) = n$. Let $\{e_1,\ldots,e_m\}$ and $\{e_{m+1},\ldots,e_n\}$ be respectively the orthonormal basis of T_pN and $T^\perp_pN$ for $p \in N$. Then, the scalar curvature σ^{∇,∇^*} of N is given by:

$$\sigma^{\nabla,\nabla^*} = \sum_{1\le i<j\le m} \mathbb{K}^{\nabla,\nabla^*}(e_i \wedge e_j). \tag{8}$$

The normalized scalar curvature ρ of N is defined as:

$$\rho^{\nabla,\nabla^*} = \frac{2\sigma^{\nabla,\nabla^*}}{m(m-1)}.$$

The mean curvature vectors $\mathcal{H}$ and $\mathcal{H}^*$ of N in $\tilde{N}$ are:

$$\mathcal{H} = \frac{1}{m}\sum_{i=1}^{m} h(e_i,e_i), \quad \mathcal{H}^* = \frac{1}{m}\sum_{i=1}^{m} h^*(e_i,e_i).$$

Furthermore, we set:

$$h_{ij}^a = \tilde{g}(h(e_i, e_j), e_a), \;\; h_{ij}^{*a} = \tilde{g}(h^*(e_i, e_j), e_a),$$

for $i, j \in \{1, \ldots, m\}$, $a \in \{m+1, \ldots, n\}$.

A statistical manifold $(\tilde{N}, \tilde{\nabla}, \tilde{g})$ is said to be *of constant curvature* $\tilde{c} \in \mathbb{R}$, denoted by $\tilde{N}(\tilde{c})$, if the following curvature equation holds:

$$\tilde{S}(E, F)G = \tilde{c}(g(F, G)E - g(E, G)F), \;\; \forall E, F, G \in \Gamma(T\tilde{N}). \tag{9}$$

3. Basics on Statistical Warped Product Manifolds

Definition 1. *[3] Let (N_1, g_1) and (N_2, g_2) be two (pseudo)-Riemannian manifolds and $\mathfrak{f} > 0$ be a differentiable function on N_1. Consider the natural projections $\pi : N_1 \times N_2 \to N_1$ and $\pi' : N_1 \times N_2 \to N_2$. Then, the warped product $N = N_1 \times_{\mathfrak{f}} N_2$ with warping function $\mathfrak{f}$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that:*

$$\tilde{g}(E, F) = g_1(\pi_* E, \pi_* F) + \mathfrak{f}^2(u) g_2(\pi'_* E, \pi'_* F), \tag{10}$$

for $E, F \in \Gamma(T_{(u,v)}N)$, $u \in N_1$, and $v \in N_2$, where $*$ denotes the tangent map.

Let $\chi(N_1)$ and $\chi(N_2)$ be the set of all vector fields on $N_1 \times N_2$, which is the horizontal lift of a vector field on N_1 and the vertical lift of a vector field on N_2, respectively. We have $T(N_1 \times N_2) = \chi(N_1) \oplus \chi(N_2)$. Thus, it can be seen that $\pi_*(\chi(N_1)) = \Gamma(TN_1)$ and $\pi'_*(\chi(N_2)) = \Gamma(TN_2)$. Therefore, $\pi_*(X) = E_1 \in \Gamma(TN_1)$, $\pi_*(Y) = F_1 \in \Gamma(TN_1)$, $\pi'_*(U) = E_2 \in \Gamma(TN_2)$ and $\pi'_*(V) = F_2 \in \Gamma(TN_2)$, for any $X, Y \in \chi(N_1)$ and $U, V \in \chi(N_2)$.

Recall the following general result from [6] for a dualistic structure on the warped product manifold $N_1 \times_{\mathfrak{f}} N_2$.

Proposition 1. *Let $(g_1, \nabla^{N_1}, \nabla^{N_1 *})$ and $(g_2, \nabla^{N_2}, \nabla^{N_2 *})$ be dualistic structures on N_1 and N_2, respectively. For $X, Y \in \chi(N_1)$ and $U, V \in \chi(N_2)$, D, D^* on $N_1 \times N_2$ satisfy:*

(a) $D_X Y = \nabla^{N_1}_{E_1} F_1,$
(b) $D_X U = D_U X = \frac{E_1 \mathfrak{f}}{\mathfrak{f}} E_2,$
(c) $D_U V = \nabla^{N_2}_{E_2} F_2 - \frac{\tilde{g}(U,V)}{\mathfrak{f}} grad \;\; \mathfrak{f},$
(d) $D^*_X Y = \nabla^{N_1 *}_{E_1} F_1,$
(e) $D^*_X U = D^*_U X = \frac{E_1 \mathfrak{f}}{\mathfrak{f}} E_2,$
(f) $D^*_U V = \nabla^{N_2 *}_{E_2} F_2 - \frac{\tilde{g}(U,V)}{\mathfrak{f}} grad \, \mathfrak{f},$

where $\nabla^{N_1}_{E_1} F_1 = \pi * (D_X Y)$, $\nabla^{N_1 *}_{E_1} F_1 = \pi * (D^*_X Y)$, $\nabla^{N_2}_{E_2} F_2 = \pi'_*(D_U V)$, and $\nabla^{N_2 *}_{E_2} F_2 = \pi'_*(D^*_U V)$. Then, $(\tilde{g}, D, D^*)$ is a dualistic structure on $N_1 \times N_2$.

Furthermore, Todjihounde [6] derived the curvature of the statistical warped product $\tilde{N} = N_1 \times_{\mathfrak{f}} N_2$ in terms of the curvature tensors R_1 and R_2 of N_1 and N_2, respectively, and its warping function $\mathfrak{f}$.

Lemma 1. *Let $(\tilde{N} = N_1 \times_{\mathfrak{f}} N_2, D, D^*, \tilde{g})$ be a statistical warped product manifold. For $X, Y, Z \in \chi(N_1)$ and $U, V, W \in \chi(N_2)$, we have:*

(a) $\tilde{R}(X, Y)Z = R_1(E_1, F_1)G_1,$
(b) $\tilde{R}(U, Y)Z = -\mathfrak{f}^{-1} Hess_{\mathfrak{f}}(Y, Z)U,$
(c) $\tilde{R}(X, Y)W = 0,$
(d) $\tilde{R}(U, V)Z = 0,$

(e) $\tilde{R}(X,V)W = -\mathfrak{f}^{-1}\tilde{g}(V,W)D_X(grad\,\mathfrak{f}),$
(f) $\tilde{R}(U,V)W = R_2(E_2,F_2)G_2 + ||grad\,\mathfrak{f}||^2[g_2(U,W)V - g_2(V,W)U],$

where $\tilde{R}$ *denotes the curvature tensor field of* $(\tilde{N} = N_1 \times_{\mathfrak{f}} N_2, D, D^*, \tilde{g})$ *and* $Hess_{\mathfrak{f}}(X,Y) = X(Y\mathfrak{f}) - (\nabla_X^{N_1}Y)\mathfrak{f}$ *is the Hessian function of* $\mathfrak{f}$ *with respect to* ∇^{N_1}.

The next result from [9] provides the Ricci tensor $\tilde{Ric}$ of the statistical warped product manifold.

Lemma 2. *Let* $(\tilde{N} = N_1 \times_{\mathfrak{f}} N_2, D, D^*, \tilde{g})$ *be a statistical warped product manifold. For* $X, Y \in \chi(N_1)$ *and* $U, V \in \chi(N_2)$, *we have:*

(a) $\tilde{Ric}(X,Y) = Ric_1(X,Y) - dim(N_2)\mathfrak{f}^{-1}Hess_{\mathfrak{f}}(X,Y),$
(b) $\tilde{Ric}(X,V) = 0,$
(c) $\tilde{Ric}(U,V) = Ric_2(U,V) - [\mathfrak{f}(\Delta\mathfrak{f}) + (dim(N_2)-1)||grad\,\mathfrak{f}||^2]g_2(U,V),$

where Ric_1 *and* Ric_2 *are the Ricci tensors of* N_1 *and* N_2, *respectively, and* $\Delta\mathfrak{f} = div(grad\,\mathfrak{f})$ *is the Laplacian of* $\mathfrak{f}$ *with respect to* D.

