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Inertial Method for Bilevel Variational Inequality Problems with Fixed Point and Minimizer Point Constraints

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Abstract: In this paper, we introduce an iterative scheme with inertial effect using Mann iterative scheme and gradient-projection for solving the bilevel variational inequality problem over the intersection of the set of common fixed points of a finite number of nonexpansive mappings and the set of solution points of the constrained optimization problem. Under some mild conditions we obtain strong convergence of the proposed algorithm. Two examples of the proposed bilevel variational inequality problem are also shown through numerical results.

Keywords: minimization problem; fixed point problem; inertial term; bilevel variational inequality

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1. Introduction

Bilevel problem is defined as a mathematical program, where the problem contains another problem as a constraint. Mathematically, bilevel problem is formulated as follows:

$$\textit{find } \bar{x} \in S \subset X \textit{ that solves problem P1 installed in space } X, \tag{1}$$

where S is the solution set of the problem

$$\text{find } x^* \in Y \subset X \text{ that solves problem P2 installed in space } X. \tag{2}$$

Usually, (1) is called the upper level problem and (2) is called the lower level problem. Many real life problems can be modeled as a bilevel problem and some studies have been performed towards solving different kinds of bilevel problems using approximation theory—see, for example, for bilevel optimization problem [1–3], for bilevel variational inequality problem [4–9], for bilevel equilibrium problems [10–12], and [13,14] for its practical applications. In [14], application of bilevel problem (bilevel optimization problem) in transportation (network design, optimal pricing), economics (Stackelberg games, principal-agent problem, taxation, policy decisions), management (network facility location, coordination of multi-divisional firms), engineering (optimal design, optimal chemical equilibria), etc. has been demonstrated. Due to the vast applications of bilevel problems, the research on approximation algorithm for bilevel problems has increased over years and is still in nascent stage.

A simple example of the practical bilevel model is a supplier and a store owner of a business chain (supply chain management), i.e., suppose the supplier will always give his/her best output of some commodities to the store owner in their business's chain. Since both want to do well in their businesses, the supplier will always give his/her best output to the store owner who in turn would like to do his/her best in the business. In some sense, both would like to minimize their loss or rather maximize their profit and thus act in the optimistic pattern. It is clear that, in this example, the store owner is the upper-level decision maker and the supplier is the lower-level decision maker. Thus, in the study of supply chain management, the bilevel problem can indeed play a fundamental role.

In this paper, our main aim is to solve a bilevel variational inequality problem over the intersection of the set of common fixed points of finite number of nonexpansive mappings, denoted by BVIPO-FM, and the set of solution points of the constrained minimization problem of real-valued convex function. To be precise, let C be closed convex subset of a real Hilbert space H, $F : H \to H$ is a mapping, $f : C \to \mathbb{R}$ is a real-valued convex function, and $U_j : C \to C$ is a nonexpansive mapping for each $j \in \{1, \ldots, M\}$. Then, BVIPO-FM is given by

$$\text{find } \bar{x} \in \Omega \text{ such that } \langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in \Omega, \tag{3}$$

where Ω is the solution set of

$$\text{find } x^* \in C \text{ such that } f(x^*) = \min_{x \in C} f(x) \text{ and } x^* \in \bigcap_{j=1}^{M} FixU_j. \tag{4}$$

The notation $FixU_j$ represents the set of fixed points of U_j, i.e., $FixU_j = \{y \in C : U_j(y) = y\}$ for $j \in \{1, \ldots, M\}$. Thus, $\Omega = (\bigcap_{j=1}^{M} FixU_j) \bigcap \Gamma$, where Γ is the solution set of constrained convex minimization problem given by

$$\text{find } x^* \in C \text{ such that } f(x^*) = \min_{x \in C} f(x). \tag{5}$$

The problem (3) is a classical variational inequality problem, denoted by VIP(Ω, F), which was studied by many authors—for example, see in [7,15–17] and references therein. The solution set of the variational inequality problem VIP(Ω, F) is denoted by SVIP(Ω, F). Therefore, BVIPO-FM is obtained by solving VIP(Ω, F), where $\Omega = (\bigcap_{j=1}^{M} FixU_j) \bigcap \Gamma$. Bilevel problem with upper-level problem is variational inequality problem, which was introduced in [18]. These problems have received significant attention from the mathematical programming community. Bilevel variational inequality problem can be used to study various bilevel models in optimization, economics, operations research, and transportation.

It is known that the gradient-projection algorithm—given by

$$x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)), \tag{6}$$

where the parameters λ_n are real positive numbers—is one of the powerful methods for solving the minimization problem (5) (see [19–21]). In general, if the gradient ∇f is Lipschitz continuous and strongly monotone, then, the sequence $\{x_n\}$ generated by recursive Formula (6) converges strongly to a minimizer of (6), where the parameters $\{\lambda_n\}$ satisfy some suitable conditions. However, if the gradient ∇f is only to be inverse strongly monotone, the sequence $\{x_n\}$ generated by (6) converges weakly.

In approximation theory, constructing iterative schemes with speedy rate of convergence is usually of great interest. For this purpose, Polyak [22] proposed an inertial accelerated extrapolation process to solve the smooth convex minimization problem. Since then, there are growing interests by authors working in this direction. Due to this reason, a lot of researchers constructed fast iterative algorithms by using inertial extrapolation, including inertial forward–backward splitting methods [23,24], inertial Douglas–Rachford splitting method [25], inertial forward-–backward-–forward method [26], inertial proximal-extragradient method [27], and others.

