Difference Mappings Associated with Nonsymmetric Monotone Types of Fejér's Inequality

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Abstract: Two mappings L_w and P_w, in connection with Fejér's inequality, are considered for the convex and nonsymmetric monotone functions. Some basic properties and results along with some refinements for Fejér's inequality according to these new settings are obtained. As applications, some special means type inequalities are given.

1. Introduction

In 1906, L. Fejér [1] proved the following integral inequalities known in the literature as Fejér's inequality:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1}$$

where $f : [a,b] \to \mathbb{R}$ is convex and $g : [a,b] \to \mathbb{R}^+ = [0,+\infty)$ is integrable and symmetric to $x = \frac{a+b}{2} \big(g(x) = g(a+b-x), \forall x \in [a,b]\big)$. If in (1) we consider $g \equiv 1$, we recapture the classic Hermite–Hadamard inequality [2,3]:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

In [4], two difference mappings L and P associated with Hermite–Hadamard's inequality have been introduced as follows:

$$L : [a,b] \to \mathbb{R}, \qquad L(t) = \frac{f(a)+f(t)}{2}(t-a) - \int_a^t f(s)ds$$

$$P : [a,b] \to \mathbb{R}, \qquad P(t) = \int_a^t f(s)ds - (t-a)f\left(\frac{a+t}{2}\right).$$

Some properties for L and P, refinements for Hermite–Hadamard's inequality and some applications were raised in [4] as well:

Theorem 1 (Theorem 1 in [4]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping on the interval I and let $a < b$ be fixed in I°. Then, we have the following:*

(i) *The mapping L is nonnegative, monotonically nondecreasing, and convex on $[a, b]$*

(ii) *The following refinement of Hadamard's inequality holds:*

$$\frac{1}{b-a} \int_a^b f(s)ds \leq \frac{1}{b-a} \int_y^b f(s)ds + \left(\frac{y-a}{b-a}\right)\frac{f(a)+f(y)}{2} \leq \frac{f(a)+f(b)}{2},$$

for each $y \in [a, b]$.

(iii) *The following inequality holds:*

$$\alpha\frac{f(t)+f(a)}{2}(t-a) + (1-\alpha)\frac{f(s)+f(a)}{2}(s-a) -$$
$$\frac{f(\alpha t + (1-\alpha)s) + f(a)}{2}[\alpha t + (1-\alpha)s - \alpha] \geq$$
$$\alpha \int_a^t f(u)du + (1-\alpha) \int_a^s f(u)du - \int_a^{\alpha t+(1-\alpha)s} f(u)du,$$

for every $t, s \in [a, b]$ and each $\alpha \in [0, 1]$.

Theorem 2 (Theorem 2 in [4]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping on the interval I and let $a < b$ be fixed in I°. Then, we have the following:*

(i) *The mapping P is nonnegative and monotonically nondecreasing on $[a, b]$.*

(ii) *The following inequality holds:*

$$0 \leq P(t) \leq L(t), \qquad \text{for all } t \in [a, b].$$

(iii) *The following refinement of Hadamard's inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \left[(b-a)f\left(\frac{a+b}{2}\right) - (y-a)f\left(\frac{a+y}{2}\right)\right] +$$
$$\frac{1}{b-a} \int_a^y f(s)ds \leq \frac{1}{b-a} \int_a^b f(s)ds,$$

for all $y \in [a, b]$.

The main results obtained in [4] (Theorems 1 and 2) are based on the facts that if $f : [a, b] \to \mathbb{R}$ is convex, then for all $x, y \in [a, b]$ with $x \neq y$ we have (see, [5,6]):

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(s)ds \leq \frac{f(x)+f(y)}{2},$$

and

$$f(x) - f(y) \geq (x-y)f'_+(y),$$

where $f'_+(y)$ is the right-derivative of f at y.

