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# Valuing Guaranteed Minimum Death Benefits by Cosine Series Expansion

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**Abstract:** Recently, the valuation of variable annuity products has become a hot topic in actuarial science. In this paper, we use the Fourier cosine series expansion (COS) method to value the guaranteed minimum death benefit (GMDB) products. We first express the value of GMDB by the discounted density function approach, then we use the COS method to approximate the valuation Equations. When the distribution of the time-until-death random variable is approximated by a combination of exponential distributions and the price of the fund is modeled by an exponential Lévy process, explicit equations for the cosine coefficients are given. Some numerical experiments are also made to illustrate the efficiency of our method.

**Keywords:** equity-linked death benefits; Fourier cosine series expansion; guaranteed minimum death benefit; option; valuation; Lévy process

**MSC:** 91B30; 60G99; 62P05; 62M05

## 1. Introduction

The guaranteed minimum death benefit (GMDB) is a common rider embedded in variable annuity products that promises a minimum payout upon the death of the insured. In this product, policyholders first pay premiums to the insurance company, and then, investment accounts are established for capital investment. When the insured dies, a payment shall be given to the designated beneficiary, and the payout amount depends on the performance of the policyholder's account. This mechanism not only provides insurance guarantee for policyholders, but also has the opportunity to benefit from the financial market, which appeals to customers.

For  $t \geq 0$ , let  $S(t) = S(0)e^{X(t)}$  denote the price of a stock fund or mutual fund at time  $t$ . For a person currently aged  $x$ , let  $T_x$  denote the remaining lifetime, called the time-until-death random variable hereafter. Moreover, we assume  $T_x$  is independent with the asset price process  $S(t)$  throughout this paper. Consider a GMDB rider that guarantees a payment of  $b(S(T_x))$  to the beneficiary when the insured dies, where  $b(\cdot)$  is an equity-linked death benefit function. For a constant force of interest  $\delta \geq 0$ , we are interested in valuing the following expectation:

$$V_x := E[e^{-\delta T_x} b(S(T_x))], \quad (1)$$

which represents a fair price for an equity-linked life-contingent payment at  $T_x$ . Since most contracts have a finite expiry date, we can modify (1) by introducing an expiry date  $T$  and consider:

$$V_{x,T} := E[e^{-\delta T_x} b(S(T_x)) \mathbf{I}(T_x \leq T)], \tag{2}$$

where  $\mathbf{I}(A)$  denotes the indicator function of event  $A$ .

We are also interested in the case when the death benefit amount depends on two stocks (or stock funds). Let  $\{S_1(t), S_2(t)\}_{t \geq 0}$  denote the price process of two stocks, and let:

$$X_i(t) = \ln(S_i(t)/S_i(0)), \quad i = 1, 2, \quad t \geq 0.$$

In the sequel, it is also assumed that  $T_x$  is independent with both  $S_1(t)$  and  $S_2(t)$ . Accordingly, we are interested in evaluating the following expectations:

$$\bar{V}_x := E[e^{-\delta T_x} b(S_1(T_x), S_2(T_x))], \quad \bar{V}_{x,T} := E[e^{-\delta T_x} b(S_1(T_x), S_2(T_x)) \mathbf{I}(T_x \leq T)].$$

Recently, the valuation of the GMDB product has drawn many researchers' attention. Under an exponential mortality law, Milevsky and Posner [1] proposed a risk-neutral framework to derive valuation equations for GMDB contracts. Besides, they conducted a numerical study under the Gompertz-Makeham law. Later, in Bauer et al. [2], they supposed that the death should only occur at the policy anniversary date, which facilitates a discrete numerical valuation approach for fairly valuing varieties of guaranteed riders, including GMDB. Gerber et al. [3] proposed a discounted density approach to value GMDB in a Brownian motion risk model, and their results were extended by Gerber et al. [4] and Siu et al. [5] to the jump diffusion model and the regime-switching jump diffusion model, respectively. Based on the fact that the combination of exponential distributions is weakly dense on the set of probability functions defined on  $[0, \infty)$  (see Dufresne [6] and Ko and Ng [7]), we notice that the density function of the time-until-death random variable was approximated by a linear combination of exponential distributions in Gerber et al. [3,4] and Siu et al. [5]. More recently, Zhang and Yong [8] and Zhang et al. [9] used two different methods to value GMDB products. Recent related literature can be found in Dai et al. [10], Bélanger et al. [11], Kang and Ziveyi [12], Asmussen et al. [13], and Zhou and Wu [14].

When the density function of the time-until-death random variable is approximated by a linear sum of exponential distributions, Gerber et al. [3,4,15] derived explicit valuation equations for GMDB contracts under various payoffs. The simplicity of using a combination of exponential distributions is excellent, but they are not representative of reality. A direct way to calibrate this is to use life table data. In Ulm [16,17], he emphasized the valuation of GMDB products under mortality laws, such as the De Moivre law of mortality and the Makeham law of mortality. A similar consideration could be found in Liang et al. [18]. They novelly introduced the piecewise constant forces of the mortality assumption to describe the time-until-death variable, then decomposed the valuation problem and presented explicit valuation equations for GMDB.

Except the aforementioned assumptions on the time-until-death random variable, the modeling of the asset process has attracted the attention of scholars. Brownian motion was widely used to model the log-asset price process, which was adopted in Milevsky and Posner [1], Bauer et al. [2] (Section 4), Gerber et al. [3], and Liang et al. [18]. In the field of financial markets, this case is basically a Black-Scholes framework. However, more processes could be also implemented. In Gerber et al. [4], Kou's jump model was used as the log-asset process. A counterpart study of Gerber et al. [3] was referred to by Gerber et al. [15], in which a random walk exponentially generates the price process. To study the valuation issue in a different perspective, scholars turned to the regime-switching model, which was used to investigate the performance of an object subject to the economic changes in financial markets. In the literature, interest readers can refer to Fan et al. [19], Siu et al. [5], Ignatieva et al. [20], and Hieber [21].