In this paper, we introduce an algorithm with inertial effect for solving BVIPO-FM using projection method for the variational inequality problem, the well-known Mann iterative scheme [28] for the nonexpansive mappings T_j's, and gradient-projection for the function f. It is proved that the sequence generated by our proposed algorithm converges strongly to the solution of BVIPO-FM.

2. Preliminary

Let H be a real Hilbert space H. The symbols "$\rightharpoonup$" and "$\to$" denote weak and strong convergence, respectively. Recall that for a nonempty closed convex subset C of H, the metric projection on C is a mapping $P_C : H \to C$, defined by

$$P_C(x) = \arg\min\{\|y - x\| : y \in C\}, \quad x \in H.$$

Lemma 1. *Let C be a closed convex subset of H. Given $x \in H$ and a point $z \in C$, then, $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C$.*

Definition 1. *For $C \subset H$, the mapping $T : C \to H$ is said to be L-Lipschitz on C if there exists $L > 0$ such that*

$$\|T(x) - T(y)\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

If $L \in (0, 1)$, then, we call T a contraction mapping on C with constant L. If $L = 1$, then, T is called a nonexpansive mapping on C.

Definition 2. *The mapping $T : H \to H$ is said to be firmly nonexpansive if*

$$\langle x - y, T(x) - T(y)\rangle \geq \|T(x) - T(y)\|^2, \quad \forall x, y \in H.$$

Alternatively, $T : H \to H$ is firmly nonexpansive if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive.

The class of firmly nonexpansive mappings belong to the class of nonexpansive mappings.

Definition 3. *The mapping $T : H \to H$ is said to be*

(a) *monotone if*

$$\langle T(x) - T(y), x - y\rangle \geq 0, \quad \forall x, y \in H;$$

(b) *β-strongly monotone if there exists a constant $\beta > 0$ such that*

$$\langle T(x) - T(y), x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in H;$$

(c) *ν-inverse strongly monotone (ν-ism) if there exists $\nu > 0$ such that*

$$\langle T(x) - T(y), x - y\rangle \geq \nu\|T(x) - T(y)\|^2, \quad \forall x, y \in H.$$

Definition 4. *The mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity mapping I and a nonexpansive mapping, that is*

$$T = (1-\alpha)I + \alpha S, \tag{7}$$

where $\alpha \in (0,1)$ and $S : H \to H$ is nonexpansive. More precisely, when (7) holds, we say that T is α-averaged.

It is easy to see that firmly nonexpansive mapping (in particular, projection) is $\frac{1}{2}$-averaged and 1-inverse strongly monotone mappings. Averaged mappings and ν-inverse strongly monotone mapping (ν-ism) have received many investigations, see [29–32]. The following propositions about averaged mappings and inverse strongly monotone mappings are some of the important facts in our discussion in this paper.

Proposition 1 ([29,30])**.** *Let the operators $S, T, V : H \to H$ be given:*

(i) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then, T is averaged.*
(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*
(iii) *If $T = (1-\alpha)S + \alpha V$, for some $\alpha \in (0,1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*
(iv) *The composition of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \ldots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then, the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Proposition 2 ([29,31])**.** *Let $T : H \to H$ be given. We have*

(a) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism;*
(b) *If T is ν-ism and $\gamma > 0$, then γT is $\frac{\nu}{\gamma}$-ism;*
(c) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Lemma 2. *(Opial's condition) For any sequence $\{x_n\}$ in the Hilbert space H with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to+\infty} \|x_n - x\| < \liminf_{n\to+\infty} \|x_n - y\|$$

holds for each $y \in H$ with $y \neq x$.

Lemma 3. *For a real Hilbert space H, we have*

(i) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \quad \forall x, y \in H;$
(ii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$

Lemma 4. *Let H be real Hilbert space. Then, $\forall x, y \in H$ and $\forall \alpha \in [0,1]$, we have*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + \alpha\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 5 ([33])**.** *Let $\{c_n\}$ and $\{\gamma_n\}$ be a sequences of non-negative real numbers, and $\{\beta_n\}$ be a sequence of real numbers such that*

$$c_{n+1} \le (1-\alpha_n)c_n + \beta_n + \gamma_n, \quad n \ge 1,$$

where $0 < \alpha_n < 1$ and $\sum \gamma_n < \infty$.

(i) *If $\beta_n \le \alpha_n M$ for some $M \ge 0$, then, $\{c_n\}$ is a bounded sequence.*
(ii) *If $\sum \alpha_n = \infty$ and $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \le 0$, then, $c_n \to 0$ as $n \to \infty$.*

Definition 5. *Let $\{\Gamma_n\}$ be a real sequence. Then, $\{\Gamma_n\}$ decreases at infinity if there exists $n_0 \in \mathbb{N}$ such that $\Gamma_{n+1} \leq \Gamma_n$ for $n \geq n_0$. In other words, the sequence $\{\Gamma_n\}$ does not decrease at infinity, if there exists a subsequence $\{\Gamma_{n_t}\}_{t\geq 1}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_t} < \Gamma_{n_t+1}$ for all $t \geq 1$.*

Lemma 6 ([34])**.** *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity. Additionally, consider the sequence of integers $\{\varphi(n)\}_{n\geq n_0}$ defined by*

$$\varphi(n) = \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Then, $\{\varphi(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \varphi(n) = 0$ and for all $n \geq n_0$, the following two estimates hold:

$$\Gamma_{\varphi(n)} \leq \Gamma_{\varphi(n)+1} \text{ and } \Gamma_n \leq \Gamma_{\varphi(n)+1}.$$