Motivated by the above concepts, inequalities and results, we introduce two difference mappings, L_w and P_w, related to Fejér's inequality:

$$L_w : [a, b] \to \mathbb{R}, \qquad L_w(t) = \frac{f(a)+f(t)}{2} \int_a^t w(s)ds - \int_a^t f(x)w(x)dx,$$

$$P_w : [a, b] \to \mathbb{R}, \qquad P_w(t) = \int_a^t f(x)w(x)dx - f\left(\frac{a+t}{2}\right) \int_a^t w(x)dx.$$

In the case that $w \equiv 1$, the mappings L_w and P_w reduce to L and P, respectively.

In this paper we obtain some properties for L_w and P_w that imply some refinements for Fejér's inequality in the case that w is a nonsymmetric monotone function. Also, our results generalize Theorems 1 and 2 from Hermite–Hadamard's type to Fejér's type. Furthermore as applications, we find some numerical and special means type inequalities.

To obtain our respective results, we need the modified version of Theorem 5 in [7] which includes the left and right part of Fejér's inequality in the monotone nonsymmetric case.

Theorem 3. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on the interval I and differentiable on* I°. *Consider* $a, b \in I^\circ$ *with* $a < b$ *such that* $w : [a, b] \to \mathbb{R}$ *is a nonnegative, integrable and monotone function. Then*

(1) If $w'(x) \le 0$ $(w'(x) \ge 0)$, $a \le x \le b$ *and* $f(a) \le f(b)$ $(f(a) \ge f(b))$, *then*

$$\int_a^b f(x)w(x)dx \le \frac{f(a) + f(b)}{2} \int_a^b w(x)dx. \tag{2}$$

(2) If $w'(x) \ge 0$ $(w'(x) \le 0)$, $a \le x \le b$ *and* $f(a) \le f(\frac{a+b}{2})$ $(f(a) \ge f(\frac{a+b}{2}))$, *then*

$$f\left(\frac{a+b}{2}\right) \int_a^b f(x)w(x)dx \le \int_a^b f(x)w(x)dx. \tag{3}$$

The main point in Theorem 3 (1) $(w'(x) \le 0)$, is that we have (2) for any $x, y \in [a, b]$ with $f(x) \le f(y)$ without the need for w to be symmetric with respect to $\frac{x+y}{2}$. Also similar properties hold for other parts of the above theorem.

Example 1. *Consider* $f(x) = \frac{1}{t}$ *and* $w(x) = \frac{1}{t^2}$ *for* $t > 0$. *It is clear that* f *is convex and* w *is nonsymmetric and decreasing. If we consider* $0 < x \le y$, *then from the fact that* $(y - x)^2 \ge 0$ *we obtain that*

$$\frac{2}{x+y} \le \frac{x+y}{2xy}.$$

This inequality implies that

$$\frac{2}{x+y}\left(\frac{y-x}{xy}\right) \le \frac{y^2 - x^2}{2x^2y^2}.$$

It follows that

$$\frac{2}{x+y}\left(\frac{1}{x} - \frac{1}{y}\right) \le \frac{1}{2x^2} - \frac{1}{2y^2}.$$

So

$$\left(\frac{1}{\frac{x+y}{2}}\right) \int_x^y \frac{1}{t^2}dt \le \int_x^y \frac{1}{t^3}dt,$$

shows that f *and* w *satisfy* (3) *on* $[x, y]$, *where* w *is not symmetric. Also, we can see that* f *and* w *satisfy* (2).

2. Main Results

The first result of this section is about some properties of the mapping L_w where the function w is nonincreasing.

Theorem 4. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on the interval I and differentiable on* I°. *Consider* $a, b \in I^\circ$ *with* $a < b$ *such that* $w : [a, b] \to \mathbb{R}$ *is a nonnegative and differentiable function with* $w'(x) \le 0$ *for all* $a \le x \le b$. *Then*

(i)   The mapping L_w *is nonnegative on* $[a, b]$, *if* $f(a) \le f(t)$ *for all* $t \in [a, b]$.
(ii) The mapping L_w *is convex on* $[a, b]$, *if* f *is nondecreasing. Also* L_w *is monotonically nondecreasing on* $[a, b]$.

