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On Locally and Globally Optimal Solutions in Scalar Variational Control Problems

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Abstract: In this paper, optimality conditions are studied for a new class of PDE and PDI-constrained scalar variational control problems governed by path-independent curvilinear integral functionals. More precisely, we formulate and prove a minimal criterion for a local optimal solution of the considered PDE and PDI-constrained variational control problem to be its global optimal solution. The effectiveness of the main result is validated by a two-dimensional non-convex scalar variational control problem.

Keywords: local optimal solution; global optimal solution; minimal criterion; control; PDE and PDI-constrained scalar variational control problem

MSC: 49J40; 65K10; 90C26; 90C30; 49K20

1. Introduction

As it is known, a minimizer for an optimization problem is unique if the associated objective function is strictly convex. However, the definition of a unique minimizer has been extended (see Polyak [1]) to the notion of a *unique sharp minimizer*. Moreover, *weak sharp solutions* associated with variational-type inequalities have been introduced and studied (see Burke and Ferris [2], Patriksson [3], Marcotte and Zhu [4], and Hiriart-Urruty and Lemaréchal [5]).

Over time, considerable interest has been given to achieving the necessary and sufficient conditions so that a real-valued continuous function has the property that any local extreme is also global. In this direction, we mention the works of Zang and Avriel [6], Zang et al. [7,8], Horst [9], Martin [10], Arana-Jiménez and Antczak [11], Treanță and Arana-Jiménez [12,13], and Treanță [14–16]. For other different but connected ideas to this subject, the reader is directed to Shang [17,18].

In this paper, motivated by the above mentioned works, in accordance to Giannessi [19], Clarke [20], and Mititelu and Treanță [21], we build a new mathematical framework on the equivalence between local and global optimal solutions in scalar variational control problems governed by path-independent curvilinear integral functionals (mechanical work). More exactly, under only the continuity hypotheses of the functionals involved in the considered scalar control problem, a minimal criterion is established such that any local optimal solution associated with the considered first-order PDE and PDI-constrained variational control problem is also its global optimal solution. To the best of the author's knowledge, there are no works in the literature dealing with this topic for multidimensional scalar variational control problems with constraints involving first-order partial differential equations/inequations (PDE/PDI). Thus, the ideas and techniques developed in this paper may stimulate further research in this dynamic field.

The present paper is organized as follows. Section 2 includes notations, working hypotheses, and problem description. The main results of this paper are contained in Section 3. Thus, under only continuity assumptions of the involved functionals, a minimal criterion is formulated and proved

so that any local optimal solution associated with the considered PDE and PDI-constrained scalar variational control problem is also its global optimal solution. To validate the main result, an application is provided in Section 4. Section 5 concludes the paper.

2. Problem Description

In order to introduce the scalar variational control problem for which the optimality is studied, we start with the following notations and working hypotheses: $\Theta \subset \mathbb{R}^m$ is a compact domain and $\theta = (\theta^v)$, $v = \overline{1, m}$ (i.e., $v = 1, \dots, m$), denotes a point in Θ ; for $\theta_1 = (\theta_1^1, \dots, \theta_1^m)$, $\theta_2 = (\theta_2^1, \dots, \theta_2^m)$ two different points in Θ , let $\Theta \supset Y : \theta = \theta(\rho)$, $\rho \in [a, b]$ (or $\theta \in \overline{\theta_1, \theta_2}$) be a piecewise smooth curve joining the points θ_1 and θ_2 in Θ ; let $\tilde{\mathcal{X}}$ be the space of all functions $x : \Theta \rightarrow \mathbb{R}^n$ and let $\overline{\mathcal{X}} \subset \tilde{\mathcal{X}}$ be the space of piecewise smooth *state functions* $x : \Theta \rightarrow \mathbb{R}^n$ with the norm:

$$\|x\| = \|x\|_\infty + \sum_{v=1}^m \|x_v\|_\infty, \quad \forall x \in \overline{\mathcal{X}}, \tag{1}$$

where x_v denotes $\frac{\partial x}{\partial \theta^v}$; also, denote by $\tilde{\mathcal{U}}$ the space of all functions $u : \Theta \rightarrow \mathbb{R}^k$ and by $\overline{\mathcal{U}} \subset \tilde{\mathcal{U}}$ the space of piecewise continuous *control functions* $u : \Theta \rightarrow \mathbb{R}^k$ with the uniform norm $\|\cdot\|_\infty$; for any two subsets A_1, A_2 of $\tilde{\mathcal{X}} \times \tilde{\mathcal{U}}$, denote by $int(A_1)$ the *interior* of A_1 and by $cl(A_1)$ the *closure* of A_1 ; also, consider the set:

$$A_1 \setminus A_2 = \{(x, u) \in \tilde{\mathcal{X}} \times \tilde{\mathcal{U}} : (x, u) \in A_1 \wedge (x, u) \notin A_2\}. \tag{2}$$