In Fang and Oosterlee [22], a highly-efficient option pricing method was proposed to price European options. The method was based on the Fourier cosine series expansions, and it is now called the cosine series expansion (COS) method in the literature. The COS method is quite easy to implement to approximate an integrable function as long as the objective function has a closed-form Fourier transform. The one-dimensional COS (1D COS) method in [22] was extended to the two-dimensional COS method (2D COS) by Ruijter and Oosterlee [23] to price financial options in two-dimensional asset price processes. Leitao et al. [24] proposed a data-driven COS method. Except for option pricing, this method has been adopted in insurance ruin theory. For example, Chau et al. [25,26] used the 1D COS method to compute the ruin probability and the expected discounted penalty function; Zhang [27] approximated the density function of the time to ruin by both 1D and 2D COS methods; Yang et al. [28] proposed a nonparametric estimator for the deficit at ruin by the 2D COS method; Wang et al. [29] and Huang et al. [30] used the 1D COS method to estimate the expected discounted penalty function under some risk models with stochastic premium income. The COS method has also been used by some authors to value variable annuities. For example, Deng et al. [31] used the 1D COS method for equity-indexed annuity products under general exponential Lévy models; Alonso-García et al. [32] extended the 1D COS method to the pricing and hedging of variable annuities embedded with guaranteed minimum withdraw benefit riders. The latest research on Fourier transform was given by Zhang et al. [33], Chan [34], Zhang and Liu [35], Have and Oosterlee [36], Shimizu and Zhang [37], Tour [38], Zhang [39], and Wang et al. [40].

The discounted density method proposed by Gerber et al. [3,4] can be successfully used to value GMDB products when the log-return process is the Brownian motion or Kou’s jump diffusion model. Under these two models, the density functions have some closed-form expressions. However, in practice, we cannot obtain the closed-form expression for the discounted density function. In this paper, we use the COS method to approximate the discounted density function, which is applicable since most of the widely-used Lévy processes have explicit characteristic functions. To the best of our knowledge, this is the first paper exploring the COS method on numerical valuation of the GMDB product under the general exponential Lévy models. In particular, there are few papers on GMDB valuation dependent on two stock prices in the literature, and this is the first paper dealing with this problem.

This paper aims to value the GMDB contracts in a risk-neutral framework, mainly by numerically solving Equations (1) and (2). In subsequent sections, we first briefly recall the COS method in Fang and Oosterlee [22], then adopt the method to approximate  $V_x$  and  $V_{x,T}$  in the one-dimensional framework. We define auxiliary functions to simplify deductions and display equations under different payoffs in Section 2. Motivated by Gerber et al. [3], we consider the situation where the density function of  $T_x$  is approximated by a linear combination of exponential distributions, and calculate cosine coefficients in Section 4.1. Under the multi-dimensional case, we shed light on a two-dimensional framework in Section 3. Finally, numerical examples are presented in Section 4, in which we display tables and figures to illustrate the performance of our proposed approach.

## 2. 1D COS Approximation

In the section, we shall use the 1D COS method to compute  $V_x$  and  $V_{x,T}$ . The idea of the 1D COS method is that every absolutely integrable function  $f$  can be approximated on a truncated domain  $[a_1, a_2]$  by a truncated Fourier cosine series with  $N$  terms,

$$f(y) \approx \sum_{k=0}^{N-1} ' \mathcal{A}_k(f, a_1, a_2) \cos \left( k\pi \frac{y - a_1}{a_2 - a_1} \right), \tag{3}$$

where  $\sum'$  means that the first term in the summation has half weight, and the cosine coefficients are given by:

$$\mathcal{A}_k(f, a_1, a_2) = \frac{2}{a_2 - a_1} \operatorname{Re} \left\{ \mathcal{F}f \left( \frac{k\pi}{a_2 - a_1} \right) \exp \left( -i \frac{ka_1\pi}{a_2 - a_1} \right) \right\}. \tag{4}$$

Here,  $\mathcal{F}f(s) = \int e^{isy} f(y) dy, s \in \mathbb{R}$ , is the Fourier transform of  $f$ , and  $\text{Re}(\cdot)$  means taking the real part.

Let  $f_{T_x}(\cdot)$  denote the probability density function of  $T_x$ , and for  $t > 0$ , let  $f_{X(t)}(\cdot)$  be the probability density function of  $X(t)$ . By changing the order of integrals, we can obtain:

$$\begin{aligned} V_x &= \int_0^\infty E[e^{-\delta t} b(S(t))] f_{T_x}(t) dt \\ &= \int_0^\infty e^{-\delta t} \int_{-\infty}^\infty b(S(0)e^y) f_{X(t)}(y) dy f_{T_x}(t) dt \\ &= \int_{-\infty}^\infty b(S(0)e^y) f_{X(T_x)}^\delta(y) dy, \end{aligned} \tag{5}$$

where:

$$f_{X(T_x)}^\delta(y) = \int_0^\infty e^{-\delta t} f_{X(t)}(y) f_{T_x}(t) dt \tag{6}$$

is the discounted density function of the random variable  $X(T_x)$ . Similarly, for the  $T$ -year life-contingent option, we have:

$$V_{x,T} = \int_{-\infty}^\infty b(S(0)e^y) f_{X(T_x),T}^\delta(y) dy, \tag{7}$$

where:

$$f_{X(T_x),T}^\delta(y) = \int_0^T e^{-\delta t} f_{X(t)}(y) f_{T_x}(t) dt. \tag{8}$$

We shall implement the 1D COS method to compute the integrals in (5) and (7). Instead of expanding the discounted densities  $f_{X(T_x)}^\delta$  and  $f_{X(T_x),T}^\delta$  via Fourier cosine series, we shall consider the following auxiliary functions:

$$g_n^\delta(y) = e^{ny} f_{X(T_x)}^\delta(y), \quad g_{n,T}^\delta(y) = e^{ny} f_{X(T_x),T}^\delta(y), \quad n \geq 0.$$

Suppose that both  $g_n^\delta$  and  $g_{n,T}^\delta$  belong to  $L^1(\mathbb{R})$ , then by Equations (3) and (4), we have:

$$g_n^\delta(y) \approx \sum_{k=0}^{N-1} ' \mathcal{A}_k(g_n^\delta, a_1, a_2) \cos\left(k\pi \frac{y - a_1}{a_2 - a_1}\right) \tag{9}$$

and:

$$g_{n,T}^\delta(y) \approx \sum_{k=0}^{N-1} ' \mathcal{A}_k(g_{n,T}^\delta, a_1, a_2) \cos\left(k\pi \frac{y - a_1}{a_2 - a_1}\right). \tag{10}$$