(iii)　The following refinement of (2) *holds:*

$$\int_a^b f(x)w(x) \leq$$

$$\int_y^b f(x)w(x)dx + \frac{f(a)+f(y)}{2}\int_a^y w(x)dx \leq \tag{4}$$

$$\frac{f(a)+f(b)}{2}\int_a^b w(x)dx,$$

　　　for any $y \in [a,b]$ *with* $f(a) \leq f(y)$.

(iv)　If f *is nondecreasing, then the following inequality holds:*

$$t\int_a^u f(x)w(x)dx + (1-t)\int_a^v f(x)w(x)dx - \int_a^{tu+(1-t)v} f(x)w(x)dx \leq$$

$$t\frac{f(u)+f(a)}{2}\int_a^u w(x)dx + (1-t)\frac{f(v)+f(u)}{2}\int_a^v w(x)dx \tag{5}$$

$$-\frac{f(tu+(1-t)v)+f(a)}{2}\int_a^{tu+(1-t)v} w(x)dx,$$

　　　for any $u,v \in [a,b]$ *and each* $t \in [0,1]$.

(v)　If $f' \in L([a,b])$, *then for each* $t \in [a,b]$ *we have*

$$|L_w(t)| \leq \frac{(t-a)^2}{2}\int_a^t w(x)|f'(x)|dx. \tag{6}$$

Furthermore when $|f'|$ *is convex on* $[a,b]$, *then:*

$$|L_w(t)| \leq \frac{t-a}{2}\left[|f'(a)|\int_a^t (t-x)w(x)dx + |f'(t)|\int_a^t (x-a)w(x)dx\right]. \tag{7}$$

Proof. (i) We need only the inequality

$$\int_a^t f(x)w(x)dx \leq \frac{f(a)+f(t)}{2}\int_a^t w(x)dx,$$

for all $t \in [a,b]$. This happens according to Theorem 3 (1).

　　(ii) Without loss of generality for $a \leq y < x < b$ consider the following identity:

$$L_w(x) - L_w(y) = \tag{8}$$

$$\frac{f(x)+f(a)}{2}\int_a^x w(s)ds - \frac{f(y)+f(a)}{2}\int_a^y w(s)ds - \int_y^x f(s)w(s)ds$$

Dividing with "$x-y$" and then letting $x \to y$ we obtain that

$$2L'_{+w}(y) - f(a)w(y) + f(y)w(y) = f'_+(y)\int_a^y w(s)ds. \tag{9}$$

Also from the convexity of f we have

$$f'_+(y) \leq \frac{f(x)-f(y)}{x-y},$$

which, along with the fact that w is nonincreasing, implies that

$$
\begin{aligned}
f'_+(y) \int_a^y w(s)ds &\leq \frac{f(x) - f(y)}{x - y} \int_a^y w(s)ds \\
&\leq \frac{f(x) + f(a)}{x - y} \int_a^x w(s)ds - \frac{f(y) + f(a)}{x - y} \int_a^y w(s)ds \\
&\quad + [f(y) - f(a)]w(y) - \frac{f(x) + f(y)}{x - y} \int_y^x w(s)ds.
\end{aligned}
\tag{10}
$$

So from (9) and (10) we get

$$
\begin{aligned}
L'_{+w}(y) &\leq \\
\frac{f(x) + f(a)}{2(x - y)} \int_a^x w(s)ds &- \frac{f(y) + f(a)}{2(x - y)} \int_a^y w(s)ds - \frac{f(x) + f(y)}{2(x - y)} \int_y^x w(s)ds.
\end{aligned}
\tag{11}
$$

On the other hand from (8) and Theorem 3 (1), we have

$$
\begin{aligned}
\frac{L_w(x) - L_w(y)}{x - y} &\geq \\
\frac{f(x) + f(a)}{2(x - y)} \int_a^x w(s)ds &- \frac{f(y) + f(a)}{2(x - y)} \int_a^y w(s)ds - \frac{f(x) + f(y)}{2(x - y)} \int_y^x w(s)ds,
\end{aligned}
$$

and, along with (11), we obtain that

$$
\frac{L_w(x) - L_w(y)}{x - y} \geq L'_{+w}(y).
$$

This implies the convexity of $L_w(t)$.