Let C be closed convex subset of a real Hilbert space H and given a bifunction $g : C \times C \to \mathbb{R}$. Then, the problem

$$\text{find } \bar{x} \in C \text{ such that } g(\bar{x}, y) \geq 0, \quad \forall y \in C$$

is called equilibrium problem (Fan inequality [35]) of g on C, denoted by $\text{EP}(g, C)$. The set of all solutions of the $\text{EP}(g, C)$ is denoted by $\text{SEP}(g, C)$, i.e., $\text{SEP}(g, C) = \{\bar{x} \in C : g(\bar{x}, y) \geq 0, \ \forall y \in C\}$. If $g(x, y) = \langle A(x), y - x\rangle$ for every $x, y \in C$, where A is a mapping from C into H, then, the equilibrium problem becomes the variational inequality problem.

We say that the bifunction $g : C \times C \to \mathbb{R}$ satisfies Condition CO on C if the following four assumptions are satisfied:

(i) $g(x, x) = 0$, for all $x \in C$;
(ii) g is monotone on H, i.e., $g(x, y) + g(y, x) \leq 0$, for all $x, y \in C$;
(iii) for each $x, y, z \in C$, $\limsup_{\alpha\downarrow 0} g(\alpha z + (1-\alpha)x, y) \leq g(x, y)$;
(iv) $g(x, .)$ is convex and lower semicontinuous on H for each $x \in C$.

Lemma 7 ([36])**.** *If g satisfies Condition CO on C, then, for each $r > 0$ and $x \in H$, the mapping given by*

$$T_r^g(x) = \left\{z \in C : g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\right\}$$

satisfies the following conditions:

(1) *T_r^g is single-valued;*
(2) *T_r^g is firmly nonexpansive, i.e., for all $x, y \in H$,*

$$\|T_r^g(x) - T_r^g(y)\|^2 \leq \langle T_r^g(x) - T_r^g(y), x - y\rangle;$$

(3) *$Fix(T_r^g) = \{\bar{x} \in H : g(\bar{x}, y) \geq 0, \forall y \in C\}$, where $Fix(T_r^g)$ is the fixed point set of T_r^g;*
(4) *$\{\bar{x} \in H : g(\bar{x}, y) \geq 0, \forall y \in C\}$ is closed and convex.*

3. Main Result

In this paper, we are interested in finding a solution to BVIPO-FM, where F and f satisfy the following conditions:

(A1) $F : H \to H$ is β-strongly monotone and κ-Lipschitz continuous on H.
(A2) The gradient ∇f is L-Lipschitz continuous on C.

We are now in a position to state our inertial algorithm and prove its strong convergence to the solution of BVIPO-FM assuming that F satisfies condition **(A1)**, f satisfies condition **(A2)**, and $\text{SVIP}(\Omega, F)$ is nonempty.

We have plenty of choices for $\{\alpha_n\}$, $\{\varepsilon_n\}$, and $\{\rho_n\}$ satisfying parameter restrictions (C3), (C4), and (C5). For example, if we take $\alpha_n = \frac{1}{3n}$, $\varepsilon_n = \frac{1}{n^2}$ and $\rho_n = \frac{n+1}{3n+1}$, then, $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = \lim_{n\to\infty} \frac{3}{n} = 0$ ($\varepsilon_n = o(\alpha_n)$), $0 \le \rho_n = \frac{n+1}{3n+1} \le 1 - \alpha_n = \frac{3n-1}{3n}$ and $\lim_{n\to\infty} \rho_n = \frac{1}{3}$. Therefore, (C3), (C4), and (C5) are satisfied.

Remark 1. *From (C4) and Step 1 of Algorithm 1, we have that*

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0, \quad n \to \infty.$$

Since $\{\alpha_n\}$ is bounded, we also have

$$\theta_n\|x_n - x_{n-1}\| \to 0, \quad n \to \infty.$$

Note that Step 1 of Algorithm 1 is easily implemented in numerical computation since the value of $\|x_n - x_{n-1}\|$ is a priori known before choosing β_n.

Algorithm 1—Inertial Algorithm for BVIPO-FM

Initialization: Choose $x_0, x_1 \in C$. Let a positive real constants θ, μ and the real sequences $\{\alpha_n\}$, $\{\varepsilon_n\}$, $\{\rho_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$ satisfy the following conditions:

- **(C1)** $0 \le \theta < 1$;
- **(C2)** $0 < \mu < \min\{\frac{2\beta}{\kappa^2}, \frac{1}{2\beta}\}$;
- **(C3)** $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- **(C4)** $\varepsilon_n > 0$ and $\varepsilon_n = o(\alpha_n)$;
- **(C5)** $0 \le \rho_n \le 1 - \alpha_n$ and $\lim_{n\to\infty} \rho_n = \rho < 1$;
- **(C6)** $0 < a \le \lambda_n \le b < \frac{2}{L}$ and $\lim_{n\to\infty} \lambda_n = \hat{\lambda}$;
- **(C7)** $0 < \xi \le \beta_n \le \zeta < 1$.