In the following, consider $\overline{\mathcal{X}} \times \overline{\mathcal{U}}$ as a nonempty open subset of $\tilde{\mathcal{X}} \times \tilde{\mathcal{U}}$ endowed with the inner product:

$$\begin{aligned} \langle (x, u); (y, w) \rangle &= \int_Y [x(\theta) \cdot y(\theta) + u(\theta) \cdot w(\theta)] d\theta^v \\ &= \int_Y [x(\theta) \cdot y(\theta) + u(\theta) \cdot w(\theta)] d\theta^1 + \dots + [x(\theta) \cdot y(\theta) + u(\theta) \cdot w(\theta)] d\theta^m \end{aligned} \tag{3}$$

and the induced norm. Also, for $v = \overline{1, m}$ and $\beta = \overline{1, q}$, consider the real-valued continuously differentiable functions (closed Lagrange 1-form densities) $f_v : \Theta \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}$, $g_\beta : \Theta \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^k \rightarrow \mathbb{R}$ and, for any $(x, u) \in \overline{\mathcal{X}} \times \overline{\mathcal{U}}$, define the following continuous functionals (involving path-independent curvilinear integral):

$$F(x, u) = \int_Y f_v(\theta, x(\theta), u(\theta)) d\theta^v \tag{4}$$

$$\begin{aligned} &= \int_Y f_1(\theta, x(\theta), u(\theta)) d\theta^1 + \dots + f_m(\theta, x(\theta), u(\theta)) d\theta^m, \\ G_\beta(x, u) &= g_\beta(\theta, x(\theta), x_v(\theta), u(\theta)), \quad \theta \in \Theta, \beta = \overline{1, q}. \end{aligned} \tag{5}$$

By using the previous mathematical objects, we formulate the PDE and PDI-constrained variational control problem (P) considered in the paper as follows:

$$\min_{(x, u)} F(x, u) \tag{6}$$

$$\text{subject to } G_\beta(x, u) \leq 0, \quad \beta \in Q := \{1, \dots, q\}, \tag{7}$$

$$x(\theta_1) = x_1 = \text{given}, \quad x(\theta_2) = x_2 = \text{given}. \tag{8}$$

Next, denote by:

$$\mathcal{F} = \{(x, u) \in \overline{\mathcal{X}} \times \overline{\mathcal{U}} : G_\beta(x, u) \leq 0, \beta \in Q, x(\theta_l) = x_l = \text{given}, l = 1, 2\} \tag{9}$$

the feasible set associated with the multidimensional scalar variational control problem (P) and define the index subset of active constraint functionals at (x, u) as follows:

$$Q_{(x,u)} = \{\beta \in Q \mid G_\beta(x, u) = 0\}. \tag{10}$$

Definition 1. A point $(x^0, u^0) \in \mathcal{F}$ is said to be a global optimal solution of (P) if the inequality $F(x^0, u^0) \leq F(x, u)$ holds for all $(x, u) \in \mathcal{F}$.

Definition 2. A point $(x^0, u^0) \in \mathcal{F}$ is said to be a local optimal solution of (P) if there exists a neighborhood $V_{(x^0, u^0)}$ of the point (x^0, u^0) such that the inequality $F(x^0, u^0) \leq F(x, u)$ is fulfilled for all $(x, u) \in \mathcal{F} \cap V_{(x^0, u^0)}$.

Definition 3. For $r \in \mathbb{R}, r > 0$, the set:

$$B_r(x^0, u^0) = \{(y, w) \in \overline{\mathcal{X}} \times \overline{\mathcal{U}} : \|(y, w) - (x^0, u^0)\| < r\} \tag{11}$$

is an open ball with center $(x^0, u^0) \in \overline{\mathcal{X}} \times \overline{\mathcal{U}}$ and radius r , where $\|\cdot\|$ is the induced norm by the inner product introduced in Equation (3).

Remark 1. (i) Obviously, any global optimal solution of (P) is its local optimal solution. The reverse is false, in general.