We recall the following result from [31]. This result is useful in some Riemannian problems like the study of the distance between two manifolds, of the extremes of sectional curvature and is applied successfully in the demonstration of the Chen inequality.

Let (N, g) be a Riemannian submanifold of a Riemannian manifold $(\tilde{N}, \tilde{g})$, and let $f : \tilde{N} \to \mathbb{R}$ be a differentiable function. Let:

$$\min_{x_0 \in N} f(x_0) \tag{11}$$

be the constrained extremum problem.

Theorem 1. *If* $x \in N$ *is the solution of the problem (11), then:*

(a) $(grad\ f)(x) \in T_x^{\perp}N,$
(b) *the bilinear form* $\Theta : T_xN \times T_xN \to \mathbb{R}$,

$$\Theta(E,F) = Hess_f(E,F) + \tilde{g}(h'(E,F),(grad\ f)(x))$$

is positive semi-definite, where h' *is the second fundamental form of* N *in* $\tilde{N}$ *and* $grad\ f$ *denotes the gradient of* f.

4. Statistical Solitons on Statistical Warped Product Manifolds

The Ricci solitons model the formation of singularities in the Ricci flow, and they correspond to self-similar solutions. R. Hamilton [32] introduced the study of Ricci solitons as fixed or stationary points of the Ricci flow in the space of the metrics on Riemannian manifolds modulo diffeomorphisms and scaling. Since then, many researchers studied Ricci solitons for different reasons and in different ambient spaces (for example [33–35]). A complete Riemannian manifold $(\tilde{N}, \tilde{g})$ is called a *Ricci soliton* $(\tilde{N}, \tilde{g}, \zeta, \lambda)$ if there exists a smooth vector field ζ and a constant $\lambda \in \mathbb{R}$ such that:

$$2\tilde{Ric} = 2\lambda\tilde{g} - \mathcal{L}_{\zeta}\tilde{g},$$

where $\mathcal{L}_{\zeta}$ denotes the Lie derivative along ζ and $\tilde{Ric}$ is the Ricci tensor of $\tilde{g}$.

A generalization of Ricci solitons in the framework of manifolds endowed with an arbitrary linear connection $\tilde{\nabla}$, different from the Levi–Civita connection of $\tilde{g}$, is defined in [36] as follows:

Let $(\tilde{N}, \tilde{\nabla})$ be a manifold and $\zeta \in \chi(\tilde{N})$. A triple $(\tilde{g}, \zeta, \lambda)$ is called a $\tilde{\nabla}$-*Ricci soliton* if $\tilde{\nabla}\zeta + \tilde{\mathcal{Q}} + \lambda I = 0$ holds, where $\tilde{\mathcal{Q}}$ is the Ricci operator of $\tilde{N}$ defined by $\tilde{g}(\tilde{\mathcal{Q}}E, F) = \tilde{R}ic(E, F)$, for vector fields E, F on $\tilde{N}$.

The statistical manifold $(\tilde{N}, \tilde{\nabla}, \tilde{g})$ is called *Ricci-symmetric* if the Ricci operator $\tilde{\mathcal{Q}}$ with respect to $\tilde{\nabla}$ (equivalently, the dual operator $\tilde{\mathcal{Q}}^*$ with respect to $\tilde{\nabla}^*$) is symmetric (cf. [36,37]).

Based on these, we have the following.

Definition 2. *A pair (ζ, λ) is called a statistical soliton on a Ricci-symmetric statistical manifold $(\tilde{N}, \tilde{\nabla}, \tilde{g})$ if the triple $(\tilde{g}, \zeta, \lambda)$ is $\tilde{\nabla}$-Ricci and $\tilde{\nabla}^*$-Ricci solitons, i.e., we have:*

$$\tilde{\nabla}\zeta + \tilde{\mathcal{Q}} + \lambda I = 0, \tag{12}$$

and:

$$\tilde{\nabla}^*\zeta + \tilde{\mathcal{Q}}^* + \lambda I = 0, \tag{13}$$

where $\tilde{g}(\tilde{\mathcal{Q}}E, F) = \tilde{R}ic(E, F)$ and $\tilde{g}(\tilde{\mathcal{Q}}^*E, F) = \tilde{R}ic^*(E, F)$, for all vector fields on $\tilde{N}$, and $\tilde{R}ic$ and $\tilde{R}ic^*$ denote the Ricci tensor fields with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

The main purpose of this section is to study the problem: *under what conditions does the base manifold or fiber manifold of the statistical warped product manifold become a statistical soliton?*

Let $(N_1, \nabla^{N_1}, \nabla^{N_1*}, g_1)$ and $(N_2, \nabla^{N_2}, \nabla^{N_2*}, g_2)$ be the Ricci-symmetric statistical manifolds. Denote the Ricci-symmetric statistical warped product manifold by $(\tilde{N} = N_1 \times_{\mathfrak{f}} N_2, D, D^*, \tilde{g} = g_1 + \mathfrak{f}^2 g_2)$. Let $\zeta = (\zeta_1, \zeta_2) \in \chi(\tilde{N})$ be a vector field on $\tilde{N}$. Then, the pair (ζ, λ) on $(\tilde{N}, \tilde{\nabla}, \tilde{g})$ is called a *statistical soliton* if the triple $(\tilde{g}, \zeta, \lambda)$ is both D-Ricci and D^*-Ricci solitons, given by (12) and (13).

It follows from Lemma 2 that the Ricci tensor of $\tilde{N}$ is given as below:

$$\begin{aligned} \tilde{R}ic = {} & Ric_1 - \mathfrak{f}^{-1} \dim(N_2) Hess_{\mathfrak{f}} + Ric_2 \\ & - [\mathfrak{f}(\Delta \mathfrak{f}) + (\dim(N_2) - 1)||grad\, \mathfrak{f}||^2] g_2. \end{aligned} \tag{14}$$

Thus, (12) and (13) can be rewritten as:

$$\begin{aligned} & \nabla^{N_1}\zeta_1 + \nabla^{N_2}\zeta_2 + Ric_1 - \mathfrak{f}^{-1} \dim(N_2) Hess_{\mathfrak{f}} + Ric_2 \\ & - [\mathfrak{f}(\Delta \mathfrak{f}) + (\dim(N_2) - 1)||grad\, \mathfrak{f}||^2] g_2 + \lambda g_1 + \lambda \mathfrak{f}^2 g_2 = 0, \end{aligned} \tag{15}$$

and:

$$\begin{aligned} & \nabla^{N_1*}\zeta_1 + \nabla^{N_2*}\zeta_2 + Ric_1^* - \mathfrak{f}^{-1} \dim(N_2) Hess_{\mathfrak{f}}^{D^*} + Ric_2^* \\ & - [\mathfrak{f}(\Delta^{D^*} \mathfrak{f}) + (\dim(N_2) - 1)||grad\, \mathfrak{f}||^2] g_2 + \lambda g_1 + \lambda \mathfrak{f}^2 g_2 = 0, \end{aligned} \tag{16}$$

respectively.

Throughout this section, we use the statistical warped products as Ricci-symmetric.

We give the following results by applying Lemma 2:

Lemma 3. *Let $(\tilde{N} = \mathbb{R} \times_{\mathfrak{f}} N_2, D, D^*, \tilde{g})$ be a statistical warped product manifold, where $(\mathbb{R}, \nabla^{\mathbb{R}}, dz^2)$ is a trivial statistical manifold of dimension one and* $\dim(N_2) = k$*. Then, for $U, V \in \chi(N_2)$, we have:*

(a) $\tilde{R}ic(\partial z, \partial z) = -k\mathfrak{f}^{-1}\ddot{\mathfrak{f}}$,
(b) $\tilde{R}ic(\partial z, V) = 0$,
(c) $\tilde{R}ic(U, V) = Ric_2(U, V) - [\mathfrak{f}\ddot{\mathfrak{f}} + (k-1)\dot{\mathfrak{f}}^2] g_2(U, V)$.