For the fact that L is monotonically nondecreasing, from convexity of f on $[a, b]$ we have

$$
f'_+(y) \geq \frac{f(y) - f(a)}{y - a},
$$

for all $y \in [a, b]$ and so

$$
\begin{aligned}
\frac{L_w(x) - L_w(y)}{x - y} &\geq L'_{+w}(y) = \frac{f'_+(y)}{2} \int_a^y w(s)ds + \frac{f(a)w(y)}{2} - \frac{f(y)w(y)}{2} \\
&= \frac{1}{2}\left[f'_+(y) \int_a^y w(s)ds + (f(a) - f(y))w(y) \right] \geq \\
&\frac{1}{2}\left[f'_+(y)(y - a) - (f(y) - f(a)) \right] w(y) \geq 0,
\end{aligned}
$$

for any $x > y$.

(iii) Since L_w is monotonically nondecreasing we have $0 \leq L_w(y) \leq L_w(b)$, for all $y \in [a, b]$ and so

$$
\begin{aligned}
\frac{f(y) + f(a)}{2} \int_a^y w(x)dx - \int_a^y f(x)w(x)dx &\leq \\
\frac{f(b) + f(a)}{2} \int_a^b w(x)dx - \int_a^b f(x)w(x)dx,
\end{aligned}
$$

which implies that

$$
\int_y^b f(x)w(x)dx + \frac{f(a) + f(y)}{2} \int_a^y w(x)dx \leq \frac{f(a) + f(b)}{2} \int_a^b w(x)dx.
\tag{12}
$$

Also, by the use of Theorem 3 (1) we get

$$\int_y^b f(x)w(x)dx + \frac{f(a)+f(y)}{2}\int_a^y w(x)dx \tag{13}$$

$$\geq \int_y^b f(x)w(x)dx + \int_a^y f(x)w(x)dx = \int_a^b f(x)w(x)dx.$$

Now from (12) and (13), we have the result.

(iv) Since L_w is convex, then from the fact that

$$L_w(tu + (1-t)v) \leq tL_w(u) + (1-t)L_w(v),$$

for any $u, v \in [a, b]$ and each $t \in [0, 1]$, we have the result.

(v) The following identity was obtained in [8]:

$$\frac{f(a)+f(t)}{2}\int_a^t w(x)dx - \int_a^t f(x)w(x)dx = \frac{(t-a)^2}{2}\int_0^1 p(s)f'(sa + (1-s)t)ds, \tag{14}$$

for any $t \in [a, b]$ where

$$p(s) = \int_s^1 w(ua + (1-u)t)du + \int_s^0 w(ua + (1-u)t)du, \qquad s \in [0,1].$$

Since w is nonincreasing, then we obtain

$$\int_s^1 w(ua + (1-u)t)du \leq w(sa + (1-s)t)(as + (1-s)t - a) =$$
$$w(sa + (1-s)t)(1-s)(t-a),$$

and

$$\int_s^0 w(ua + (1-u)t)du \leq w(sa + (1-s)t)(t - sa - (1-s)t) =$$
$$w(sa + (1-s)t)s(t-a).$$