**Remark 1.** The cosine coefficients  $\mathcal{A}_k(g_n^\delta, a_1, a_2)$  and  $\mathcal{A}_k(g_{n,T}^\delta, a_1, a_2)$  can be explicitly computed when  $\{X(t)\}_{t \geq 0}$  is a Lévy process and  $f_{T_x}$  is a combination of exponential density function. To this end, it suffices to specify the Fourier transforms  $\mathcal{F}g_n^\delta$  and  $\mathcal{F}g_{n,T}^\delta$ . Suppose that  $\{X(t)\}_{t \geq 0}$  (with  $X(0) = 0$ ) is a Lévy process with characteristic function:

$$\phi_{X(t)}(s) = E[e^{isX(t)}] = e^{t\Psi_X(s)}, \quad s \in \mathbb{R}, \tag{11}$$

where  $\Psi_X(s) = \ln(E[e^{isX(1)}])$  is called the characteristic exponent. Furthermore, suppose that:

$$f_{T_x}(t) = \sum_{j=1}^m A_j \alpha_j e^{-\alpha_j t}, \quad t > 0, \tag{12}$$

where  $\alpha_j > 0$  and  $\sum_{j=1}^m A_j = 1$ . Under these assumptions, one easily obtains:

$$\mathcal{F}g_n^\delta(s) = \sum_{j=1}^m \frac{A_j \alpha_j}{\delta + \alpha_j - \Psi_X(s - in)} \tag{13}$$

and:

$$\mathcal{F}_{g_{n,T}^\delta}(s) = \sum_{j=1}^m A_j \alpha_j \frac{1 - e^{-(\delta + \alpha_j - \Psi_X(s-in))T}}{\delta + \alpha_j - \Psi_X(s-in)}. \tag{14}$$

In the sequel, we shall consider three payoff functions.

Case 1.  $b(s) = s$ .

In this case, Equations (5) and (7) become:

$$V_x = S(0) \int_{-\infty}^{\infty} g_1^\delta(y) dy, \quad V_{x,T} = S(0) \int_{-\infty}^{\infty} g_{1,T}^\delta(y) dy.$$

For small number  $c_1$  and large number  $c_2$ , using Equation (9), we can approximate  $V_x$  as follows,

$$\begin{aligned} V_x &\approx S(0) \int_{c_1}^{c_2} g_1^\delta(y) dy \\ &\approx S(0) \int_{c_1}^{c_2} \sum_{k=0}^{N-1} ' \mathcal{A}_k(g_1^\delta, a_1, a_2) \cos\left(k\pi \frac{y - a_1}{a_2 - a_1}\right) dy \\ &= S(0) \sum_{k=0}^{N-1} ' \mathcal{A}_k(g_1^\delta, a_1, a_2) \cdot \chi_k(a_1, a_2, c_1, c_2), \end{aligned} \tag{15}$$

where:

$$\chi_k(a_1, a_2, c_1, c_2) = \begin{cases} \frac{a_2 - a_1}{k\pi} \left[ \sin\left(k\pi \frac{c_2 - a_1}{a_2 - a_1}\right) - \sin\left(k\pi \frac{c_1 - a_1}{a_2 - a_1}\right) \right], & k \neq 0, \\ c_2 - c_1, & k = 0. \end{cases}$$

Similarly,  $V_{x,T}$  can be approximated as follows,

$$V_{x,T} \approx S(0) \sum_{k=0}^{N-1} ' \mathcal{A}_k(g_{1,T}^\delta, a_1, a_2) \cdot \chi_k(a_1, a_2, c_1, c_2). \tag{16}$$

Case 2.  $b(s) = s^n \mathbf{I}_{(s>K)}$  with  $n \geq 0$  and  $K > 0$ .

Here, the positive constant  $K$  denotes the strike price. Put  $\kappa = \ln(K/S(0))$ . It follows from Equation (5) that:

$$V_x = \int_{\kappa}^{\infty} [S(0)]^n e^{ny} f_{X(T_x)}^\delta(y) dy = \int_{\kappa}^{\infty} [S(0)]^n g_n^\delta(y) dy.$$

Then, using the 1D COS method, we obtain for large number  $c_2$ ,

$$V_x \approx [S(0)]^n \sum_{k=0}^{N-1} ' \mathcal{A}_k(g_n^\delta, a_1, a_2) \cdot \chi_k(a_1, a_2, \kappa, c_2). \tag{17}$$

Similarly, for  $V_{x,T}$ , we have:

$$V_{x,T} \approx [S(0)]^n \sum_{k=0}^{N-1} ' \mathcal{A}_k(g_{n,T}^\delta, a_1, a_2) \cdot \chi_k(a_1, a_2, \kappa, c_2). \tag{18}$$

**Remark 2.** For the call option, the payoff is given by:

$$b(s) = (s - K)_+ = s \mathbf{I}_{(s>K)} - K \mathbf{I}_{(s>K)}.$$

By applying Equations (5) and (17), we obtain:

$$\begin{aligned}
 V_x &= \int_{\kappa}^{\infty} S(0)e^y f_{X(T_x)}^{\delta}(y)dy - \int_{\kappa}^{\infty} K f_{X(T_x)}^{\delta}(y)dy \\
 &= S(0) \int_{\kappa}^{\infty} g_1^{\delta}(y)dy - K \int_{\kappa}^{\infty} g_0^{\delta}(y)dy \\
 &\approx S(0) \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_1^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, \kappa, c_2) - K \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_0^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, \kappa, c_2). \tag{19}
 \end{aligned}$$

For the T-year life-contingent option, we have:

$$V_{x,T} \approx S(0) \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_{1,T}^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, \kappa, c_2) - K \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_{0,T}^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, \kappa, c_2). \tag{20}$$

Case 3.  $b(s) = s^n \mathbf{I}_{(s < K)}$  with  $n \geq 0$  and  $K > 0$ .

By Equation (5) and the 1D COS method, we have for small number  $c_1$ ,

$$\begin{aligned}
 V_x &= \int_{\infty}^{\kappa} [S(0)]^n g_n^{\delta}(y)dy \approx [S(0)]^n \int_{c_1}^{\kappa} g_n^{\delta}(y)dy \\
 &\approx [S(0)]^n \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_n^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, c_1, \kappa). \tag{21}
 \end{aligned}$$

Similarly, for  $V_{x,T}$ , we have:

$$V_{x,T} \approx [S(0)]^n \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_{n,T}^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, c_1, \kappa). \tag{22}$$

**Remark 3.** For the put option, the payoff function is given by:

$$b(s) = (K - s)_+ = K \mathbf{I}_{(s < K)} - s \mathbf{I}_{(s < K)},$$

which together with Equations (21) and (22) gives:

$$V_x \approx K \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_0^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, c_1, \kappa) - S(0) \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_1^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, c_1, \kappa) \tag{23}$$

and:

$$V_{x,T} \approx K \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_{0,T}^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, c_1, \kappa) - S(0) \sum_{k=0}^{N-1} {}' \mathcal{A}_k(g_{1,T}^{\delta}, a_1, a_2) \cdot \chi_k(a_1, a_2, c_1, \kappa). \tag{24}$$

### 3. 2D COS Approximation

In this section, we use the 2D COS method to compute  $\bar{V}_x$  and  $\bar{V}_{x,T}$ . For a bivariate integrable function  $f$ , we denote its Fourier transform by:

$$\mathcal{F}f(s_1, s_2) = \int \int e^{is_1y + is_2z} f(y, z)dydz, \quad s_1, s_2 \in \mathbb{R}.$$