Proposition 2. *Let* (ζ, λ) *be a statistical soliton on statistical warped product manifold* $(\tilde{N} = \mathbb{R} \times_{\mathfrak{f}} N_2, D, D^*, \tilde{g} = dz^2 + \mathfrak{f}^2 g_2)$ *with* $\dim(\mathbb{R}) = 1$ *and* $\dim(N_2) = k$. *Then:*

$$Hess_{\mathfrak{f}} \;=\; \frac{\mathfrak{f}\lambda}{k}.$$

Proof. Since $\tilde{N}$ is a statistical soliton, then from (6), we have:

$$\tilde{g}(\tilde{\nabla}_{\partial z}\zeta, \partial z) + \tilde{R}ic(\partial z, \partial z) + \lambda\tilde{g}(\partial z, \partial z) \;=\; 0.$$

By taking into account Lemma 3 and $Ric_1(\partial z, \partial z) \;=\; 0$, we get:

$$-\tilde{g}(\zeta, \tilde{\nabla}^*_{\partial z}\partial z) - k\mathfrak{f}^{-1}Hess_{\mathfrak{f}}(\partial z, \partial z) + \lambda\tilde{g}(\partial z, \partial z) \;=\; 0,$$

which gives $Hess_{\mathfrak{f}}(\partial z, \partial z) \;=\; (\frac{\mathfrak{f}\lambda}{k})\tilde{g}(\partial z, \partial z)$. □

Theorem 2. *Let* $\zeta = (\partial z, \zeta_2) \in \chi(\tilde{N})$ *be a vector field on statistical warped product manifold* $(\tilde{N} = \mathbb{R} \times_{\mathfrak{f}} N_2, D, D^*, \tilde{g} = dz^2 + \mathfrak{f}^2 g_2)$ *with* $\dim(\mathbb{R}) = 1$ *and* $\dim(N_2) = k$. *If* (ζ, λ) *is a statistical soliton on* $\tilde{N}$, *then:*

(a) $(N_2, g_2, \zeta_2, \lambda_2)$ *is a statistical soliton on* $(N_2, \nabla^{N_2}, \nabla^{N_2 *}, g_2)$, *where* $\lambda_2 \;=\; (k-1)[\mathfrak{f}\ddot{\mathfrak{f}} - \dot{\mathfrak{f}}^2]$,
(b) $\mathfrak{f}(z) \;=\; az + b$ *if* $\lambda \;=\; 0$,
(c) $\mathfrak{f}(z) \;=\; \cosh(az + b)$ *if* $\lambda \neq 0$,

where $a, b \in \mathbb{R}$.

Proof. From Equation (15) and Lemma 3, we have:

$$\begin{aligned}&\nabla^{N_1}\partial z + \nabla^{N_2}\zeta_2 + Ric_1 - k\mathfrak{f}^{-1}\ddot{\mathfrak{f}} + Ric_2\\ &-(\mathfrak{f}\ddot{\mathfrak{f}} + (k-1)\dot{\mathfrak{f}}^2)g_2 + \lambda g_1 + \lambda\mathfrak{f}^2 g_2 \;=\; 0.\end{aligned}$$

Note $g_1(\nabla^{N_1}_{\partial z}\partial z, \partial z) \;=\; 0$ and $Ric_1(\partial z, \partial z) \;=\; 0$. Thus, the above equation becomes:

$$\nabla^{N_2}\zeta_2 - k\mathfrak{f}^{-1}\ddot{\mathfrak{f}} + Ric_2 - (\mathfrak{f}\ddot{\mathfrak{f}} + (k-1)\dot{\mathfrak{f}}^2)g_2 + \lambda g_1 + \lambda\mathfrak{f}^2 g_2 \;=\; 0,$$

from which we get:

$$\lambda \;=\; k\mathfrak{f}^{-1}\ddot{\mathfrak{f}}, \tag{17}$$

$$\nabla^{N_2}\zeta_2 + Ric_2 + [\lambda\mathfrak{f}^2 - (\mathfrak{f}\ddot{\mathfrak{f}} + (k-1)\dot{\mathfrak{f}}^2)]g_2 \;=\; 0. \tag{18}$$

Putting (17) into the Equation (18), we arrive at:

$$\nabla^{N_2}\zeta_2 + Ric_2 + (k-1)[\mathfrak{f}\ddot{\mathfrak{f}} - \dot{\mathfrak{f}}^2]g_2 = 0.$$

Similarly, by using (16), we derive:

$$\nabla^{N_2 *}\zeta_2 + Ric^*_2 + (k-1)[\mathfrak{f}\ddot{\mathfrak{f}} - \dot{\mathfrak{f}}^2]g_2 \;=\; 0.$$

Thus, $(N_2, g_2, \zeta_2, (k-1)[\mathfrak{f}\ddot{\mathfrak{f}} - \dot{\mathfrak{f}}^2])$ is a statistical soliton provided that $(k-1)[\mathfrak{f}\ddot{\mathfrak{f}} - \dot{\mathfrak{f}}^2]$ is constant. On the other hand, by using (17), we have the following cases:

(a) if $\lambda \;=\; 0$, then $\mathfrak{f}(z) \;=\; az + b$, and
(b) if $\lambda \neq 0$, then $\mathfrak{f}(z) \;=\; \cosh(az + b)$ [9],

where a, b are real constants. □

Before proving the next result, we state the following:

Lemma 4. *Let* $(\tilde{N} = N_1 \times_{\mathfrak{f}} \mathbb{R}, D, D^*, \tilde{g})$ *be a statistical warped product manifold, where* $(\mathbb{R}, \nabla^{\mathbb{R}}, dz^2)$ *is a trivial statistical manifold of dimension one and* $\dim(N_1) = k$. *For* $X, Y \in \chi(N_1)$, *we have:*

(a) $\tilde{R}ic(X, Y) = Ric_1(X, Y) - \mathfrak{f}^{-1} Hess_{\mathfrak{f}}(X, Y)$,
(b) $\tilde{R}ic(X, \partial z) = 0$,
(c) $\tilde{R}ic(\partial z, \partial z) = -\mathfrak{f}(\Delta \mathfrak{f}) g_2(\partial z, \partial z)$.

Theorem 3. *Let* $\zeta = (\zeta_1, \partial z) \in \chi(\tilde{N})$ *be a vector field on statistical warped product manifold* $(\tilde{N} = N_1 \times_{\mathfrak{f}} \mathbb{R}, D, D^*, \tilde{g} = g_1 + \mathfrak{f}^2 dz^2)$ *with* $\dim(\mathbb{R}) = 1$ *and* $\dim(N_1) = k$. *Suppose that* $Hess_{\mathfrak{f}} = 0$. *Then,* (ζ, λ) *is a statistical soliton on* $\tilde{N}$ *if and only if* $(\zeta_1, \lambda = \mathfrak{f}^{-1}(\Delta \mathfrak{f}))$ *is a statistical soliton on* N_1.