So

$$|p(s)| \leq w(sa + (1-s)t)(t-a), \qquad s \in [0,1]. \tag{15}$$

Now by the use of (15) in (14) we get

$$|L_w(t)| \leq \frac{(t-a)^3}{2}\int_0^1 w(sa + (1-s)t)|f'(sa + (1-s)t)|ds, \tag{16}$$

for any $t \in [a, b]$. Using the change of variable $x = sa + (1-s)t$ and some calculations imply that

$$|L_w(t)| \leq \frac{(t-a)^2}{2}\int_a^t w(x)|f'(x)|dx,$$

for any $t \in [a, b]$. Furthermore if $|f'|$ is convex on $[a, b]$, then from (16) and by the use of the change of variable $x = sa + (1-s)t$ we get

$$|L_w(t)| \leq \frac{(t-a)^3}{2}\left[|f'(a)|\int_a^t \frac{t-x}{t-a}w(x)\frac{dx}{t-a} + |f'(t)|\int_a^t \frac{x-a}{t-a}w(x)\frac{dx}{t-a}\right],$$

which implies that

$$|L_w(t)| \leq \frac{(t-a)}{2}\left[|f'(a)|\int_a^t (t-x)w(x)dx + |f'(t)|\int_a^t (x-a)w(x)dx\right],$$

for any $t \in [a,b]$.　□

Remark 1. *(i) By the use of Theorem 3 (1), it is not hard to see that if w is nondecreasing on $[a,b]$, then some properties of L_w and corresponding results obtained in Theorem 4 may change. However the argument of proof is similar. The details are omitted.*

(ii) Theorem 4 gives a generalization of Theorem 1, along with some new results.

The following result is including some properties of the mapping P_w in the case that w is nondecreasing.

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and differentiable on I°. Consider $a, b \in I^\circ$ with $a < b$ such that $w : [a,b] \to \mathbb{R}$ is a nonnegative and continuous function with $w'(x) \geq 0$ for all $a \leq x \leq b$. Then*

(i)　P_w is nonnegative, if $f(a) \leq f\left(\frac{a+t}{2}\right)$ for any $t \in [a,b]$.
(ii)　If for any $x < y$ we have $f(x) \leq f\left(\frac{x+y}{2}\right)$, then P_w is nondecreasing on $[a,b]$.
(iii)　If $f' \in L([a,b])$, then for each $t \in [a,b]$ we have

$$|P_w(t)| \leq (t-a)\left[\int_a^{\frac{a+t}{2}} w(x)(x-a)|f'(x)|dx + \int_{\frac{a+t}{2}}^t w(x)(t-x)|f'(x)|dx\right]. \tag{17}$$

Furthermore when $|f'|$ is convex on $[a,b]$, then:

$$|P_w(t)| \leq \left[\int_a^{\frac{a+t}{2}} w(x)(t-x)(x-a)dx + \int_{\frac{a+t}{2}}^t w(x)(t-x)^2 dx\right]|f'(a)| + \tag{18}$$

$$\left[\int_a^{\frac{a+t}{2}} w(x)(x-a)^2 dx + \int_{\frac{a+t}{2}}^t w(x)(t-x)(x-a)dx\right]|f'(t)|.$$

(iv)　The following inequality holds:

$$P_w(t) - L_w(t) \leq \int_a^t f(x)w(x)dx, \tag{19}$$

provided that $f(a) \leq f\left(\frac{a+t}{2}\right)$ for all $t \in [a,b]$.
(v)　If for any $x < y$ we have $f(x) \leq f\left(\frac{x+y}{2}\right)$, then the following refinement of (3) holds:

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq$$

$$\int_a^t f(x)w(x)dx + f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx - f\left(\frac{a+t}{2}\right)\int_a^t w(x)dx \leq \tag{20}$$

$$\int_a^b f(x)w(x)dx,$$

for all $t \in [a,b]$.

Proof. (i) It follows from Theorem 3 (2).