It follows from Ruijter and Oosterlee [23] that  $f$  can be approximated on a truncated domain  $[a_1, a_2] \times [b_1, b_2]$  by truncated Fourier cosine series expansions with  $N_1 \times N_2$  terms,

$$f(y, z) \approx \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \mathcal{B}_{k_1, k_2}(f) \cos\left(k_1 \pi \frac{y - a_1}{a_2 - a_1}\right) \cos\left(k_2 \pi \frac{z - b_1}{b_2 - b_1}\right), \tag{25}$$

where the cosine coefficients are given by:

$$\mathcal{B}_{k_1, k_2}(f) = \frac{1}{2} \left( \mathcal{B}_{k_1, k_2}^+(f) + \mathcal{B}_{k_1, k_2}^-(f) \right)$$

with:

$$\mathcal{B}_{k_1, k_2}^\pm(f) = \frac{2}{a_2 - a_1} \frac{2}{b_2 - b_1} \operatorname{Re} \left\{ \mathcal{F}f \left( \frac{k_1 \pi}{a_2 - a_1}, \pm \frac{k_2 \pi}{b_2 - b_1} \right) \exp \left( -ik_1 \pi \frac{a_1}{a_2 - a_1} \mp ik_2 \pi \frac{b_1}{b_2 - b_1} \right) \right\}. \tag{26}$$

For  $t > 0$ , we denote the joint probability density function of  $(X_1(t), X_2(t))$  by  $f_{X_1(t), X_2(t)}(y, z)$ , and for  $\delta \geq 0$ , define the following discounted density functions:

$$\begin{aligned} f_{X_1(T_x), X_2(T_x)}^\delta(y, z) &= \int_0^\infty e^{-\delta t} f_{X_1(t), X_2(t)}(y, z) f_{T_x}(t) dt, \\ f_{X_1(T_x), X_2(T_x), T}^\delta(y, z) &= \int_0^T e^{-\delta t} f_{X_1(t), X_2(t)}(y, z) f_{T_x}(t) dt. \end{aligned}$$

For  $\bar{V}_x$ , by changing the order of integrals, we have:

$$\begin{aligned} \bar{V}_x &= \int_0^\infty E[e^{-\delta t} b(S_1(t), S_2(t))] f_{T_x}(t) dt \\ &= \int_0^\infty e^{-\delta t} \int_{-\infty}^\infty \int_{-\infty}^\infty b(S_1(0)e^y, S_2(0)e^z) f_{X_1(t), X_2(t)}(y, z) dy dz f_{T_x}(t) dt \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty b(S_1(0)e^y, S_2(0)e^z) f_{X_1(T_x), X_2(T_x)}^\delta(y, z) dy dz. \end{aligned} \tag{27}$$

For  $\bar{V}_{x, T}$ , we have:

$$\bar{V}_{x, T} = \int_{-\infty}^\infty \int_{-\infty}^\infty b(S_1(0)e^y, S_2(0)e^z) f_{X_1(T_x), X_2(T_x), T}^\delta(y, z) dy dz. \tag{28}$$

As in Section 2, we shall pay attention to the following auxiliary functions,

$$g_{m, n}^\delta(y, z) = e^{my+nz} f_{X_1(T_x), X_2(T_x)}^\delta(y, z), \quad g_{m, n, T}^\delta(y, z) = e^{my+nz} f_{X_1(T_x), X_2(T_x), T}^\delta(y, z).$$

Suppose that both  $g_{m, n}^\delta$  and  $g_{m, n, T}^\delta$  are absolutely integrable. Then, by Equation (25), we obtain:

$$g_{m, n}^\delta(y, z) \approx \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \mathcal{B}_{k_1, k_2}(g_{m, n}^\delta) \cos\left(k_1 \pi \frac{y - a_1}{a_2 - a_1}\right) \cos\left(k_2 \pi \frac{z - b_1}{b_2 - b_1}\right) \tag{29}$$

and:

$$g_{m, n, T}^\delta(y, z) \approx \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \mathcal{B}_{k_1, k_2}(g_{m, n, T}^\delta) \cos\left(k_1 \pi \frac{y - a_1}{a_2 - a_1}\right) \cos\left(k_2 \pi \frac{z - b_1}{b_2 - b_1}\right). \tag{30}$$

To calculate cosine coefficients in Equations (29) and (30), we suppose  $(X_1(t), X_2(t))$  is a two-dimensional Lévy process with  $X_1(0) = X_2(0) = 0$ . The characteristic exponent is defined by:

$$\Psi_{X_1, X_2}(s_1, s_2) = \ln(E[e^{is_1 X_1(1) + is_2 X_2(1)}]).$$

Furthermore, suppose that  $f_{T_x}$  is a combination of the exponential density function given by Equation (12). By Fubini theorem, we have:

$$\begin{aligned}
 \mathcal{F}g_{m,n}^\delta(s_1, s_2) &= \int \int e^{is_1y+is_2z} e^{my+nz} f_{X_1(T_x), X_2(T_x)}^\delta(y, z) dydz \\
 &= \int \int e^{(is_1+m)y+(is_2+n)z} \int_0^\infty e^{-\delta t} f_{X_1(t), X_2(t)}(y, z) f_{T_x}(t) dt dydz \\
 &= \int_0^\infty e^{-(\delta - \Psi_{X_1, X_2}(s_1 - mi, s_2 - ni))t} f_{T_x}(t) dt \\
 &= \sum_{j=1}^m \frac{A_j \alpha_j}{\delta + \alpha_j - \Psi_{X_1, X_2}(s_1 - mi, s_2 - ni)}
 \end{aligned}
 \tag{31}$$

and:

$$\begin{aligned}
 \mathcal{F}g_{m,n,T}^\delta(s_1, s_2) &= \int_0^T e^{-(\delta - \Psi_{X_1, X_2}(s_1 - mi, s_2 - ni))t} f_{T_x}(t) dt \\
 &= \sum_{j=1}^m A_j \alpha_j \frac{1 - e^{-(\delta + \alpha_j - \Psi_{X_1, X_2}(s_1 - mi, s_2 - ni))T}}{\delta + \alpha_j - \Psi_{X_1, X_2}(s_1 - mi, s_2 - ni)}.
 \end{aligned}
 \tag{32}$$