Step 1. Given the iterates x_{n-1} and x_n ($n \ge 1$), choose θ_n such that $0 \le \theta_n \le \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\varepsilon_n}{\|x_{n-1} - x_n\|}\right\}, & \text{if } x_{n-1} \ne x_n \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. Evaluate $z_n = x_n + \theta_n(x_n - x_{n-1})$.
Step 3. Evaluate $y_n = P_C(z_n - \lambda_n \nabla f(z_n))$.
Step 4. Evaluate $t_n^j = (1 - \beta_n)y_n + \beta_n U_j(y_n)$ for each $j \in \{1, \ldots, M\}$.
Step 5. Evaluate $t_n = \arg\max\{\|v - y_n\| : v \in \{t_n^1, \ldots, t_n^M\}\}$.
Step 6. Compute

$$x_{n+1} = \rho_n z_n + \Psi_{\mu,\alpha_n,\rho_n}(t_n),$$

where $\Psi_{\mu,\alpha_n,\rho_n} := (1 - \rho_n)I - \alpha_n \mu F$.

Remark 2. *Note that the point $\bar{x} \in C$ solves the minimization problem (5) if and only if*

$$P_C(\bar{x} - \lambda\nabla f(\bar{x})) = \bar{x},$$

where $\lambda > 0$ is any fixed positive number. Therefore, the solution set Γ of the problem (5) is closed and convex subset of H, because for $0 < \lambda < \frac{2}{L}$ the mapping $P_C(I - \lambda\nabla f)$ is nonexpansive mapping and solution points of (5) are fixed points of $P_C(I - \lambda\nabla f)$. Moreover, U_j is nonexpansive and hence $FixU_j$ is closed and convex for each $j \in \{1, \ldots, M\}$.

Lemma 8. *For a real number $\lambda > 0$ with $0 < a \leq \lambda \leq b < \frac{2}{L}$, the mapping $T_\lambda := P_C(I - \lambda \nabla f)$ is $\frac{2+\lambda L}{4}$-averaged.*

Proof. Since ∇f is L-Lipschitz, the gradient ∇f is $\frac{1}{L}$-ism [37], which then implies that $\lambda \nabla f$ is $\frac{1}{\lambda L}$-ism. So by Proposition 2 (c), $I - \lambda \nabla f$ is $\frac{\lambda L}{2}$-averaged. Now since the projection P_C is $\frac{1}{2}$-averaged, we see from Proposition 2 (iv) that the composite $P_C(I - \lambda \nabla f)$ is $\frac{2+\lambda L}{4}$-averaged. Therefore, for some nonexpansive mapping T, T_λ can written as

$$T_\lambda := P_C(I - \lambda \nabla f) = (1 - \delta)I + \delta T, \tag{8}$$

where $\frac{1}{2} < a_1 = \frac{2+aL}{4} \leq \delta = \frac{2+\lambda L}{4} \leq b_1 = \frac{2+bL}{4} < 1$. Note that, in view of Remark 2 and (8), the point $\bar{x} \in C$ solves the minimization problem (5) if and only if $T(\bar{x}) = \bar{x}$. □

Lemma 9. *For each n, the mapping $\Psi_{\mu,\alpha_n,\rho_n}$ defined in Step 6 of Algorithm 1 satisfies the inequality*

$$\|\Psi_{\mu,\alpha_n,\rho_n}(x) - \Psi_{\mu,\alpha_n,\rho_n}(y)\| \leq (1 - \eta - \alpha_n \tau)\|x - y\|, \quad \forall x, y \in H,$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu\kappa^2)} \in (0, 1)$.

Proof. From (C2), it is easy to see that

$$0 < 1 - 2\mu\beta < 1 - \mu(2\beta - \mu\kappa^2) < 1.$$

This implies that $0 < \sqrt{1 - \mu(2\beta - \mu\kappa^2)} < 1$.
Then,

$$\begin{aligned} \|\Psi_{\mu,\alpha_n,\rho_n}(x) - \Psi_{\mu,\alpha_n,\rho_n}(y)\| &= \|[(1 - \rho_n)x - \alpha_n \mu F(x)] - [(1 - \rho_n)y - \alpha_n \mu F(y)]\| \\ &= \|(1 - \rho_n - \alpha_n)(x - y) + \alpha_n[(x - y) - \mu(F(x) - F(y))]\| \\ &\leq (1 - \rho_n - \alpha_n)\|x - y\| + \alpha_n\|(x - y) - \mu(F(x) - F(y))\|. \end{aligned} \tag{9}$$

By the strong monotonicity and the Lipschitz continuity of F, we have

$$\begin{aligned} \|(x - y) - \mu(F(x) - F(y))\|^2 &= \|x - y\|^2 + \mu^2\|F(x) - F(y)\|^2 \\ &\quad -2\mu\langle x - y, F(x) - F(y)\rangle \\ &\leq \|x - y\|^2 + \mu^2\kappa^2\|x - y\|^2 - 2\mu\beta\|x - y\|^2 \\ &= (1 + \mu^2\kappa^2 - 2\mu\beta)\|x - y\|^2. \end{aligned} \tag{10}$$

From (9) and (10), we have

$$\begin{aligned} &\|\Psi_{\mu,\alpha_n,\rho_n}(x) - \Psi_{\mu,\alpha_n,\rho_n}(y)\| \\ &\leq (1 - \rho_n - \alpha_n)\|x - y\| + \alpha_n\sqrt{(1 + \mu^2\kappa^2 - 2\mu\beta)\|x - y\|^2} \\ &= (1 - \rho_n - \alpha_n)\|x - y\| + \alpha_n\sqrt{1 - \mu(2\beta - \mu\kappa^2)}\|x - y\| \\ &= (1 - \rho_n - \alpha_n\tau)\|x - y\|, \end{aligned}$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu\kappa^2)} \in (0, 1)$. □

Theorem 1. *The sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the unique solution of BVIPO-FM.*

Proof. Let $\bar{x} \in \mathrm{SVIP}(\Omega, F)$.