(ii) Suppose that $a \leq x < y < b$. So from Theorem 3 (2) and the facts that w is nondecreasing and f is convex, we get

$$P_w(y) - P_w(x) =$$

$$\int_a^y f(t)w(t)dt - f\left(\frac{a+y}{2}\right)\int_a^y w(t)dt - \int_a^x f(t)w(t)dt + f\left(\frac{a+x}{2}\right)\int_a^x w(t)dt =$$

$$\int_x^y f(t)w(t)dt + f\left(\frac{a+x}{2}\right)\int_a^x w(t)dt - f\left(\frac{a+y}{2}\right)\int_a^y w(t)dt \geq$$

$$f\left(\frac{x+y}{2}\right)\int_x^y w(t)dt + f\left(\frac{a+x}{2}\right)\int_a^x w(t)dt - f\left(\frac{a+y}{2}\right)\int_a^y w(t)dt \geq$$

$$f\left(\frac{x+y}{2}\right)(y-x)w(x) + f\left(\frac{a+x}{2}\right)(x-a)w(a) - f\left(\frac{a+y}{2}\right)(y-a)w(y) \geq$$

$$\left[f\left(\frac{x+y}{2}\right)(y-x) + f\left(\frac{a+x}{2}\right)(x-a) - f\left(\frac{a+y}{2}\right)(y-a)\right]w(a) \geq 0.$$

This completes the proof.

(iii) The following identity is obtained in [8]:

$$\int_a^t f(x)w(x)dx - f\left(\frac{a+t}{2}\right)\int_a^t w(x)dx = (t-a)^2\int_0^1 k(s)f'(sa + (1-s)t)ds,$$

for any $t \in [a, b]$, where

$$k(s) = \begin{cases} \int_0^s w(ua + (1-u)t)du, & s \in [0, \frac{1}{2}); \\ -\int_s^1 w(ua + (1-u)t)du, & s \in [\frac{1}{2}, 1]. \end{cases}$$

By similar method used to prove part (v) of Theorem 4, we can obtain the results. We omitted the details here.

(iv) By Theorem 3 (1), for any $t \in (a, b]$ we have

$$\int_a^{\frac{a+t}{2}} f(x)w(x)dx \leq \frac{f\left(\frac{a+t}{2}\right) + f(a)}{2}\int_a^{\frac{a+t}{2}} w(x)dx \leq \frac{f\left(\frac{a+t}{2}\right) + f(a)}{2}\int_a^t w(x)dx, \qquad (21)$$

and

$$\int_{\frac{a+t}{2}}^t f(x)w(x)dx \leq \frac{f\left(\frac{a+t}{2}\right) + f(t)}{2}\int_{\frac{a+t}{2}}^t w(x)dx \leq \frac{f\left(\frac{a+t}{2}\right) + f(t)}{2}\int_a^t w(x)dx. \qquad (22)$$

If we add (21) to (22), we obtain

$$\int_a^t f(x)w(x)dx \leq \left[f\left(\frac{a+t}{2}\right) + \frac{f(a) + f(t)}{2}\right]\int_a^t w(x)dx,$$

which is equivalent with

$$\int_a^t f(x)w(x)dx \leq -P_w(t) + L_w(t) + 2\int_a^t f(x)w(x)dx.$$

This implies the desired result.

(v) The left side of (20) is a consequence of assertion (i) and the following inequality:

$$\int_a^t f(x)w(x)dx - f\left(\frac{a+t}{2}\right)\int_a^t w(x)dx \geq 0,$$

for all $t \in [a, b]$.

Since P_w is nondecreasing we have $P_w(t) \le P_w(b)$ for all $t \in [a,b]$, i. e.

$$\int_a^t f(x)w(x)dx - f\left(\frac{a+t}{2}\right)\int_a^t w(x)dx \le$$
$$\int_a^b f(x)w(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx.$$

Then we have the right side of (20). □

Remark 2. *(i) By the use of Theorem 3 (2) (w is nonincreasing on $[a,b]$) in the proof of Theorem 5, we can obtain some different properties for P_w with new corresponding results. The details are omitted.*
(ii) Theorem 5 gives a generalization of Theorem 2, along with some new results.