In the sequel, we study the numerical approximation of the value of life-contingent two-asset options. We shall consider the Margrabe option, maximum/minimum option, and geometric option. First, the contingent Margrabe option, also called the exchange option, is considered. Note that its payoff function is:

$$[S_1(T_x) - S_2(T_x)]_+.$$

Then, we have:

$$\begin{aligned}
 &E[e^{-\delta T_x} [S_1(T_x) - S_2(T_x)]_+ | S_1(0) < S_2(0)] \\
 &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-\delta t} [S_1(0)e^y - S_2(0)e^z]_+ \cdot f_{X_1(t), X_2(t)}(y, z) f_{T_x}(t) dydzdt \\
 &= \int_{-\infty}^\infty \int_{-\infty}^\infty [S_1(0)e^y - S_2(0)e^z]_+ \cdot f_{X_1(T_x), X_2(T_x)}^\delta(y, z) dydz \\
 &= \int \int_{(y,z) \in \mathcal{D}_1} (S_1(0)e^y - S_2(0)e^z) f_{X_1(T_x), X_2(T_x)}^\delta(y, z) dydz \\
 &= S_1(0) \int \int_{(y,z) \in \mathcal{D}_1} e^y f_{X_1(T_x), X_2(T_x)}^\delta(y, z) dydz - S_2(0) \int \int_{(y,z) \in \mathcal{D}_1} e^z f_{X_1(T_x), X_2(T_x)}^\delta(y, z) dydz \\
 &= S_1(0) \int \int_{(y,z) \in \mathcal{D}_1} g_{1,0}^\delta(y, z) dydz - S_2(0) \int \int_{(y,z) \in \mathcal{D}_1} g_{0,1}^\delta(y, z) dydz,
 \end{aligned}
 \tag{33}$$

where  $\mathcal{D}_1$  denotes the region  $\{(y, z) : y - z > \ln \frac{S_2(0)}{S_1(0)}\}$ . Set  $\mathcal{D}'_1 = \mathcal{D}_1 \cap ([a_1, a_2] \times [b_1, b_2])$ . Utilizing Equation (29), we obtain the following approximation formula:

$$\begin{aligned}
 &E[e^{-\delta T_x} [S_1(T_x) - S_2(T_x)]_+ | S_1(0) < S_2(0)] \\
 &\approx S_1(0) \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \mathcal{B}_{k_1, k_2}(g_{1,0}^\delta) \int \int_{(y,z) \in \mathcal{D}'_1} \cos\left(k_1 \pi \frac{y - a_1}{a_2 - a_1}\right) \cos\left(k_2 \pi \frac{z - b_1}{b_2 - b_1}\right) dydz \\
 &\quad - S_2(0) \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \mathcal{B}_{k_1, k_2}(g_{0,1}^\delta) \int \int_{(y,z) \in \mathcal{D}'_1} \cos\left(k_1 \pi \frac{y - a_1}{a_2 - a_1}\right) \cos\left(k_2 \pi \frac{z - b_1}{b_2 - b_1}\right) dydz,
 \end{aligned}$$

where the double integrals in both summations can be analytically computed.

What follows next are the approximation procedures for maximum/minimum options. The payoff functions for maximum and minimum options are given by:

$$\max\{S_1(T_x), S_2(T_x)\}, \quad \min\{S_1(T_x), S_2(T_x)\}.$$

With a trivial mathematical change, we can turn them into the following forms:

$$\begin{aligned}
 \max\{S_1(T_x), S_2(T_x)\} &= S_2(T_x) + [S_1(T_x) - S_2(T_x)]_+, \\
 \min\{S_1(T_x), S_2(T_x)\} &= S_1(T_x) - [S_1(T_x) - S_2(T_x)]_+.
 \end{aligned}$$



Hence, the valuation equations can be obtained by taking discounted expectations on both sides of the above equations,

$$E[e^{-\delta T_x} \max\{S_1(T_x), S_2(T_x)\}] = E[e^{-\delta T_x} S_2(T_x)] + E[e^{-\delta T_x} [S_1(T_x) - S_2(T_x)]_+],$$

$$E[e^{-\delta T_x} \min\{S_1(T_x), S_2(T_x)\}] = E[e^{-\delta T_x} S_1(T_x)] - E[e^{-\delta T_x} [S_1(T_x) - S_2(T_x)]_+],$$

which imply that we can take advantage of the deductions of exchange option and basic option taking the form  $b(s) = s$  to compute the above expectations.

Finally, we pay attention to the geometric option. The payoff function with strike price  $K$  is given by:

$$\left[ \sqrt{S_1(T_x)S_2(T_x)} - K \right]_+$$

When condition  $\sqrt{S_1(0)S_2(0)} < K$  holds, we can develop the following approximation.

$$E[e^{-\delta T_x} [\sqrt{S_1(T_x)S_2(T_x)} - K]_+ \mid \sqrt{S_1(0)S_2(0)} < K]$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\sqrt{S_1(T_x)S_2(T_x)} - K]_+ \cdot f_{X_1(T_x), X_2(T_x)}^\delta(y, z) dydz$$

$$= \sqrt{S_1(0)S_2(0)} \int \int_{(y,z) \in \mathcal{D}_2} e^{\frac{1}{2}(y+z)} f_{X_1(T_x), X_2(T_x)}^\delta(y, z) dydz - K \int \int_{(y,z) \in \mathcal{D}_2} f_{X_1(T_x), X_2(T_x)}^\delta(y, z) dydz$$

$$= \sqrt{S_1(0)S_2(0)} \int \int_{(y,z) \in \mathcal{D}_2} g_{\frac{1}{2}, \frac{1}{2}}^\delta(y, z) dydz - K \int \int_{(y,z) \in \mathcal{D}_2} g_{0,0}^\delta(y, z) dydz, \tag{34}$$

where  $\mathcal{D}_2$  denotes the region  $\{(y, z) : y + z > \ln \frac{K^2}{S_1(0)S_2(0)}\}$ . Set  $\mathcal{D}'_2 = \mathcal{D}_2 \cap ([a_1, a_2] \times [b_1, b_2])$ . Then, Equation (34) can be approximated by:

$$E[e^{-\delta T_x} [\sqrt{S_1(T_x)S_2(T_x)} - K]_+ \mid \sqrt{S_1(0)S_2(0)} < K]$$

$$\approx \sqrt{S_1(0)S_2(0)} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \mathcal{B}_{k_1, k_2}(g_{\frac{1}{2}, \frac{1}{2}}^\delta) \int \int_{(y,z) \in \mathcal{D}'_2} \cos\left(k_1 \pi \frac{y - a_1}{a_2 - a_1}\right) \cos\left(k_2 \pi \frac{z - b_1}{b_2 - b_1}\right) dydz$$

$$- K \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \mathcal{B}_{k_1, k_2}(g_{0,0}^\delta) \int \int_{(y,z) \in \mathcal{D}'_2} \cos\left(k_1 \pi \frac{y - a_1}{a_2 - a_1}\right) \cos\left(k_2 \pi \frac{z - b_1}{b_2 - b_1}\right) dydz,$$

where analytical expressions of the double integrals in both summations exist.