Now, from the definition of z_n, we get

$$\begin{aligned} \|z_n - \bar{x}\| &= \|x_n + \theta_n(x_n - x_{n-1}) - \bar{x}\| \\ &\leq \|x_n - \bar{x}\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{11}$$

Note that for each n, there is a nonexpansive mapping T_n such that $y_n = (1-\delta_n)z_n + \delta_n T_n(z_n)$, where $\delta_n = \frac{2+\lambda_n L}{4} \in [a_1, b_1] \subset (0,1)$ for $a_1 = \frac{2+aL}{4}$ and $b_1 = \frac{2+bL}{4}$. Now, using Lemma 4 and the fact that $T_n(\bar{x}) = \bar{x}$, we have

$$\begin{aligned} \|y_n - \bar{x}\|^2 &= \|(1-\delta_n)z_n + \delta_n T_n(z_n) - \bar{x}\|^2 \\ &= (1-\delta_n)\|z_n - \bar{x}\|^2 + \delta_n\|T_n(z_n) - \bar{x}\|^2 \\ &\qquad - \delta_n(1-\delta_n)\|T_n(z_n) - z_n\|^2 \\ &\leq \|z_n - \bar{x}\|^2 - \delta_n(1-\delta_n)\|T_n(z_n) - z_n\|^2. \end{aligned} \tag{12}$$

Let $\{j_n\}_{n=1}^{\infty}$ be the sequence of natural numbers such that $1 \leq j_n \leq M$ where $j_n \in \arg\max\{\|t_n^j - x_n\| : j \in \{1, \ldots, M\}\}$. This means that $t_n = (1-\beta_n)y_n + \beta_n U_{j_n}(y_n)$. Thus, by Lemma 4

$$\begin{aligned} \|t_n - \bar{x}\|^2 &= (1-\beta_n)\|y_n - \bar{x}\|^2 + \beta_n\|U_{j_n}(y_n) - \bar{x}\|^2 \\ &\qquad - \beta_n(1-\beta_n)\|U_{j_n}(y_n) - y_n\|^2 \\ &\leq \|y_n - \bar{x}\|^2 - \beta_n(1-\beta_n)\|U_{j_n}(y_n) - y_n\|^2. \end{aligned} \tag{13}$$

From (11)–(13) we have

$$\|t_n - \bar{x}\| \leq \|y_n - \bar{x}\| \leq \|z_n - \bar{x}\|. \tag{14}$$

Using the definition of x_{n+1}, (14) and Lemma 9, we get

$$\begin{aligned} &\|x_{n+1} - \bar{x}\| = \|\rho_n z_n + \Psi_{\mu,\alpha_n,\rho_n}(t_n) - \bar{x}\| \\ &= \|\Psi_{\mu,\alpha_n,\rho_n}(t_n) - \Psi_{\mu,\alpha_n,\rho_n}(\bar{x}) + \rho_n(z_n - \bar{x}) - \alpha_n\mu F(\bar{x})\| \\ &\leq \|\Psi_{\mu,\alpha_n,\rho_n}(t_n) - \Psi_{\mu,\alpha_n,\rho_n}(\bar{x})\| + \rho_n\|z_n - \bar{x}\| + \alpha_n\mu\|F(\bar{x})\| \\ &\leq (1 - \rho_n - \alpha_n\tau)\|t_n - \bar{x}\| + \rho_n\|z_n - \bar{x}\| + \alpha_n\mu\|F(\bar{x})\| \\ &\leq (1 - \alpha_n\tau)\|z_n - \bar{x}\| + \alpha_n\mu\|F(\bar{x})\| \\ &\leq (1 - \alpha_n\tau)\|x_n - \bar{x}\| + (1 - \alpha_n\tau)\theta_n\|x_n - x_{n-1}\| + \alpha_n\mu\|F(\bar{x})\| \\ &\leq (1 - \alpha_n\tau)\|x_n - \bar{x}\| + \alpha_n\tau\left\{\frac{(1-\alpha_n\tau)}{\tau}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \frac{\mu\|F(\bar{x})\|}{\tau}\right\}. \end{aligned} \tag{15}$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0,1)$. Observe that by condition (C3) and by Remark 1, we see that

$$\lim_{n\to\infty} \frac{(1-\alpha_n\tau)}{\tau}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$$

Let

$$\bar{M} = 2\max\left\{\frac{\mu\|F(\bar{x})\|}{\tau}, \sup_{n\geq 1}\frac{(1-\alpha_n\tau)}{\tau}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\right\}.$$

Then, (15) becomes

$$\|x_{n+1} - \bar{x}\| \leq (1 - \alpha_n\tau)\|x_n - \bar{x}\| + \alpha_n\tau\bar{M}.$$

Thus, by Lemma 5 the sequence $\{x_n\}$ is bounded. As a consequence, $\{z_n\}$, $\{y_n\}$, $\{t_n\}$, and $\{F(t_n)\}$ are also bounded.