Proof. Since $g_2(\nabla^{N_1}_{\partial z} \partial z, \partial z) = 0$ and $Ric_2(\partial z, \partial z) = 0$, then by using Equation (15) and Lemma 4, we get:

$$\nabla^{N_1} \zeta_1 + Ric_1 - \mathfrak{f}(\Delta \mathfrak{f}) g_2 + \lambda g_1 + \lambda \mathfrak{f}^2 g_2 = 0.$$

Therefore, we have:

$$\nabla^{N_1} \zeta_1 + Ric_1 + \lambda g_1 = 0. \tag{19}$$

Furthermore, $\mathfrak{f}^{-1}(\Delta \mathfrak{f}) = \lambda = constant$. Putting this into (19), we get:

$$\nabla^{N_1} \zeta_1 + Ric_1 + \mathfrak{f}^{-1}(\Delta \mathfrak{f}) g_1 = 0.$$

Similarly, by using (16), we obtain:

$$\nabla^{N_1 *} \zeta_1 + Ric_1^* + \mathfrak{f}^{-1}(\Delta^* \mathfrak{f}) g_1 = 0.$$

Since $\mathfrak{f}^{-1}(\Delta \mathfrak{f})$ is constant, $(N_1, g_1, \zeta_1, \lambda = \mathfrak{f}^{-1}(\Delta \mathfrak{f}))$ is a statistical soliton.
Conversely, if $(\zeta_1, \lambda = \mathfrak{f}^{-1}(\Delta \mathfrak{f}))$ is a statistical soliton on N_1, then:

$$\begin{aligned}
\nabla^{N_1} \zeta_1 + \nabla^{N_2} \partial z + Ric_1 - \mathfrak{f}^{-1} k_2 Hess_{\mathfrak{f}} + Ric_2 \\
- [\mathfrak{f}(\Delta \mathfrak{f}) + (k_2 - 1) ||grad\, \mathfrak{f}||^2] g_2 \\
= \nabla^{N_1} \zeta_1 + Ric_1 + \mathfrak{f}^{-1}(\Delta \mathfrak{f}) g_1 - \mathfrak{f}^{-1}(\Delta \mathfrak{f}) g_1 - \mathfrak{f}(\Delta \mathfrak{f}) g_2 \\
= -\mathfrak{f}^{-1}(\Delta \mathfrak{f}) g_1 - \mathfrak{f}(\Delta \mathfrak{f}) g_2 = -\mathfrak{f}^{-1}(\Delta \mathfrak{f})(g_1 + g_2) \\
= -\lambda \tilde{g}.
\end{aligned}$$

Thus, $D\zeta + \tilde{\mathcal{Q}} + \lambda I = 0$. Similarly, $D^* \zeta + \tilde{\mathcal{Q}}^* + \lambda I = 0$. Hence, (ζ, λ) is a statistical soliton on $\tilde{N}$. □

An immediate consequence of Theorem 3 is as follows:

Corollary 1. *Let* $(\tilde{N}, \tilde{g}, \zeta, \lambda)$ *be a Statistical soliton on statistical manifold* $(\tilde{N} = N_1 \times_{\mathfrak{f}} \mathbb{R}, D, D^*, \tilde{g} = g_1 + \mathfrak{f}^2 dz^2)$ *with* $dim(\mathbb{R}) = 1$ *and* $dim(N_1) = k$. *If* $Hess_{\mathfrak{f}} = \varrho g_1$, $\varrho \in C^\infty(N_1)$, *then* $(N_1, g_1, \zeta_1, \mathfrak{f}^{-1}(\Delta \mathfrak{f}) - \mathfrak{f}^{-1} \varrho)$ *is a statistical soliton.*

5. B.Y. Chen Inequality

A universal sharp inequality for submanifolds in a Riemannian manifold of constant sectional curvature was established in [38], known as the first Chen inequality. The main purpose of this section is to establish the corresponding inequality for statistical warped product manifolds statistically immersed in a statistical manifold of constant curvature.

Let $\varphi : N = N_1 \times_{\mathfrak{f}} N_2 \to \tilde{N}(\tilde{c})$ be an isometric statistical immersion of a warped product $N_1 \times_{\mathfrak{f}} N_2$ into a statistical manifold of constant sectional curvature $\tilde{c}$. We denote by r, k, and $m = r + k$ the dimensions of N_1, N_2, and $N_1 \times N_2$, respectively. Since $N_1 \times_{\mathfrak{f}} N_2$ is a statistical warped product, we have:

$$\nabla_{E_1} E_2 = \nabla_{E_2} E_1 = (E_1 \ln \mathfrak{f}) E_2,$$

for unit vector fields E_1 and E_2 tangent to N_1 and N_2, respectively. Hence, we derive:

$$\mathbb{K}(E_1 \wedge E_2) = \frac{1}{\mathfrak{f}}\{(\nabla_{E_1} E_1)\mathfrak{f} - E_1^2 \mathfrak{f}\}. \tag{20}$$

If we choose a local orthonormal frame $\{e_1, \ldots, e_m\}$ such that $\{e_1, \ldots, e_r\}$ are tangent to N_1 and $\{e_{r+1}, \ldots, e_{r+k} = e_m\}$ are tangent to N_2, then we have:

$$\frac{\Delta \mathfrak{f}}{\mathfrak{f}} = \sum_{i=1}^{r} \mathbb{K}(e_i \wedge e_j), \tag{21}$$

for each $j = r+1, \ldots, m$.

On the other hand, let E_1 and E_2 be two unit local vector fields tangent to N_1 and N_2, respectively, such that $e_1 = E_1$ and $e_{r+1} = E_2$. By taking into account Equations (3), (6), and (9), we derive (7) as follows:

$$\begin{aligned}
\mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) &= \frac{\tilde{c}}{2}\{2g(e_{r+1}, e_{r+1})g(e_1, e_1) - 2g(e_1, e_{r+1})g(e_{r+1}, e_1)\} \\
&\quad + \frac{1}{2}\{g(h^*(e_1, e_1), h(e_{r+1}, e_{r+1})) \\
&\quad + g(h(e_1, e_1), h^*(e_{r+1}, e_{r+1})) - 2g(h(e_1, e_{r+1}), h^*(e_1, e_{r+1}))\} \\
&= \tilde{c} + \frac{1}{2}\sum_{a=m+1}^{n}\{h_{11}^{*a} h_{r+1,r+1}^{a} + h_{11}^{a} h_{r+1,r+1}^{*a} - 2h_{1,r+1}^{a} h_{1,r+1}^{*a}\}.
\end{aligned}$$

We rewrite the terms of the RHS of the previous equation as:

$$\begin{aligned}
\mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) = \tilde{c} + \frac{1}{2}\sum_{a=m+1}^{n} &\{(h_{11}^{a} + h_{11}^{*a})(h_{r+1,r+1}^{a} + h_{r+1,r+1}^{*a}) \\
&- (h_{1,r+1}^{a} + h_{1,r+1}^{*a})^2 + (h_{1,r+1}^{a})^2 + (h_{1,r+1}^{*a})^2 \\
&- h_{11}^{a} h_{r+1,r+1}^{a} - h_{11}^{*a} h_{r+1,r+1}^{*a}\}.
\end{aligned}$$

Since, $2h^0 = h + h^*$, we get:

$$\begin{aligned}
\mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) = \tilde{c} + \frac{1}{2}\sum_{a=m+1}^{n} &\{4h_{11}^{0a} h_{r+1,r+1}^{0a} \\
&- (h_{11}^{a} h_{r+1,r+1}^{a} - (h_{1,r+1}^{a})^2) \\
&- (h_{11}^{*a} h_{r+1,r+1}^{*a} - (h_{1,r+1}^{*a})^2) - 4(h_{1,r+1}^{0a})^2\}.
\end{aligned}$$

Thus, we have:

$$\mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) = \tilde{c} + \sum_{a=m+1}^{n} \{2(h_{11}^{0a} h_{r+1,r+1}^{0a} - (h_{1,r+1}^{0a})^2) - \frac{1}{2}(h_{11}^{a} h_{r+1,r+1}^{a} - (h_{1,r+1}^{a})^2) - \frac{1}{2}(h_{11}^{*a} h_{r+1,r+1}^{*a} - (h_{1,r+1}^{*a})^2)\}. \tag{22}$$

Using the Gauss equation for the Levi–Civita connection, we arrive at:

$$\mathbb{K}^{0}(e_1 \wedge e_{r+1}) = \tilde{c} - \sum_{a=m+1}^{n} \{(h_{1,r+1}^{0a})^2 - h_{11}^{0a} h_{r+1,r+1}^{0a}\},$$

which can be rewritten as:

$$\sum_{a=m+1}^{n} \{(h_{1,r+1}^{0a})^2 - h_{11}^{0a} h_{r+1,r+1}^{0a}\} = \mathbb{K}^{0}(e_1 \wedge e_{r+1}) - \tilde{c}. \tag{23}$$