(ii) Since $(x^0, u^0) \in \text{int}(V_{(x^0, u^0)})$, there exists $r \in \mathbb{R}, r > 0$, such that $B_r(x^0, u^0) \subseteq V_{(x^0, u^0)}$.

(iii) For any $\beta \in Q_{(x,u)}$, it results:

$$G_\beta(x, u) = g_\beta(\theta, x(\theta), x_v(\theta), u(\theta)) = 0, \quad \theta \in \Theta. \tag{12}$$

If, for any $\beta \in Q_{(x,u)}$, the PDEs formulated in Equation (12) can be rewritten in normal form (m-flow type PDEs):

$$\frac{\partial x^i}{\partial \theta^v}(\theta) = Z_v^i(\theta, x(\theta), u(\theta)), \quad i = \overline{1, n}, v = \overline{1, m}, \theta \in \Theta, \tag{13}$$

then we assume that the continuously differentiable functions:

$$Z_v = (Z_v^i) : \Theta \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n, \quad i = \overline{1, n}, v = \overline{1, m} \tag{14}$$

fulfill the closeness (integrability) conditions:

$$D_\zeta Z_v^i = D_v Z_\zeta^i, \quad v, \zeta = \overline{1, m}, v \neq \zeta, i = \overline{1, n}, \tag{15}$$

where D_ζ is the total derivative operator.

3. Main Results

In this section, a minimal criterion is formulated and proved such that any local optimal solution associated with the considered PDE and PDI-constrained variational control problem (P) is also a global optimal solution of (P).

A necessary and sufficient condition for a local optimal solution of the scalar variational control problem (P) to be its global minimizer is established in the next result. The following notation:

$$S = \{s : [0, 1] \rightarrow \overline{\mathcal{X}} \times \overline{\mathcal{U}} \mid (\exists) \lim_{\tau \rightarrow 0^+} \|s(\tau) - s(0)\| = 0\} \tag{16}$$

is used throughout the paper.

Theorem 1. Any local optimal solution associated with the considered constrained variational control problem (P) is also its global optimal solution if and only if, for any $(x, u), (y, w) \in \mathcal{F}$, with $F(x, u) - F(y, w) < 0$, there exists $h : (\bar{\mathcal{X}} \times \bar{\mathcal{U}})^2 \rightarrow S$ such that:

$$(y, w) + h((x, u), (y, w))(\tau) \in \bar{\mathcal{X}} \times \bar{\mathcal{U}} \tag{17}$$

and, for all $\tau \in (0, 1)$, it is verified:

$$h((x, u), (y, w))(\tau)|_{\theta=\theta_1, \theta_2} = (0, \underline{u}), \quad \underline{u} \in \bar{\mathcal{U}}, \tag{18}$$

$$F((y, w) + h((x, u), (y, w))(\tau)) < F(y, w), \tag{19}$$

$$G_\beta((y, w) + h((x, u), (y, w))(\tau)) \leq 0, \quad \beta \in Q. \tag{20}$$

Proof. " \Leftarrow " Let $(x^0, u^0) \in \mathcal{F}$ be a local optimal solution of (P). We proceed by contradiction. Suppose that there exists $(x, u) \in \mathcal{F}$ such that $F(x, u) - F(x^0, u^0) < 0$. By hypothesis, there exists $h : (\bar{\mathcal{X}} \times \bar{\mathcal{U}})^2 \rightarrow S$ such that:

$$(x^0, u^0) + h((x, u), (x^0, u^0))(\tau) \in \bar{\mathcal{X}} \times \bar{\mathcal{U}}, \tag{21}$$

$$h((x, u), (x^0, u^0))(\tau)|_{\theta=\theta_1, \theta_2} = (0, \underline{u}), \quad \underline{u} \in \bar{\mathcal{U}}, \tag{22}$$

$$F((x^0, u^0) + h((x, u), (x^0, u^0))(\tau)) < F(x^0, u^0), \tag{23}$$

for any $\tau \in (0, 1)$.