Now, using the definition of z_n and Lemma 3 (i), we obtain

$$\begin{aligned}\|z_n - \bar{x}\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - \bar{x}\|^2 \\ &= \|x_n - \bar{x}\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - \bar{x}, x_n - x_{n-1}\rangle. \end{aligned} \tag{16}$$

Again, by Lemma 3 (i), we have

$$\langle x_n - \bar{x}, x_n - x_{n-1}\rangle = \frac{1}{2}\|x_n - \bar{x}\|^2 - \frac{1}{2}\|x_{n-1} - \bar{x}\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2. \tag{17}$$

From (16) and (17), and since $0 \le \theta_n < 1$, we get

$$\begin{aligned}\|z_n - \bar{x}\|^2 &= \|x_n - \bar{x}\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\quad +\theta_n(\|x_n - \bar{x}\|^2 - \|x_{n-1} - \bar{x}\|^2 + \|x_n - x_{n-1}\|^2) \\ &\le \|x_n - \bar{x}\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 \\ &\quad +\theta_n(\|x_n - \bar{x}\|^2 - \|x_{n-1} - \bar{x}\|^2). \end{aligned} \tag{18}$$

Using the definition of x_{n+1} together with (14) and Lemma 9, we have

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= \|\rho_n z_n + \Psi_{\mu,\alpha_n,\rho_n}(t_n) - \bar{x}\|^2 \\ &= \|\Psi_{\mu,\alpha_n,\rho_n}(t_n) - \Psi_{\mu,\alpha_n,\rho_n}(\bar{x}) + \rho_n(z_n - \bar{x}) - \alpha_n\mu F(\bar{x})\|^2 \\ &= \|\Psi_{\mu,\alpha_n,\rho_n}(t_n) - \Psi_{\mu,\alpha_n,\rho_n}(\bar{x}) + \rho_n(z_n - \bar{x})\|^2 \\ &\quad -2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\ &= \left\{\|\Psi_{\mu,\alpha_n,\rho_n}(t_n) - \Psi_{\mu,\alpha_n,\rho_n}(\bar{x})\| + \rho_n\|z_n - \bar{x}\|\right\}^2 \\ &\quad -2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\ &\le \left\{(1 - \rho_n - \alpha_n\tau)\|t_n - \bar{x}\| + \rho_n\|z_n - \bar{x}\|\right\}^2 \\ &\quad -2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\ &\le (1 - \rho_n - \alpha_n\tau)\|t_n - \bar{x}\|^2 + \rho_n\|z_n - \bar{x}\|^2 \\ &\quad -2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle. \end{aligned} \tag{19}$$

From (12) and (13), we obtain

$$\begin{aligned}\|t_n - \hat{x}\|^2 &\le \|z_n - \hat{x}\|^2 - \delta_n(1 - \delta_n)\|T_n(z_n) - z_n\|^2 \\ &\quad -\beta_n(1 - \beta_n)\|U_{j_n}(y_n) - y_n\|^2. \end{aligned} \tag{20}$$

In view of (19) and (20), we get

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\le (1 - \rho_n - \alpha_n\tau)\|t_n - \bar{x}\|^2 + \rho_n\|z_n - \bar{x}\|^2 \\ &\quad -2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\ &\le (1 - \rho_n - \alpha_n\tau)\|z_n - \bar{x}\|^2 + \rho_n\|z_n - \bar{x}\|^2 - 2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\ &\quad -\delta_n(1 - \rho_n - \alpha_n\tau)(1 - \delta_n)\|T_n(z_n) - z_n\|^2 \\ &\quad -\beta_n(1 - \rho_n - \alpha_n\tau)(1 - \beta_n)\|U_{j_n}(y_n) - y_n\|^2 \\ &= (1 - \alpha_n\tau)\|z_n - \bar{x}\|^2 - 2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\ &\quad -\delta_n(1 - \rho_n - \alpha_n\tau)(1 - \delta_n)\|T_n(z_n) - z_n\|^2 \\ &\quad -\beta_n(1 - \rho_n - \alpha_n\tau)(1 - \beta_n)\|U_{j_n}(y_n) - y_n\|^2. \end{aligned} \tag{21}$$

Since the sequence $\{x_n\}$ is bounded, there exists $\bar{M}$ such that $-2\alpha_n\mu\langle F(\bar{x}), x_{n+1}-\bar{x}\rangle \leq \bar{M}$ for all $n \geq 1$. Thus, from (18) and (21), we get

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 \leq & (1-\alpha_n\tau)\|x_n-\bar{x}\|^2 + 2(1-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\|^2 \\ & +(1-\alpha_n\tau)\theta_n(\|x_n-\bar{x}\|^2 - \|x_{n-1}-\bar{x}\|^2) + \alpha_n\bar{M} \\ & -\delta_n(1-\rho_n-\alpha_n\tau)(1-\delta_n)\|T_n(z_n)-z_n\|^2 \\ & -\beta_n(1-\rho_n-\alpha_n\tau)(1-\beta_n)\|U_{j_n}(y_n)-y_n\|^2.\end{aligned}$$

Let us distinguish the following two cases related to the behavior of the sequence $\{\Gamma_n\}$, where $\Gamma_n = \|x_n-\bar{x}\|^2$.

Case 1. Suppose the sequence $\{\Gamma_n\}$ decrease at infinity. Thus, there exists $n_0 \in \mathbb{N}$ such that $\Gamma_{n+1} \leq \Gamma_n$ for $n \geq n_0$. Then, $\{\Gamma_n\}$ converges and $\Gamma_n - \Gamma_{n+1} \to 0$ as $n \to 0$.