Substituting (23) into (22), we get:

$$\mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) = 2\mathbb{K}^{0}(e_1 \wedge e_{r+1}) - \tilde{c} - \frac{1}{2} \sum_{a=m+1}^{n} \{h_{11}^{a} h_{r+1,r+1}^{a} - (h_{1,r+1}^{a})^2 + h_{11}^{*a} h_{r+1,r+1}^{*a} - (h_{1,r+1}^{*a})^2\}. \tag{24}$$

Furthermore, we derive (8) as:

$$\begin{aligned}\sigma^{\nabla,\nabla^*} &= \frac{m(m-1)\tilde{c}}{2} + \frac{1}{2} \sum_{a=m+1}^{n} \sum_{i<j} \{h_{ii}^{*a} h_{jj}^{a} + h_{ii}^{a} h_{jj}^{*a} - 2h_{ij}^{a} h_{ij}^{*a}\} \\ &= \frac{m(m-1)\tilde{c}}{2} + \frac{1}{2} \sum_{a=m+1}^{n} \sum_{i<j} \{(h_{ii}^{a} + h_{ii}^{*a})(h_{jj}^{a} + h_{jj}^{*a}) \\ &\quad - h_{ii}^{a} h_{jj}^{a} - h_{ii}^{*a} h_{jj}^{*a} - (h_{ij}^{a} + h_{ij}^{*a})^2 + (h_{ij}^{a})^2 + (h_{ij}^{*a})^2\}.\end{aligned}$$

By a similar argument as above, we deduce that:

$$\sigma^{\nabla,\nabla^*} = \frac{m(m-1)\tilde{c}}{2} + \frac{1}{2} \sum_{a=m+1}^{n} \sum_{i<j} \{2(h_{ii}^{0a} h_{jj}^{0a} - (h_{ij}^{0a})^2) - \frac{1}{2}(h_{ii}^{a} h_{jj}^{a} - (h_{ij}^{a})^2) - \frac{1}{2}(h_{ii}^{*a} h_{jj}^{*a} - (h_{ij}^{*a})^2)\}. \tag{25}$$

Again by the Gauss equation for the Levi–Civita connection, we find that:

$$\sigma^{0} = \frac{m(m-1)\tilde{c}}{2} + \sum_{a=m+1}^{n} \sum_{i<j} \{h_{ii}^{0a} h_{jj}^{0a} - (h_{ij}^{0a})^2\},$$

or:

$$\sum_{a=m+1}^{n} \sum_{i<j} \{h_{ii}^{0a} h_{jj}^{0a} - (h_{ij}^{0a})^2\} = \sigma^{0} - \frac{m(m-1)\tilde{c}}{2}. \tag{26}$$

Inserting (26) into (25), we have:

$$\sigma^{\nabla,\nabla^*} = 2\sigma^0 - \frac{m(m-1)\tilde{c}}{2} - \frac{1}{2}\sum_{a=m+1}^{n}\sum_{i<j}\{h^a_{ii}h^a_{jj} - (h^a_{ij})^2 + h^{*a}_{ii}h^{*a}_{jj} - (h^{*a}_{ij})^2\}. \quad (27)$$

By subtracting (24) from (27), we can state the following result:

Lemma 5. *Let $N = N_1 \times_{\mathfrak{f}} N_2$ be an m-dimensional statistical warped product submanifold immersed into an n-dimensional statistical manifold of constant sectional curvature $\tilde{c}$. Then:*

$$\begin{aligned}\sigma^{\nabla,\nabla^*} - \mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) = {} & 2(\sigma^0 - \mathbb{K}^0(e_1 \wedge e_{r+1})) - \frac{(m-2)(m+1)\tilde{c}}{2} \\ & - \frac{1}{2}\sum_{a=m+1}^{n}\sum_{i<j}\{h^a_{ii}h^a_{jj} - (h^a_{ij})^2 + h^{*a}_{ii}h^{*a}_{jj} \\ & - (h^{*a}_{ij})^2\} + \frac{1}{2}\sum_{a=m+1}^{n}\{h^a_{11}h^a_{r+1,r+1} - (h^a_{1,r+1})^2 \\ & + h^{*a}_{11}h^{*a}_{r+1,r+1} - (h^{*a}_{1,r+1})^2\}.\end{aligned}$$

Further, we have:

$$\begin{aligned}\sigma^{\nabla,\nabla^*} - \mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) \geq {} & 2(\sigma^0 - \mathbb{K}^0(e_1 \wedge e_{r+1})) - \frac{(m-2)(m+1)\tilde{c}}{2} \\ & - \frac{1}{2}\sum_{a=m+1}^{n}\sum_{i<j}\{h^a_{ii}h^a_{jj} + h^{*a}_{ii}h^{*a}_{jj}\} \\ & + \frac{1}{2}\sum_{a=m+1}^{n}\{h^a_{11}h^a_{r+1,r+1} + h^{*a}_{11}h^{*a}_{r+1,r+1}\},\end{aligned}$$

or we write it as:

$$\begin{aligned}2(\sigma^0 - \mathbb{K}^0(e_1 \wedge e_{r+1})) \leq {} & \sigma^{\nabla,\nabla^*} - \mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) + \frac{(m-2)(m+1)\tilde{c}}{2} \\ & + \frac{1}{2}\sum_{a=m+1}^{n}\{\sum_{i<j}\{h^a_{ii}h^a_{jj}\} - h^a_{11}h^a_{r+1,r+1}\} \\ & + \frac{1}{2}\sum_{a=m+1}^{n}\{\sum_{i<j}\{h^{*a}_{ii}h^{*a}_{jj}\} - h^{*a}_{11}h^{*a}_{r+1,r+1}\}. \quad (28)\end{aligned}$$

We use an optimization technique: For $a \in [m+1, n]$, we consider the quadratic forms:

$$\phi_a : \mathbb{R}^m \to \mathbb{R}, \ \ \phi^*_a : \mathbb{R}^m \to \mathbb{R}$$

given by:

$$\phi_a(h^a_{11}, \ldots, h^a_{mm}) = \sum_{i<j}\{h^a_{ii}h^a_{jj}\} - h^a_{11}h^a_{r+1,r+1}, \quad (29)$$

and:

$$\phi^*_a(h^{*a}_{11}, \ldots, h^{*a}_{mm}) = \sum_{i<j}\{h^{*a}_{ii}h^{*a}_{jj}\} - h^{*a}_{11}h^{*a}_{r+1,r+1}. \quad (30)$$

The constrained extremum problem is max ϕ_a subject to:

$$Q : h^a_{11} + \cdots + h^a_{mm} = t^a, \quad (t^a \text{ is any constant}).$$

The partial derivatives of ϕ_a are:

$$\begin{aligned}
\frac{\partial \phi_a}{\partial h^a_{11}} &= \sum_{i=2}^{m} h^a_{ii} - h^a_{r+1,r+1}, \\
\frac{\partial \phi_a}{\partial h^a_{r+1,r+1}} &= \sum_{i \in \overline{1,m}\ r+1} h^a_{ii} - h^a_{11}, \\
\frac{\partial \phi_a}{\partial h^a_{ll}} &= \sum_{i \in \overline{1,m}\ \{l\}} h^a_{ii}, \quad l \in [r+2, m].
\end{aligned}$$

For an optimal solution $(h^a_{11}, \ldots, h^a_{mm})$ of the above problem and grad (ϕ_a) normal at Q, we obtain:

$$(h^a_{11}, h^a_{22}, \ldots, h^a_{mm}) = (0, \alpha^a, \ldots, \alpha^a). \tag{31}$$