Furthermore, since $\bar{\mathcal{X}} \times \bar{\mathcal{U}}$ is an open set and $(x^0, u^0) \in \bar{\mathcal{X}} \times \bar{\mathcal{U}}$, there exists $r_0 \in \mathbb{R}, r_0 > 0$, such that $B_{r_0}(x^0, u^0) \subset \bar{\mathcal{X}} \times \bar{\mathcal{U}}$. Let $\beta \in Q \setminus Q_{(x^0, u^0)}$. Consequently, it follows that $G_\beta(x^0, u^0) < 0$ and, by using the continuity property of the functionals G_β , there exist $r_\beta \in \mathbb{R}, r_\beta > 0$, such that $B_{r_\beta}(x^0, u^0) \subset \bar{\mathcal{X}} \times \bar{\mathcal{U}}$ and

$$G_\beta(y, w) < 0, \quad \forall (y, w) \in B_{r_\beta}(x^0, u^0). \tag{24}$$

Since,

$$\lim_{\tau \rightarrow 0^+} \|h((x, u), (x^0, u^0))(\tau)\| = \|h((x, u), (x^0, u^0))(0)\| = 0, \tag{25}$$

therefore, for any $r \in \mathbb{R}, r > 0$, there exists $\tau_0 \in (0, 1)$ such that:

$$\|h((x, u), (x^0, u^0))(\tau)\| < \min\{r, r_0, r_\beta\}, \quad \beta \in Q \setminus Q_{(x^0, u^0)}, \tau \in [0, \tau_0]. \tag{26}$$

For $(z, \mu) = (x^0, u^0) + h((x, u), (x^0, u^0))(\frac{\tau_0}{2})$, we get $(z, \mu) \in B_{r_\beta}(x^0, u^0) \subset \bar{\mathcal{X}} \times \bar{\mathcal{U}}, \beta \in Q \setminus Q_{(x^0, u^0)}$, and, by using (24), it follows:

$$G_\beta(z, \mu) < 0, \quad \beta \in Q \setminus Q_{(x^0, u^0)}. \tag{27}$$

By hypothesis (20), we obtain:

$$G_\beta(z, \mu) = G_\beta((x^0, u^0) + h((x, u), (x^0, u^0))(\frac{\tau_0}{2})) = 0, \quad \beta \in Q_{(x^0, u^0)}. \tag{28}$$

Taking into account Equations (22), (24), and (28), it follows that $(z, \mu) \in \mathcal{F}$ and $F(z, \mu) < F(x^0, u^0)$, that is, (x^0, u^0) is not a local optimal solution of (P). Consequently, we obtain a contradiction.

“ \implies ” Assume that any local optimal solution of (P) is also global. In accordance with Equation (19), consider $(x, u), (y, w) \in \mathcal{F}$ such that:

$$F(x, u) - F(y, w) < 0. \tag{29}$$

Since $\overline{\mathcal{X}} \times \overline{\mathcal{U}}$ is an open set, there exists $r_1 \in \mathbb{R}, r_1 > 0$, such that: $B_{r_1}(y, w) \subset \overline{\mathcal{X}} \times \overline{\mathcal{U}}$. By using the continuity property of F on $\overline{\mathcal{X}} \times \overline{\mathcal{U}}$, for $\epsilon := \frac{F(y, w) - F(x, u)}{2} > 0$, there exists $r_2 \in (0, r_1)$ such that

$$\| F(z, \mu) - F(y, w) \| < \epsilon, \quad \forall (z, \mu) \in B_{r_2}(y, w). \tag{30}$$

Furthermore, for any $r \in (0, r_2)$, define the following variational control problem (P'_r) associated to (P) :

$$\min_{(z, \mu)} F(z, \mu) \tag{31}$$

$$\text{subject to } (z, \mu) \in cl(B_r(y, w)) \cap \mathcal{F}. \tag{32}$$

Denote by \mathcal{F}'_r the set of optimal solutions associated to (P'_r) . Since,

$$(cl(B_r(y, w)) \cap \mathcal{F}) \subset \overline{\mathcal{X}} \times \overline{\mathcal{U}} \tag{33}$$

is a compact subset of $\tilde{\mathcal{X}} \times \tilde{\mathcal{U}}$, it follows that $\mathcal{F}'_r \neq \emptyset$. Now, let us consider the following two cases.

(i) There exists $r^* \in (0, r_2)$ such that $(z^*, \mu^*) \in int(cl(B_{r^*}(y, w)) \cap \mathcal{F})$ and $(z^*, \mu^*) \in \mathcal{F}'_{r^*}$. This case involves (z^*, μ^*) is a local optimal solution of (P) and, by hypothesis, it is a global optimal solution for (P) . But, in accordance with Equation (30) and using the inequality (29), we get:

$$F(x, u) < F(y, w) - \epsilon = F(y, w) - F(z^*, \mu^*) + F(z^*, \mu^*) - \epsilon \tag{34}$$

$$< \epsilon + F(z^*, \mu^*) - \epsilon = F(z^*, \mu^*),$$

which means that (z^*, μ^*) is not a global optimal solution of (P) and, therefore, we obtain a contradiction. In consequence, we shall consider the next case.