3. Applications

The following means for real numbers $a,b \in \mathbb{R}$ are well known:

$$A(a,b) = \frac{a+b}{2} \qquad\qquad \text{arithmetic mean,}$$

$$L_n(a,b) = \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} \qquad \text{generalized log$-$mean, } n \in \mathbb{R},\ a < b.$$

The following result holds between the two above special means:

Theorem 6. *For any $a,b \in \mathbb{R}$ with $0 < a < b$ and $n \in \mathbb{N}$ we have*

$$A^n(a,b) \le L_n^n(a,b) \le A(a^n,b^n). \tag{23}$$

In this section as applications of our results in previous section, we give some refinements for the inequalities mentioned in (23).

Consider $a,b \in (0,\infty)$ with $a < b$. Define

$$\begin{cases} f(x) = x^n, & x \in [a,b] \text{ and } n \ge 1; \\ w(x) = x^{-s}, & x \in [a,b] \text{ and } s \in [0,1)\cup(1,\infty). \end{cases}$$

From (4) with some calculations we have

$$\frac{b^{n-s+1}-a^{n-s+1}}{n-s+1} \le$$
$$\frac{b^{n-s+1}-t^{n-s+1}}{n-s+1} + \frac{a^n+t^n}{2}\left(\frac{t^{1-s}-a^{1-s}}{1-s}\right) \le$$
$$\frac{a^n+b^n}{2}\left(\frac{b^{1-s}-a^{1-s}}{1-s}\right),$$

for all $t \in [a,b]$, which implies that

$$(b-a)L_{n-s}^{n-s}(a,b) \le$$
$$(b-t)L_{n-s}^{n-s}(t,b) + A(a^n,t^n)\left(\frac{t^{1-s}-a^{1-s}}{1-s}\right) \le \tag{24}$$
$$A(a^n,b^n)\left(\frac{b^{1-s}-a^{1-s}}{1-s}\right).$$

Inequality (24) gives a refinement for the right part of (23).

In the case that $s = 1$ we have

$$(b - a)L_{n-1}^{n-1}(a, b) \leq (b - t)L_{n-1}^{n-1}(t, b) + ln\frac{t}{a}A(a^n, t^n) \leq ln\frac{t}{a}A(a^n, b^n).$$

In the case that $s = 0$ we get

$$L_n^n(a, b) \leq \left(\frac{b - t}{b - a}\right)L_n^n(t, b) + \left(\frac{t - a}{b - a}\right)A(a^n, t^n) \leq A(a^n, b^n), \tag{25}$$

for all $t \in [a, b]$. In fact inequality (25) is equivalent with the first inequality obtained in the applications section of [4].

Now with the same assumption for f and w as was used to obtain (24), by the use of (20) we get:

$$\begin{aligned}
&A^n(a, b)\left(\frac{b^{1-s} - a^{1-s}}{1 - s}\right) \leq \\
&A^n(a, b)\left(\frac{b^{1-s} - a^{1-s}}{1 - s}\right) + (t - a)L_{n-s}^{n-s}(t, a) - A^n(a, t)\left(\frac{t^{1-s} - a^{1-s}}{1 - s}\right) \leq \\
&(b - a)L_{n-s}^{n-s}(b, a),
\end{aligned} \tag{26}$$

for all $t \in [a, b]$ and $s \in [0, 1) \cup (1, \infty)$. Inequality (26) gives a refinement for the left part of (23). Also if we consider $s = 1$, then we obtain

$$ln\frac{b}{a}A^n(a, b) \leq ln\frac{b}{a}A^n(a, b) + (t - a)L_{n-1}^{n-1}(t, a) - ln\frac{t}{a}A^n(a, t) \leq (b - a)L_{n-1}^{n-1}(b, a),$$

for all $t \in [a, b]$. In a more special case, if we set $s = 0$, then we get:

$$A^n(a, b) \leq A^n(a, b) + \left(\frac{t - a}{b - a}\right)\left[L_n^n(t, a) - A^n(a, t)\right] \leq L_n^n(b, a),$$

for all $t \in [a, b]$.

Finally we encourage interested readers to use inequalities (4)–(7) and inequalities (17)–(20), for appropriate functions f and w to obtain some new special means types and numerical inequalities.

Conflicts of Interest: The authors declare no conflict of interest.

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