As $t^a = \sum_{i=1}^{m} h^a_{ii} = (m-1)\alpha^a$, then we have:

$$\alpha^a = \frac{t^a}{m-1}. \tag{32}$$

As ϕ_a is obtained from the similar function studied in [39] by subtracting some square terms, $\phi_a|Q$ will have the Hessian semi-negative definite. Consequently, the point in (31), together with (32) is a global maximum point, and hence, we calculate:

$$\begin{aligned}
\phi_a &\leq \frac{(m-1)(m-2)(\alpha^a)^2}{2} \\
&= \frac{(m-2)(t^a)^2}{2(m-1)} = \frac{m^2(m-2)}{2(m-1)}(\mathcal{H}^a)^2.
\end{aligned}$$

Similarly, one gets:

$$\phi^*_a \leq \frac{m^2(m-2)}{2(m-1)}(\mathcal{H}^{*a})^2,$$

by considering (30) and the constrained extremum problem max ϕ^*_a subject to:

$$Q^* : h^{*a}_{11} + \cdots + h^{*a}_{mm} = t^{*a}, \quad (t^{*a} \text{ is any constant}).$$

Thus, (28) becomes:

$$\begin{aligned}
2(\sigma^0 - \mathbb{K}^0(e_1 \wedge e_{r+1})) \leq \sigma^{\nabla,\nabla^*} - \mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) + \frac{(m-2)(m+1)\tilde{c}}{2} \\
+ \frac{m^2(m-2)}{4(m-1)}(||\mathcal{H}||^2 + ||\mathcal{H}^*||^2).
\end{aligned}$$

By summarizing, we state the following:

Theorem 4. *Let $N = N_1 \times_{\mathfrak{f}} N_2$ be an m-dimensional statistical warped product submanifold immersed into an n-dimensional statistical manifold of constant sectional curvature $\tilde{c}$. Then:*

$$\sigma^{\nabla,\nabla^*} - \mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) \geq 2(\sigma^0 - \mathbb{K}^0(e_1 \wedge e_{r+1})) - \frac{(m-2)(m+1)\tilde{c}}{2} - \frac{m^2(m-2)}{4(m-1)}(||\mathcal{H}||^2 + ||\mathcal{H}^*||^2).$$

By using (20), we obtain:

$$\begin{aligned}\mathbb{K}^{\nabla,\nabla^*}(e_1 \wedge e_{r+1}) &= \frac{1}{2}(\mathbb{K}(e_1 \wedge e_{r+1}) + \mathbb{K}^*(e_1 \wedge e_{r+1})) \\ &= \frac{1}{2\mathfrak{f}}\{(\nabla_{e_1}e_1)\mathfrak{f} - e_1^2\mathfrak{f} + (\nabla^*_{e_1}e_1)\mathfrak{f} - e_1^2\mathfrak{f}\}.\end{aligned}$$

For $b = 1, 2, \ldots, r$, we also have:

$$\mathbb{K}^{\nabla,\nabla^*}(e_b \wedge e_{r+1}) = \frac{1}{2\mathfrak{f}}\{(\nabla_{e_b}e_b)\mathfrak{f} - e_b^2\mathfrak{f} + (\nabla^*_{e_b}e_b)\mathfrak{f} - e_b^2\mathfrak{f}\}.$$

By summing up b from one to r, we find that:

$$\sum_{b=1}^{r} \frac{1}{2\mathfrak{f}}\{(\nabla_{e_b}e_b)\mathfrak{f} - e_b^2\mathfrak{f} + (\nabla^*_{e_b}e_b)\mathfrak{f} - e_b^2\mathfrak{f}\} = \frac{1}{2}(\frac{\Delta^{N_1}\mathfrak{f}}{\mathfrak{f}} + \frac{\Delta^{N_1*}\mathfrak{f}}{\mathfrak{f}}) = \frac{\Delta^{N_1 0}\mathfrak{f}}{\mathfrak{f}},$$

where Δ^{N_1} and Δ^{N_1*} are dual Laplacians of N_1 and $\Delta^{N_1 0}$ denotes the Laplacian operator of N_1 for the Levi–Civita connection [37]. Thus, we have:

Theorem 5. *Let $N = N_1 \times_{\mathfrak{f}} N_2$ be an m-dimensional statistical warped product submanifold immersed into an n-dimensional statistical manifold of constant sectional curvature $\tilde{c}$. Then, the scalar curvature σ^{∇,∇^*} of N satisfies:*

$$\sigma^{\nabla,\nabla^*} \geq 2\sigma^0 - \frac{\Delta^{N_1 0}\mathfrak{f}}{r\mathfrak{f}} - \frac{(m-2)(m+1)\tilde{c}}{2} - \frac{m^2(m-2)}{4(m-1)}(||\mathcal{H}||^2 + ||\mathcal{H}^*||^2).$$

6. Optimal Casorati Inequality

Let $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_n\}$ be respectively the orthonormal basis of T_pN and $T_p^{\perp}N$, $p \in N$. Then, the squared norm of second fundamental forms h and h^* is denoted by $\mathcal{C}$ and $\mathcal{C}^*$, respectively, called the Casorati curvatures of N in $\tilde{N}$. Therefore, we have:

$$\mathcal{C} = \frac{1}{m}||h||^2, \quad \mathcal{C}^* = \frac{1}{m}||h^*||^2, \tag{33}$$

where:

$$||h||^2 = \sum_{a=m+1}^{n} \sum_{i,j=1}^{m} (h_{ij}^a)^2, \quad ||h^*||^2 = \sum_{a=m+1}^{n} \sum_{i,j=1}^{m} (h_{ij}^{*a})^2.$$

If W is a q-dimensional subspace of TN, $q \geq 2$, and $\{e_1, \ldots, e_q\}$ an orthonormal basis of W. Then, the scalar curvature of the q-plane section W is:

$$\sigma^{\nabla,\nabla^*}(W) = \sum_{1 \leq i < j \leq q} S(e_i, e_j, e_j, e_i),$$

and the Casorati curvatures of the subspace W are as follows:

$$\mathcal{C}(W) = \frac{1}{q}\sum_{a=m+1}^{n}\sum_{i,j=1}^{q}(h_{ij}^{a})^2, \ \ \mathcal{C}^*(W) = \frac{1}{q}\sum_{a=m+1}^{n}\sum_{i,j=1}^{q}(h_{ij}^{*a})^2.$$

(1) The normalized Casorati curvatures $\delta_{\mathcal{C}}(m-1)$ and $\delta_{\mathcal{C}}^*(m-1)$ are defined as:

$$[\delta_{\mathcal{C}}(m-1)]_p = \frac{1}{2}\mathcal{C}_p + (\frac{m+1}{2m})\inf\{\mathcal{C}(W)|W : \text{a hyperplane of } T_pN\},$$
$$\text{and } [\delta_{\mathcal{C}}^*(m-1)]_p = \frac{1}{2}\mathcal{C}_p^* + (\frac{m+1}{2m})\inf\{\mathcal{C}^*(W)|W : \text{a hyperplane of } T_pN\}.$$

(2) The normalized Casorati curvatures $\widehat{\delta}_{\mathcal{C}}(m-1)$ and $\widehat{\delta}_{\mathcal{C}}^*(m-1)$ are defined as:

$$[\widehat{\delta}_{\mathcal{C}}(m-1)]_p = 2\mathcal{C}_p - (\frac{2m-1}{2m})\sup\{\mathcal{C}(W)|W : \text{a hyperplane of } T_pN\},$$
$$\text{and } [\widehat{\delta}_{\mathcal{C}}^*(m-1)]_p = 2\mathcal{C}_p^* - (\frac{2m-1}{2m})\sup\{\mathcal{C}^*(W)|W : \text{a hyperplane of } T_pN\}.$$