From (22) we have

$$\begin{aligned}&\delta_n(1-\rho_n-\alpha_n\tau)(1-\delta_n)\|T_n(z_n)-z_n\|^2 \\ &\quad \leq (\Gamma_n-\Gamma_{n+1}) + \alpha_n M_1 + (1-\alpha_n\tau)\theta_n(\Gamma_n-\Gamma_{n-1}) \\ &\qquad +2(1-\alpha_n\tau)\theta_n\|x_n-x_{n-1}\|^2.\end{aligned} \tag{22}$$

Since $\Gamma_n - \Gamma_{n+1} \to 0$ ($\Gamma_{n-1}-\Gamma_n \to 0$) and using condition (C3) and Remark 1 (noting $\alpha_n \to 0$, $0 < \alpha_n < 1$, $\theta_n\|x_n - x_{n-1}\| \to 0$ and $\{x_n\}$ is bounded), from (22) we have

$$\delta_n(1-\rho_n-\alpha_n\tau)(1-\delta_n)\|T_n(z_n)-z_n\|^2 \to 0, \quad n \to \infty. \tag{23}$$

The conditions (C2) and (C5) (i.e., $0 < \alpha_n < 1$, $\alpha_n \to 0$ and $0 < \rho_n \leq 1-\alpha_n$), together with (23) and the fact that $\delta_n = \frac{2+\lambda_n L}{4} \in [a_1, b_1] \subset (0,1)$, we obtain

$$\|T_n(z_n)-z_n\| \to 0, \quad n \to \infty. \tag{24}$$

Similarly, from (23) and the restriction condition imposed on β_n in (C6), together with conditions (C2) and (C5), we have

$$\|U_{j_n}(y_n)-y_n\| \to 0, \quad n \to \infty. \tag{25}$$

Thus, using the definition of y_n together with (24) gives

$$\|y_n-z_n\| = \|(1-\delta_n)z_n + \delta_n T_n(z_n) - z_n\| = \delta_n\|T_n(z_n)-z_n\| \to 0, \quad n \to \infty. \tag{26}$$

Moreover, using the definition of z_n and Remark 1, we have

$$\|x_n-z_n\| = \|x_n - x_n - \theta_n(x_n-x_{n-1})\| = \theta_n\|x_n-x_{n-1}\| \to 0, \quad n \to \infty. \tag{27}$$

By (26) and (27), we get

$$\|x_n-y_n\| \leq \|x_n-z_n\| + \|y_n-z_n\| \to 0, \quad n \to \infty. \tag{28}$$

By the definition of t_n together with (25) gives

$$\|t_n-y_n\| = \|(1-\beta_n)y_n + \beta_n U_{j_n}(y_n) - y_n\| = \beta_n\|U_{j_n}(y_n)-y_n\| \to 0, \quad n \to \infty. \tag{29}$$

By (28) and (29), we get

$$\|x_n-t_n\| \leq \|x_n-y_n\| + \|y_n-t_n\| \to 0, \quad n \to \infty. \tag{30}$$

Again, from (26) and (29), we obtain

$$\|z_n - t_n\| \le \|z_n - y_n\| + \|y_n - t_n\| \to 0, \quad n \to \infty. \tag{31}$$

By the definition of x_{n+1}, with the parameter restriction conditions (C2) and (C6) together with (31) and boundedness of $\{F(t_n)\}$, we have

$$\begin{aligned} \|x_{n+1} - t_n\| &= \|\rho_n z_n + \Psi_{\mu,\alpha_n,\rho_n}(t_n) - t_n\| \\ &= \|\rho_n z_n + (1-\rho_n)t_n - \alpha_n \mu F(t_n) - t_n\| \\ &\le \rho_n \|z_n - t_n\| + \alpha_n \mu \|F(t_n)\| \to 0, \quad n \to \infty. \end{aligned} \tag{32}$$

Results from (30) and (32) give

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - t_n\| + \|t_n - x_n\| \to 0, \quad n \to \infty. \tag{33}$$

By definition of t_n^j and t_n, and using (30), for all $j \in \{1, \dots, M\}$, we have

$$\|t_n^j - x_n\| \le \|t_n - x_n\| \to 0, \quad n \to \infty$$

and this together with (28), yields

$$\|t_n^j - y_n\| \le \|t_n^j - x_n\| + \|y_n - x_n\| \to 0, \quad n \to \infty$$

for all $j \in \{1, \dots, M\}$. Thus,

$$\|U_j(y_n) - y_n\| = \frac{1}{\beta_n}\|t_n^j - y_n\| \to 0, \quad n \to \infty \tag{34}$$

for all $j \in \{1, \dots, M\}$. Therefore, from (28) and (34)

$$\begin{aligned} \|U_j(x_n) - x_n\| &= \|U_j(x_n) - U_j(y_n)\| + \|U_j(y_n) - y_n\| + \|y_n - x_n\| \\ &\le \|U_j(y_n) - y_n\| + 2\|y_n - x_n\| \to 0, \quad n \to \infty \end{aligned} \tag{35}$$

for all $j \in \{1, \dots, M\}$. Moreover, from (24) and (27)

$$\begin{aligned} \|T_n(x_n) - x_n\| &= \|T_n(x_n) - T_n(z_n)\| + \|T_n(z_n) - z_n\| + \|z_n - x_n\| \\ &\le \|T_n(z_n) - z_n\| + 2\|z_n - x_n\| \to 0, \quad n \to \infty. \end{aligned} \tag{36}$$