(ii) For any $r^* \in (0, r_2)$, if $(z^*, \mu^*) \in \mathcal{F}'_{r^*}$, then,

$$(z^*, \mu^*) \in (cl(B_{r^*}(y, w)) \cap \mathcal{F}) \setminus int(cl(B_{r^*}(y, w)) \cap \mathcal{F}), \tag{35}$$

involving,

$$0 < \| (y, w) - (z^*, \mu^*) \| \leq r^*. \tag{36}$$

Furthermore, we shall construct the function h . Consider the function $a : [0, 1] \rightarrow \mathbb{R}, a(\tau) = \tau \cdot r^*$. Also, for each $\tau \in (0, 1)$, introduce the following optimization problem $(P'_{a(\tau)})$:

$$\min_{(z, \mu)} F(z, \mu) \tag{37}$$

$$\text{subject to } (z, \mu) \in cl(B_{a(\tau)}(y, w)) \cap \mathcal{F}. \tag{38}$$

As in the previous case (see (i)), we get:

$$\emptyset \neq \mathcal{F}'_{a(\tau)} \subset (cl(B_{a(\tau)}(y, w)) \cap \mathcal{F}) \setminus int(cl(B_{a(\tau)}(y, w)) \cap \mathcal{F}), \tag{39}$$

where $\mathcal{F}'_{a(\tau)}$ denotes the set of optimal solutions for $(P'_{a(\tau)})$. Define:

$$h((x, u), (y, w)) : [0, 1] \rightarrow \overline{\mathcal{X}} \times \overline{\mathcal{U}}, \quad h((x, u), (y, w))(\tau) = (z, \mu)_\tau - (y, w), \tag{40}$$

where $(z, \mu)_\tau$ is an optimal solution in $(P'_{a(\tau)})$. We get:

$$\| h((x, u), (y, w))(\tau) \| = \| (z, \mu)_\tau - (y, w) \| \leq a(\tau) = \tau \cdot r^*, \tag{41}$$

which implies:

$$\lim_{\tau \rightarrow 0^+} \| h((x, u), (y, w))(\tau) \| = \| h((x, u), (y, w))(0) \| = 0. \tag{42}$$

As well, for any $\tau \in (0, 1)$, the relation (39) is fulfilled and, moreover, $(y, w) \notin \mathcal{F}'_{a(\tau)}$ (see Equation (36)). Thus,

$$\begin{aligned} F((y, w) + h((x, u), (y, w))(\tau)) &= F((y, w) + (z, \mu)_\tau - (y, w)) \\ &= F((z, \mu)_\tau) < F(y, w). \end{aligned} \tag{43}$$

By Equation (40), it follows $(y, w) + h((x, u), (y, w))(\tau) \in \mathcal{F}$ and, in consequence, the condition (20) is satisfied for all $\tau \in (0, 1)$. Also, Equation (40) involves:

$$h((x, u), (y, w))(\tau)|_{\theta=\theta_1, \theta_2} = (0, \underline{u}), \quad \underline{u} \in \overline{U}, \tag{44}$$

for all $\tau \in (0, 1)$. \square

4. Illustrative Application

In order to illustrate the effectiveness of the aforementioned result, for $m = 2, n = k = 1, q = 3$ (see Section 2) and Θ a square fixed by the diagonally opposite points $\theta_1 = (0, 0)$ and $\theta_2 = (2, 2)$ in \mathbb{R}^2 , consider the following two-dimensional variational control problem (BP):

$$\min_{(x, u)} \left\{ F(x, u) = \int_Y [u(\theta) - 5]^2 d\theta^1 + [u(\theta) - 5]^2 d\theta^2 \right\} \tag{45}$$

$$\text{subject to } \frac{\partial x}{\partial \theta^1}(\theta) = \frac{\partial x}{\partial \theta^2}(\theta) = 2 - u(\theta), \tag{46}$$

$$16 - x^2(\theta) < 0, \tag{47}$$

$$x(0, 0) = 4, \quad x(2, 2) = 10, \tag{48}$$

where $\theta = (\theta^1, \theta^2) \in \Theta$.