Let $\varphi : N = N_1 \times_{\mathfrak{f}} N_2 \to \tilde{N}(\tilde{c})$ be an isometric statistical immersion of a warped product $N_1 \times_{\mathfrak{f}} N_2$ into a statistical manifold of constant sectional curvature $\tilde{c}$. If we chose a local orthonormal frame $\{e_1, \ldots, e_m\}$ such that $\{e_1, \ldots, e_r\}$ are tangent to N_1 and $\{e_{r+1}, \ldots, e_{r+k} = e_m\}$ are tangent to N_2, then the two partial mean curvature vectors $\mathcal{H}_1$ (resp. $\mathcal{H}_1^*$) and $\mathcal{H}_2$ (resp. $\mathcal{H}_2^*$) of N are given by:

$$\mathcal{H}_1 = \frac{1}{r}\sum_{i=1}^{r} h(e_i, e_i), \ \ \mathcal{H}_1^* = \frac{1}{r}\sum_{i=1}^{r} h^*(e_i, e_i),$$

and:

$$\mathcal{H}_2 = \frac{1}{k}\sum_{j=1}^{k} h(e_{r+j}, e_{r+j}), \ \ \mathcal{H}_2^* = \frac{1}{k}\sum_{j=1}^{k} h^*(e_{r+j}, e_{r+j}).$$

Furthermore, the Casorati curvatures are:

$$\mathcal{C}_1 = \frac{1}{r}\sum_{a=m+1}^{n}\sum_{i,j=1}^{r}(h_{ij}^{a})^2, \ \ \ \mathcal{C}_1^* = \frac{1}{r}\sum_{a=m+1}^{n}\sum_{i,j=1}^{r}(h_{ij}^{*a})^2, \tag{34}$$

and:

$$\mathcal{C}_2 = \frac{1}{k}\sum_{a=m+1}^{n}\sum_{i,j=1}^{k}(h_{r+ir+j}^{a})^2, \ \ \ \mathcal{C}_2^* = \frac{1}{k}\sum_{a=m+1}^{n}\sum_{i,j=1}^{k}(h_{r+ir+j}^{*a})^2. \tag{35}$$

Equation (21) implies:

$$\frac{k\Delta^{N_1 0}\mathfrak{f}}{\mathfrak{f}} = \sigma^{\nabla,\nabla^*} - \sum_{1\leq i\leq j\leq r}\mathbb{K}^{\nabla,\nabla^*}(e_i \wedge e_j) - \sum_{r+1\leq l\leq s\leq m}\mathbb{K}^{\nabla,\nabla^*}(e_l \wedge e_s).$$

By using (8), the previous equation becomes:

$$\begin{aligned}2\sigma^{\nabla,\nabla^*} =& \frac{k\Delta^{N_10}\mathfrak{f}}{\mathfrak{f}} + r(r-1)\tilde{c} + k(k-1)\tilde{c} + 2r^2||\mathcal{H}_1^0||^2 \\ &- \frac{r^2}{2}(||\mathcal{H}_1||^2 + ||\mathcal{H}_1^*||^2) - \frac{k^2}{2}(||\mathcal{H}_2||^2 + ||\mathcal{H}_2^*||^2) \\ &+ 2k^2||\mathcal{H}_2^0||^2 - 2r\mathcal{C}_1^0 + \frac{r}{2}(\mathcal{C}_1 + \mathcal{C}_1^*) \\ &- 2k\mathcal{C}_2^0 + \frac{k}{2}(\mathcal{C}_2 + \mathcal{C}_2^*). \end{aligned} \tag{36}$$

We define a polynomial P in terms of the components of the second fundamental form h^0 (with respect to the Levi–Civita connection) of N.

$$\begin{aligned}P =& 2r(r-1)\mathcal{C}_1^0 + (r^2-1)\mathcal{C}_1^0(W_1) + \frac{r}{2}(\mathcal{C}_1 + \mathcal{C}_1^*) \\ &+ 2k(k-1)\mathcal{C}_2^0 + (k^2-1)\mathcal{C}_2^0(W_2) + \frac{k}{2}(\mathcal{C}_2 + \mathcal{C}_2^*) \\ &+ \frac{k\Delta^{N_10}\mathfrak{f}}{\mathfrak{f}} + r(r-1)\tilde{c} + k(k-1)\tilde{c} - \frac{r^2}{2}(||\mathcal{H}_1||^2 + ||\mathcal{H}_1^*||^2) \\ &- \frac{k^2}{2}(||\mathcal{H}_2||^2 + ||\mathcal{H}_2^*||^2) - 2\sigma^{\nabla,\nabla^*}. \end{aligned} \tag{37}$$

Without loss of generality, we assume that W_1 and W_2 are respectively spanned by $\{e_1, \ldots, e_{r-1}\}$ and $\{e_{r+1}, \ldots, e_{r+k-1}\}$. Then, by (36) and (37), we derive:

$$\begin{aligned}P =& \sum_{a=m+1}^{n} \left\{ \sum_{i,j=1}^{r} \frac{r+3}{2}(h_{ij}^{0a})^2 + \frac{r+1}{2}\sum_{i,j=1}^{r-1}(h_{ij}^{0a})^2 - 2(\sum_{i=1} h_{ii}^{0a})^2 \right\} \\ &+ \sum_{a=m+1}^{n} \left\{ \sum_{l,s=1}^{k} \frac{k+3}{2}(h_{ls}^{0a})^2 + \frac{k+1}{2}\sum_{l,s=1}^{k-1}(h_{ls}^{0a})^2 - 2(\sum_{l=1} h_{ll}^{0a})^2 \right\} \\ =& \sum_{a=m+1}^{n} \Big\{ 2(r+2)\sum_{1\le i<j\le r-1}(h_{ij}^{0a})^2 + (r+3)\sum_{i=1}^{r-1}(h_{ir}^{0a})^2 \\ &+ r\sum_{i=1}^{r-1}(h_{ii}^{0a})^2 - 4\sum_{1\le i<j\le r}(h_{ii}^{0a}h_{jj}^{0a}) + \frac{r-1}{2}(h_{rr}^{0a})^2 \Big\} \\ &+ \sum_{a=m+1}^{n} \Big\{ 2(k+2)\sum_{1\le l<s\le k-1}(h_{ls}^{0a})^2 + (k+3)\sum_{l=1}^{k-1}(h_{lk}^{0a})^2 \\ &+ k\sum_{l=1}^{k-1}(h_{ll}^{0a})^2 - 4\sum_{1\le l<s\le k}(h_{ll}^{0a}h_{ss}^{0a}) + \frac{k-1}{2}(h_{kk}^{0a})^2 \Big\} \\ \ge& \sum_{a=m+1}^{n} \left\{ \sum_{i=1}^{r-1} r(h_{ii}^{0a})^2 + \frac{r-1}{2}(h_{rr}^{0a})^2 - 4\sum_{1\le i<j\le r} h_{ii}^{0a}h_{jj}^{0a} \right\} \\ &+ \sum_{a=m+1}^{n} \left\{ \sum_{l=1}^{k-1} k(h_{ll}^{0a})^2 + \frac{k-1}{2}(h_{kk}^{0a})^2 - 4\sum_{1\le l<s\le k} h_{ll}^{0a}h_{ss}^{0a} \right\}. \end{aligned}$$

For any $a \in \{m+1, \ldots, n\}$, we define two quadratic forms $\phi_a : \mathbb{R}^r \to \mathbb{R}$ and $\varphi_a : \mathbb{R}^k \to \mathbb{R}$ by:

$$\begin{aligned}\phi_a(h^{0a}_{11}, h^{0a}_{22}, \ldots, h^{0a}_{r-1,r-1}, h^{0a}_{rr}) \\ = \sum_{i=1}^{r-1} r(h^{0a}_{ii})^2 + \frac{r-1}{2}(h^{0a}_{rr})^2 - 4\sum_{1\le i<j\le r} h^{0a}_{ii}h^{0a}_{jj},\end{aligned} \tag{38}$$

and:

$$\begin{aligned}\varphi_a(h^{0a}_{11}, h^{0a}_{22}, \ldots, h^{0a}_{k-1,k-1}, h^{0a}_{kk}) \\ = \sum_{l=1}^{k-1} k(h^{0a}_{ll})^2 + \frac{k-1}{2}(h^{0a}_{kk})^2 - 4\sum_{1\le l<s\le k} h^{0a}_{ll}h^{0a}_{ss}.\end{aligned} \tag{39}$$