**Remark 4.** When we study  $\bar{V}_{x,T}$ , the approximation Equations are similar to those of  $\bar{V}_x$ , and the only modification is replacing  $\mathcal{F}g_{m,n}^\delta$  by  $\mathcal{F}g_{m,n,T}^\delta$ .

### 4. Numerical Illustrations

In this section, we present some numerical examples to show the performance of the proposed approach. For the linear combination of exponential distributions, which is used to approximate the density function of  $T_x$ , we use the case considered in Gerber et al. [3], which is given by:

$$f_{T_x}(t) = 3 \times 0.08e^{-0.08t} - 2 \times 0.12e^{-0.12t}, \quad t > 0. \tag{35}$$

In the following subsections, we shall show that the COS method is very efficient for valuing GMDB.

#### 4.1. The 1D COS Results

In this subsection, we use some Lévy processes to model the log asset process  $X(t)$  (see Remark 1). Note that the probability law of the Lévy process  $X(t)$  is determined by its characteristic exponent  $\Psi_X(s)$ . In this section, all computations were performed in MATLAB 2016b with Intel Core i7 at 3.4 GHz and RAM of 8 GB. We shall consider the following five models.

- Black-Scholes model (BS):  $\Psi_X(s) = i\mu s - \frac{\sigma^2}{2}s^2$ ;
- Kou’s jump diffusion model (Kou):  $\Psi_X(s) = i\mu s - \frac{\sigma^2}{2}s^2 + \lambda_K \left( \frac{(1-p)\eta_2}{\eta_2 + is} + \frac{p\eta_1}{\eta_1 - is} - 1 \right)$ ;
- Merton’s jump diffusion model (MJD):  $\Psi_X(s) = i\mu s - \frac{\sigma^2}{2}s^2 + \lambda_J \left( e^{is\mu_J - \frac{\sigma_J^2}{2}s^2} - 1 \right)$ ;
- Variance Gammamodel (VG):  $\Psi_X(s) = i\mu s - \frac{\sigma^2}{2}s^2 - \frac{1}{\nu} \ln \left( 1 - i\nu\mu_\nu s + \nu \frac{\sigma_\nu^2}{2}s^2 \right)$ ;
- Normal inverse Gaussian model (NIG):  $\Psi_X(s) = i\mu s - \delta_{NIG} \left( \sqrt{\alpha^2 - (\beta + is)^2} - \sqrt{\alpha^2 - \beta^2} \right)$ .

In the sequel, let  $\delta = 0.05$  and  $\mu$  be chosen to satisfy the risk-neutral requirement. Here, we need to assume that  $T_x$  is still independent with the stock price process even under the risk-neutral measure. Thus,  $\mu$  is set such that  $\Psi_X(-i) = \delta$ . The values of other parameters are listed in Table 1. Furthermore, set  $a_1 = -100, a_2 = 100$ . For the death benefit function  $b$ , we consider the following two cases:

$$\text{call option : } b(s) = (s - K)_+; \quad \text{put option : } b(s) = (K - s)_+.$$

**Table 1.** Parameter setting for the log-asset process X.

Model	Abbreviation	Parameter Sets
Black-Scholes model	BS	$\sigma = 0.25, S(0) = 100$ ;
Kou’s jump diffusion model	Kou	$\sigma = 0.25, S(0) = 100, \lambda_K = 0.6, \eta_1 = 4, \eta_2 = 1, p = 0.5$ ;
Merton’s jump diffusion model	Merton	$\sigma = 0.25, S(0) = 100, \lambda_J = 0.6, \mu_J = 0.01, \sigma_J = 0.13$ ;
Variance Gamma model	VG	$\sigma = 0.25, S(0) = 100, \nu = 2, \mu_\nu = 0.01, \sigma_\nu = 0.05$ ;
Normal inverse Gaussian model	NIG	$S(0) = 100, \alpha = 2, \beta = 0.5, \delta_{NIG} = 0.05$ .

First, in Tables 2 and 3, we display some results when X is the Black-Scholes model and Kou’s jump diffusion model where the strike price  $K = 80, 90, 110, 120$ , respectively. Note that if X is Black-Scholes or Kou’s jump diffusion, reference values can be obtained from Gerber et al. [3,4]. In Tables 2 and 3, we calculate the relative errors and average running time to show the performance of our method, where the average running time (in seconds) is reported based on 1000 operations. From a horizontal perspective of Tables 2 and 3, our approximation results performed better as the expansion term  $N$  increased. When  $N = 2^{12}$ , relative errors were around  $10^{-8}$ . Furthermore, we present Monte Carlo simulation results with sample size  $10^7$ . We found that the COS method can result in smaller relative errors for each case, and this method requires less time than MC. When X is Merton’s jump diffusion, VG, and NIG, true reference values are not available. In these cases, we present MC simulation results with sample size  $10^7$ . From the simulation results given in Table 4, we see that the differences between approximation results and simulation results were small and our method required less time than MC.

Next, we consider valuation with a finite expiry date. In Tables 5 and 6, we assume  $T = 20$  and display some GMDB valuation results. In finite-time cases, reference values are available only when X is BS. For Kou’s jump diffusion, we display MC results with sample size  $10^7$ . With the call payoff function, it is observed that as  $K$  increases, the results decrease, which is opposite those with put payoff. Compared with MC, again, we found that the COS method required less time. Now, we further display the valuation results in Table 7 under different expiry dates in the BS model. As  $T$  grew from five to infinity, which denotes a whole life insurance type, valuation results increased as expected. If the expiry date is too far from the present, the probability that the insured dies becomes larger; hence, the insurance company is likely to make a payment. Besides, the dynamics of the asset is driven by a Brownian motion, with a positive drift; hence, the account accumulates from present to future. From a vertical perspective, it is also observed that relative errors decrease obviously as  $N$  increases.

Finally, to illustrate the accuracy of the proposed method, denary logarithms of relative errors are displayed in Figures 1 and 2 with different expansion terms and strike prices. For a fixed  $K$ , the curve tended to descend as expansion term  $N$  increased.