Furthermore, we assume that we have interest only for affine state functions. In the previous application, we have:

$$f_\nu : \Theta \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad g_\beta : J^1(\Theta, \mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R}, \quad \nu = 1, 2, \beta = \overline{1, 3}, \tag{49}$$

with:

$$f_1(\theta, x(\theta), u(\theta)) = f_2(\theta, x(\theta), u(\theta)) = [u(\theta) - 5]^2, \tag{50}$$

$$g_1(\theta, x(\theta), x_\nu(\theta), u(\theta)) = 2 - u(\theta) - \frac{\partial x}{\partial \theta^1}(\theta), \tag{51}$$

$$g_2(\theta, x(\theta), x_\nu(\theta), u(\theta)) = 2 - u(\theta) - \frac{\partial x}{\partial \theta^2}(\theta), \tag{52}$$

$$g_3(\theta, x(\theta), x_\nu(\theta), u(\theta)) = 16 - x^2(\theta), \tag{53}$$

accompanied by the boundary conditions $x(0, 0) = 4, x(2, 2) = 10$.

With the above mathematical objects and $\bar{\mathcal{X}} \times \bar{\mathcal{U}}$, \mathcal{F} having the same meaning as in Section 2, for $(x, u), (y, w) \in \mathcal{F}$ with $F(x, u) - F(y, w) < 0$, define:

$$h((x, u), (y, w)) : [0, 1] \rightarrow \bar{\mathcal{X}} \times \bar{\mathcal{U}}, \quad h((x, u), (y, w))(\tau) = (z, \mu)_\tau - (y, w), \tag{54}$$

where $(z, \mu)_\tau = \left(\frac{3}{2}(\theta^1 + \theta^2) + 4, \frac{1}{2}\right)$ is an optimal solution in (BP_τ) :

$$\min_{(z, \mu)} F(z, \mu) \tag{55}$$

$$\text{subject to } (z, \mu) \in \text{cl}(B_\tau(y, w)) \cap \mathcal{F}, \tag{56}$$

with $\tau \in (0, 1)$. We get:

$$0 < \|h((x, u), (y, w))(\tau)\| = \|(z, \mu)_\tau - (y, w)\| \leq \tau, \tag{57}$$

which implies:

$$\lim_{\tau \rightarrow 0^+} \|h((x, u), (y, w))(\tau)\| = \|h((x, u), (y, w))(0)\| = 0. \tag{58}$$

Also, for any $\tau \in (0, 1)$, it results (see Theorem 1) that the following relation:

$$\emptyset \neq \mathcal{F}'_\tau \subset (\text{cl}(B_\tau(y, w)) \cap \mathcal{F}) \setminus \text{int}(\text{cl}(B_\tau(y, w)) \cap \mathcal{F}) \tag{59}$$

is fulfilled, where \mathcal{F}'_τ denotes the set of optimal solutions for (BP_τ) . Moreover, by using Equation (57), we have $(y, w) \notin \mathcal{F}'_\tau$. In consequence,

$$\begin{aligned} F((y, w) + h((x, u), (y, w))(\tau)) &= F((y, w) + (z, \mu)_\tau - (y, w)) \\ &= F((z, \mu)_\tau) < F(y, w). \end{aligned} \tag{60}$$

By direct computation, it follows $(y, w) + h((x, u), (y, w))(\tau) \in \mathcal{F}$ and, in consequence, the condition:

$$G_\beta((y, w) + h((x, u), (y, w))(\tau)) \leq 0, \quad \beta \in Q = \{1, 2, 3\}, \tag{61}$$

is satisfied for all $\tau \in (0, 1)$. Also, Equation (54) involves:

$$h((x, u), (y, w))(\tau)|_{\theta=\theta_1, \theta_2} = (0, \underline{u}), \quad \tau \in (0, 1), \tag{62}$$

where $\underline{u} = \frac{1}{2} - w|_{\theta=\theta_1, \theta_2}$.

In the case when $(x, u), (y, w) \in \mathcal{F}$ and $F(x, u) - F(y, w) < 0$ is not verified, then we set $h((x, u), (y, w)) : [0, 1] \rightarrow \bar{\mathcal{X}} \times \bar{\mathcal{U}}, h((x, u), (y, w))(\tau) = 0$.

5. Conclusions

In this paper, we have investigated the optimality conditions associated with PDE and PDI-constrained control problems involving path-independent curvilinear integral functionals. More concretely, we have established a minimal criterion such that any local optimal solution of the considered PDE and PDI-constrained scalar variational control problem is also its global optimal solution. The effectiveness of the main result is validated by a two-dimensional nonconvex scalar variational control problem. As future directions and open problems, the main result presented in this paper can be extended for new classes of optimization problems governed by vector cost functionals.

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