First, we consider the constrained extremum problem $\min \phi_a$ subject to:

$$Q : h^{0a}_{11} + \cdots + h^{0a}_{rr} = t^a, \quad (t^a \text{ is any constant}).$$

From (38), we find that the critical points

$$h^{0c} = (h^{0a}_{11}, h^{0a}_{22}, \ldots, h^{0a}_{r-1,r-1}, h^{0a}_{rr})$$

of Q are the solutions of the following system of linear homogeneous equations.

$$\left.\begin{aligned}\frac{\partial\phi_a}{\partial h^{0a}_{ii}} &= 2(r+2)(h^{0a}_{ii}) - 4\textstyle\sum_{j=1}^{r} h^{0a}_{jj} = 0, \\ \frac{\partial\phi_a}{\partial h^{0a}_{rr}} &= (r-1)h^{0a}_{rr} - 4\textstyle\sum_{j=1}^{r-1} h^{0a}_{jj} = 0,\end{aligned}\right\} \tag{40}$$

for $i \in \{1, 2, \ldots, r-1\}$ and $a \in \{m+1, \ldots, n\}$. Hence, every solution h^{0c} has:

$$h^{0a}_{ii} = \frac{1}{r+1}t^a, \quad h^{0a}_{rr} = \frac{4}{r+3}t^a,$$

for $i \in \{1, 2, \ldots, r-1\}$ and $a \in \{m+1, \ldots, n\}$.

Now, we fix $x \in Q$. The bilinear form $\Theta : T_xQ \times T_xQ \to \mathbb{R}$ has the following expression (cf. Theorem 1):

$$\Theta(E, F) = Hess_{\phi_a}(E, F) + \langle h'(E, F), grad(\phi_a)(x)\rangle,$$

where h' denotes the second fundamental form of Q in $\mathbb{R}^r$ and $< \cdot, \cdot >$ denotes the standard inner product on $\mathbb{R}^r$. The Hessian matrix of ϕ_a is given by:

$$Hess_{\phi_a} = \begin{pmatrix} 2(r+2) & -4 & \ldots & -4 & -4 \\ -4 & 2(r+2) & \ldots & -4 & -4 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -4 & -4 & \ldots & 2(r+2) & -4 \\ -4 & -4 & \ldots & -4 & (r-1) \end{pmatrix}.$$

Take a vector $E \in T_xQ$, which satisfies a relation $\sum_{i=1}^{r} E_i = 0$. As the hyperplane is totally geodesic, i.e., $h' = 0$ in $\mathbb{R}^r$, we get:

$$\begin{aligned}\Theta(E,E) &= Hess_{\phi_a}(E,E) \\ &= 2(r+2)\sum_{i=1}^{r-1} E_i^2 + (r-1)E_r^2 - 8\sum_{i\neq j=1}^{r} E_iE_j \\ &= 2(r+2)\sum_{i=1}^{r-1} E_i^2 + (r-1)E_r^2 - 4\left\{(\sum_{i=1}^{r} E_i)^2 - \sum_{i=1}^{r} E_i^2\right\} \\ &= 2(r+4)\sum_{i=1}^{r-1} E_i^2 + (r+3)E_r^2 \\ &\geq 0.\end{aligned}$$

However, the point h^{0c} is the only optimal solution, i.e., the global minimum point of problem, and reaches a minimum $Q(h^{0c}) = 0$ by considering (39) and the constrained extremum problem $\min \varphi_a$ subject to:

$$Q' : h_{11}^{0a} + \cdots + h_{kk}^{0a} = \alpha^a, \quad (\alpha^a \text{ is any constant}).$$

Thus, we have:

$$\begin{aligned}2\sigma^{\nabla,\nabla^*} \leq\ & r(r-1)\mathcal{C}_1^0 + (r^2-1)\mathcal{C}_1^0(W_1) + \frac{r}{2}(\mathcal{C}_1 + \mathcal{C}_1^*) \\ &+ k(k-1)\mathcal{C}_2^0 + (k^2-1)\mathcal{C}_2^0(W_2) + \frac{k}{2}(\mathcal{C}_2 + \mathcal{C}_2^*) \\ &+ \frac{k\Delta^{N_1 0}\mathfrak{f}}{\mathfrak{f}} + r(r-1)\tilde{c} + k(k-1)\tilde{c} \\ &- \frac{r^2}{2}(||\mathcal{H}_1||^2 + ||\mathcal{H}_1^*||^2) - \frac{k^2}{2}(||\mathcal{H}_2||^2 + ||\mathcal{H}_2^*||^2).\end{aligned}$$

Consequently, we get immediately the following theorem from the above relation:

Theorem 6. *Let $N = N_1 \times_{\mathfrak{f}} N_2$ be an m-dimensional statistical warped product submanifold immersed into an n-dimensional statistical manifold of constant sectional curvature $\tilde{c}$. Then, the Casorati curvatures satisfy:*

$$\begin{aligned}2\sigma^{\nabla,\nabla^*} \leq\ & r(r-1)\mathcal{C}_1^0 + (r^2-1)\mathcal{C}_1^0(W_1) + r\mathcal{C}_1^0 \\ &+ k(k-1)\mathcal{C}_2^0 + (k^2-1)\mathcal{C}_2^0(W_2) + k\mathcal{C}_2^0 \\ &+ \frac{k\Delta^{N_1 0}\mathfrak{f}}{\mathfrak{f}} + r(r-1)\tilde{c} + k(k-1)\tilde{c} \\ &- \frac{r^2}{2}(||\mathcal{H}_1||^2 + ||\mathcal{H}_1^*||^2) - \frac{k^2}{2}(||\mathcal{H}_2||^2 + ||\mathcal{H}_2^*||^2),\end{aligned}$$

where W_1 and W_2 are respectively the hyperplanes of T_pN_1 and T_pN_2, $\mathcal{C}_1^0 = \frac{1}{2}(\mathcal{C}_1 + \mathcal{C}_1^*)$, and $\mathcal{C}_2^0 = \frac{1}{2}(\mathcal{C}_2 + \mathcal{C}_2^*)$.

7. Examples

We provide examples of statistical warped product submanifolds as follows:

Example 1. *By generalizing Example 2.7 of [10] to higher dimensions, we see that:*

$$\left(\mathbb{R} \times_{e^z} \mathbb{R}^n,\ \tilde{g} = dz^2 + e^{2z}(dx_1^2 + \cdots + dx_n^2), \nabla, \nabla^*\right)$$

is a statistical warped product manifold. Furthermore, the hyperbolic space:

$$\mathbb{H}^{n+1}(-1) = \left(\{(x_0, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_0 > 0\}, \tilde{g} = \frac{dx_0^2 + \cdots + dx_{n+1}^2}{x_0^2}, \tilde{\nabla}, \tilde{\nabla}^* \right)$$

is the statistical manifold of constant sectional curvature -1*. Thus, with respect to the Levi–Civita connection,* $\mathbb{R} \times_{e^z} \mathbb{R}^{n-1}$ *admits an isometric minimal immersion into* $\mathbb{H}^{n+1}(-1)$*.*

Example 2. $(\mathbb{R} \times_z \mathbb{R}^n, \tilde{g} = dt^2 + t^2(dx_1^2 + \cdots + dx_n^2), \nabla, \nabla^*)$ *is a statistical warped product manifold, and it is isometric to the Euclidean* $(n+1)$*-space* $\mathbb{E}^{n+1}$*. Let* N *be a minimal submanifold of the unit hypersphere* $S^n(1) \subset \mathbb{E}^{n+1}$ *center at the origin* $o \in \mathbb{E}^{n+1}$*, and let* $C(N)$ *be the cone over* N *with the vertex at* o*.*

The metric of $C(N)$ *is the warped product metric* $g_{C(N)} = dt^2 + t^2 g_N$*, where* g_N *denotes the metric of* N*. Any open submanifold* M *of* $C(N)$ *is a warped product manifold, which admits an isometric minimal immersion into the statistical manifold* $\mathbb{E}^{n+1}$ *of constant sectional curvature zero.*

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