**Table 2.** Approximation results for guaranteed minimum death benefit (GMDB), where  $X$  is BS,  $b(s) = (K - s)_+$ .

K	Reference Value	BS						MC		
		$N = 2^4$		$N = 2^8$		$N = 2^{12}$		Relative Errors	Time	Std
		Relative Errors	Time	Relative Errors	Time	Relative Errors	Time			
80	3.6161	6.37	0.1009	$3.10 \times 10^{-3}$	0.1010	$1.41 \times 10^{-8}$	0.0977	$1.02 \times 10^{-3}$	1215.7831	7.5637
90	4.9871	4.59	0.0986	$1.24 \times 10^{-2}$	0.1002	$4.54 \times 10^{-8}$	0.1021	$8.96 \times 10^{-4}$	1215.2156	9.4440
110	8.4402	2.72	0.0970	$1.54 \times 10^{-2}$	0.1041	$3.13 \times 10^{-8}$	0.0987	$7.48 \times 10^{-4}$	1215.4682	13.5673
120	10.4920	2.21	0.0973	$1.37 \times 10^{-2}$	0.0999	$1.66 \times 10^{-8}$	0.1018	$7.31 \times 10^{-4}$	1215.6826	15.7678

**Table 3.** Approximation results for GMDB, where  $X$  is Kou,  $b(s) = (K - s)_+$ .

K	Reference Value	Kou						MC		
		$N = 2^4$		$N = 2^8$		$N = 2^{12}$		Relative Errors	Time	Std
		Relative Errors	Time	Relative Errors	Time	Relative Errors	Time			
80	18.0238	1.06	0.1106	$3.34 \times 10^{-4}$	0.1217	$3.45 \times 10^{-9}$	0.1184	$4.81 \times 10^{-4}$	1043.5026	16.7433
90	20.9370	$9.43 \times 10^{-1}$	0.1208	$4.60 \times 10^{-4}$	0.1173	$1.04 \times 10^{-8}$	0.1166	$5.18 \times 10^{-4}$	1043.4856	18.9326
110	27.0526	$7.74 \times 10^{-1}$	0.1157	$1.31 \times 10^{-3}$	0.1208	$9.50 \times 10^{-9}$	0.1139	$5.89 \times 10^{-4}$	1043.8940	23.2865
120	30.2424	$7.13 \times 10^{-1}$	0.1222	$1.48 \times 10^{-3}$	0.1151	$5.28 \times 10^{-9}$	0.1205	$6.17 \times 10^{-4}$	1043.7561	25.4520

**Table 4.** Approximation results for GMDB, where  $X$  is Merton’s jump diffusion model (MJD), the Variance Gamma model (VG), or the normal inverse Gaussian model (NIG),  $b(s) = (K - s)_+$ .

K	Merton (Time)		MC (Time)	VG (Time)		MC (Time)	NIG (Time)		MC (Time)
	$N = 2^8$	$N = 2^{12}$		$N = 2^8$	$N = 2^{12}$		$N = 2^8$	$N = 2^{12}$	
	80	4.4349 (0.0877)	4.4514 (0.0863)	4.4508 (1027.4452)	3.8263 (0.0870)	3.8395 (0.0886)	3.8412 (1237.0955)	6.1072 (0.0891)	6.1399 (0.0879)
90	5.9255 (0.0866)	5.9823 (0.0884)	5.9822 (1027.4338)	5.1947 (0.0853)	5.2556 (0.0890)	5.2574 (1237.0935)	7.9234 (0.0870)	7.9881 (0.0901)	7.9932 (1695.8093)
110	9.6156 (0.0854)	9.7228 (0.0861)	9.7250 (1027.4386)	8.6668 (0.0852)	8.7901 (0.0856)	8.7909 (1237.0956)	12.2390 (0.0876)	12.3349 (0.0879)	12.3326 (1695.8200)
120	11.7834 (0.0879)	11.8986 (0.0872)	11.9015 (1027.4401)	10.7420 (0.0863)	10.8770 (0.0885)	10.8771 (1237.1015)	14.6968 (0.0881)	14.7924 (0.0894)	14.8006 (1695.8699)

**Table 5.** Approximation results for finite GMDB, where  $X$  is BS or Kou,  $b(s) = (s - K)_+$ ,  $T = 20$ .

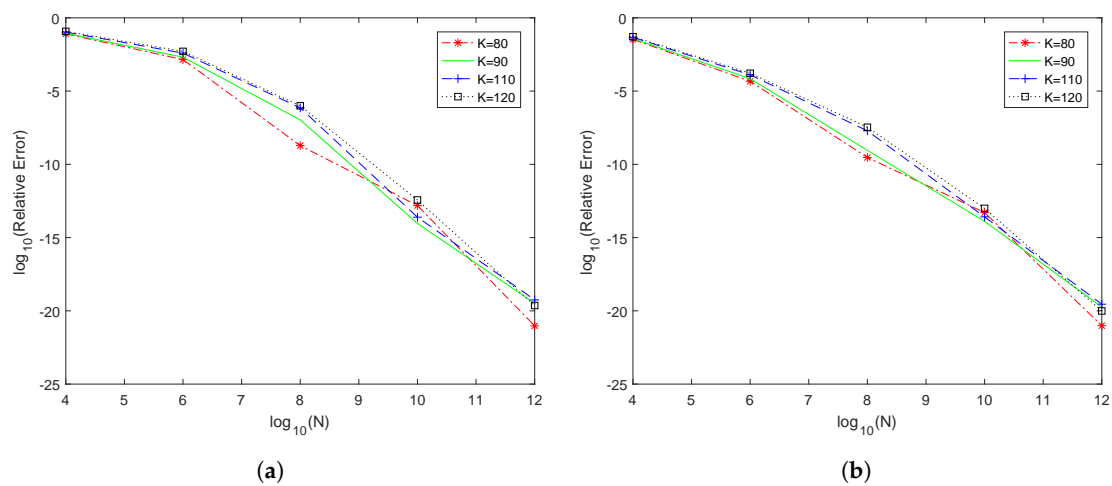
K	BS (Relative Errors)			Kou (Time)			MC (Time)
	$N = 2^4$	$N = 2^8$	$N = 2^{12}$	$N = 2^4$	$N = 2^8$	$N = 2^{12}$	
	80	19.1403 (0.414)	32.6564 ( $3.43 \times 10^{-4}$ )	32.6676 ( $1.56 \times 10^{-9}$ )	27.4259 (0.1194)	42.7130 (0.1173)	42.7070 (0.1173)
90	17.0844 (0.436)	30.2620 ( $2.04 \times 10^{-3}$ )	30.3241 ( $7.46 \times 10^{-9}$ )	25.7974 (0.1194)	41.4205 (0.1175)	41.4301 (0.1183)	41.4293 (1091.1883)
110	13.2010 (0.497)	26.1378 ( $4.95 \times 10^{-3}$ )	26.2680 ( $1.00 \times 10^{-8}$ )	22.7491 (0.1183)	39.1093 (0.1174)	39.1448 (0.1197)	39.1418 (1091.1751)
120	11.3513 (0.537)	24.3848 ( $5.86 \times 10^{-3}$ )	24.5286 ( $7.08 \times 10^{-9}$ )	21.3090 (0.1176)	38.0807 (0.1172)	38.1253 (0.1184)	38.1210 (1091.1742)

**Table 6.** Approximation results for finite GMDB, where  $X$  is Merton, VG, or NIG,  $b(s) = (s - K)_+$  and  $T = 20$ .

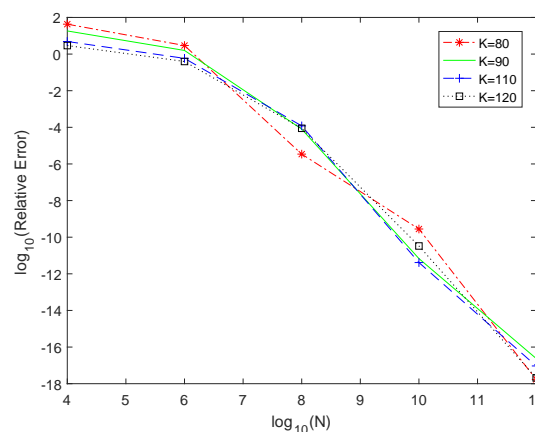
K	Merton (Time)		MC (Time)	VG (Time)		MC (Time)	NIG (Time)		MC (Time)
	$N = 2^8$	$N = 2^{12}$		$N = 2^8$	$N = 2^{12}$		$N = 2^8$	$N = 2^{12}$	
80	33.2207 (0.1128)	33.2371 (0.1112)	33.2582 (1033.1533)	32.8072 (0.1109)	32.8204 (0.1105)	32.8216 (1242.9794)	34.3088 (0.1145)	34.3415 (0.1140)	34.3638 (1701.5796)
90	30.9514 (0.1121)	31.0082 (0.1111)	31.0310 (1033.2315)	30.4485 (0.1126)	30.5094 (0.1104)	30.5123 (1242.8595)	32.2714 (0.1167)	32.3360 (0.1140)	32.3611 (1701.3483)
110	27.0436 (0.1103)	27.1508 (0.1133)	27.1778 (1033.0862)	26.3865 (0.1090)	26.5099 (0.1139)	26.5150 (1242.7859)	28.8046 (0.1141)	28.9006 (0.1160)	28.9307 (1701.5626)
120	25.3774 (0.1104)	25.4925 (0.1122)	25.5214 (1033.0618)	24.6586 (0.1095)	24.7936 (0.1129)	24.7996 (1242.9656)	27.3386 (0.1139)	27.4342 (0.1182)	27.4663 (1701.6632)

**Table 7.** Approximation results for finite GMDB, where  $X$  is BS,  $b(s) = (s - K)_+$  and  $K = 120$ .

$T$	5	10	30	60	$\infty$
$N = 2^8$	1.2988 ( $8.60 \times 10^{-2}$ )	7.0082 ( $2.01 \times 10^{-2}$ )	39.2337 ( $3.65 \times 10^{-3}$ )	55.9713 ( $2.56 \times 10^{-3}$ )	58.2215 ( $2.46 \times 10^{-3}$ )
$N = 2^{12}$	1.4211 ( $1.22 \times 10^{-7}$ )	7.1521 ( $2.42 \times 10^{-8}$ )	39.3774 ( $4.41 \times 10^{-9}$ )	56.1150 ( $3.09 \times 10^{-9}$ )	58.3653 ( $2.97 \times 10^{-9}$ )



**Figure 1.** Relative errors with payoff  $b(s) = (s - K)_+$ : (a) BS model; (b) Kou model.



**Figure 2.** Relative errors for BS model with payoff  $b(s) = (K - s)_+$  and finite expiry time  $T = 20$ .

#### 4.2. The 2D COS Results

In this subsection, the valuation results involving two assets in a GMDB contract are presented. For illustrative purpose, we assumed that the density function of  $T_x$  is represented by the combination of exponential distribution Equation (35). Besides, we considered a bivariate normal distribution. In addition, suppose that  $(S_1(t), S_2(t))$  is a bivariate geometric Brownian motion, where the log-asset price at time  $t$  is bivariate normally distributed:

$$(X_1(t), X_2(t)) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}t),$$

where:

$$\boldsymbol{\mu} = [0.02, -0.005], \boldsymbol{\Sigma} = [0.04, 0.015; 0.015, 0.09].$$

Set  $(S_1(0), S_2(0)) = (90, 110)$ , and the strike price  $K = 100$ . We considered the cosine expansion over a symmetric interval  $[-100, 100] \times [-100, 100]$ .

In Gerber et al. [3], the valuation equations for GMDB with the exchange option type were explicitly obtained by the Esscher transform, from which we can compute reference values and relative errors. Motivated by the idea in Gerber et al. [3], we further considered some other option types. Approximation results are given in Table 8, and the relative errors confirmed the accuracy of the proposed procedure. It was observed that as  $N$  increased, the relative errors decreased.

Table 8. Approximation results for 2D GMDB.

Option Type	$N = 2^6$	$N = 2^8$	$N = 2^{10}$	$N = 2^{12}$
Exchange Option	140.8768 ( $8.31 \times 10^{-2}$ )	153.5866 ( $3.55 \times 10^{-4}$ )	153.6412 ( $6.38 \times 10^{-7}$ )	153.6411 ( $6.74 \times 10^{-10}$ )
Geometric Option	111.4024 ( $2.30 \times 10^{-2}$ )	114.0135 ( $1.29 \times 10^{-4}$ )	114.0281 ( $1.54 \times 10^{-7}$ )	114.0281 ( $1.27 \times 10^{-10}$ )
Maximum Option	470.8768 ( $2.64 \times 10^{-2}$ )	483.5866 ( $1.12 \times 10^{-4}$ )	483.6412 ( $2.03 \times 10^{-7}$ )	483.6411 ( $2.14 \times 10^{-10}$ )
Minimum Option	129.1232 ( $1.10 \times 10^{-1}$ )	116.4134 ( $4.68 \times 10^{-4}$ )	116.3588 ( $8.43 \times 10^{-7}$ )	116.3589 ( $8.90 \times 10^{-10}$ )

### 5. Conclusions

In this paper, we used the COS method to value GMDB products under a general Lévy framework. When the death benefit payment depended on only one stock fund, we used the 1D COS method to value the products; when it depended on two stock funds, the 2D COS method was used for valuation. Various numerical results illustrated the accuracy and efficiency of the proposed method.

Our COS method can only be used to value GMDB contracts that depend on the terminal value of the stock price; however, we note that some products are also dependent on the running maximum or minimum of the stock price; for example, life-contingent lookback options and barrier options. Hence, we have to develop the COS method or search for another numerical method to solve this problem. We leave this for future research.

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