From (C6), we have $0 < \hat{\lambda} < \frac{2}{L}$. Thus, let $T := P_C(I - \hat{\lambda}\nabla f)$. Then, using the nonexpansiveness of projection mapping and (C6) of assumption 1 together with (28) and boundedness of $\{\|\nabla f(z_n)\|\}$ ($\{z_n\}$ is bouded and ∇f is Lipschitz continuous), we get

$$\begin{aligned} \|T(z_n) - x_n\| &= \|T(z_n) - y_n + y_n - x_n\| \\ &\le \|T(z_n) - y_n\| + \|y_n - x_n\| \\ &= \|P_C(z_n - \hat{\lambda}\nabla f(z_n)) - P_C(z_n - \lambda_n \nabla f(z_n))\| + \|y_n - x_n\| \\ &\le |\hat{\lambda} - \lambda_n|\|\nabla f(z_n)\| + \|y_n - x_n\| \to 0, \quad n \to \infty. \end{aligned} \tag{37}$$

Hence, in view of (27), (37), and the nonexpansiveness of T, we get

$$\begin{aligned} \|T(x_n) - x_n\| &= \|T(x_n) - T(z_n) + T(z_n) - x_n\| \\ &\le \|x_n - z_n\| + \|T(z_n) - x_n\| \to 0, \quad n \to \infty. \end{aligned} \tag{38}$$

Let p be a weak cluster point of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ as $k \to \infty$. We observe that $p \in C$ because $\{x_{n_k}\} \subset C$ and C is weakly closed. Assume $p \notin Fix(U_{j_0})$

for some $j_0 \in \{1, \dots, M\}$. Since $x_{n_k} \rightharpoonup p$ and U_{j_0} is a nonexpansive mapping, from (35) and Opial's condition, one has

$$\begin{aligned}
\liminf_{k\to+\infty} \|x_{n_k} - p\| &< \liminf_{k\to+\infty} \|x_{n_k} - U_{j_0}(p)\| \\
&= \liminf_{k\to+\infty} \|x_{n_k} - U_{j_0}(x_{n_k}) + U_{j_0}(x_{n_k}) - U_{j_0}(p)\| \\
&\le \liminf_{k\to+\infty} (\|x_{n_k} - U_{j_0}(x_{n_k})\| + \|U_{j_0}(x_{n_k}) - U_{j_0}(p)\|) \\
&= \liminf_{k\to+\infty} \|U_{j_0}(x_{n_k}) - U_{j_0}(p)\| \\
&\le \liminf_{k\to+\infty} \|x_{n_k} - p\|
\end{aligned}$$

which is a contradiction. It must be the case that $p \in Fix(U_j)$ for all $j \in \{1, \dots, M\}$. Similarly, using Opial's condition and (38), we can show that $p \in Fix(T)$, i.e., $p \in \Gamma$. Therefore, $p \in \Omega = (\bigcap_{j=1}^{M} FixU_j) \bigcap \Gamma$.

Next, we show that $\limsup_{n\to\infty} \langle F(\bar{x}), \bar{x} - x_{n+1} \rangle \le 0$. Indeed, since $\bar{x} \in \mathrm{SVI}(\Omega, F)$ and $p \in \Omega$, we obtain that

$$\limsup_{n\to\infty} \langle F(\bar{x}), \bar{x} - x_n \rangle = \lim_{k\to\infty} \langle F(\bar{x}), \bar{x} - x_{n_k} \rangle = \langle F(\bar{x}), \bar{x} - p \rangle \le 0.$$

Since $\|x_{n+1} - x_n\| \to 0$ from (33), by (39), we have

$$\limsup_{n\to\infty} \langle F(\bar{x}), \bar{x} - x_{n+1} \rangle \le 0.$$

From (11), (14) and (19), we have

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\le (1 - \rho_n - \alpha_n\tau)\|t_n - \bar{x}\|^2 + \rho_n\|z_n - \bar{x}\|^2 \\
&\qquad -2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\
&\le (1 - \alpha_n\tau)\|z_n - \bar{x}\|^2 - 2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\
&\le (1 - \alpha_n\tau)(\|x_n - \bar{x}\| + \theta_n\|x_n - x_{n-1}\|)^2 - 2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\
&\le (1 - \alpha_n\tau)\big(\|x_n - \bar{x}\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\qquad +2\theta_n\|x_n - x_{n-1}\|\|x_n - \bar{x}\|\big) - 2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle \\
&\le (1 - \alpha_n\tau)\|x_n - \bar{x}\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\qquad +2\theta_n\|x_n - x_{n-1}\|\|x_n - \bar{x}\| - 2\alpha_n\mu\langle F(\bar{x}), x_{n+1} - \bar{x}\rangle.
\end{aligned} \tag{39}$$

Since $\{x_n\}$ is bounded, there exists $M_2 > 0$ such that $\|x_n - \bar{x}\| \le M_2$ for all $n \ge 1$. Thus, in view of (39), we have

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\le (1 - \alpha_n\tau)\|x_n - \bar{x}\|^2 + \theta_n\|x_n - x_{n-1}\|(\theta_n\|x_n - x_{n-1}\| + 2M_2) \\
&\qquad +2\alpha_n\mu\langle F(\bar{x}), \bar{x} - x_{n+1}\rangle.
\end{aligned} \tag{40}$$

Therefore, from (41), we get

$$\Gamma_{n+1} \le (1 - \omega_n)\Gamma_n + \omega_n\vartheta_n, \tag{41}$$

where $\omega_n = \alpha_n\tau$ and

$$\vartheta_n = \frac{1}{\tau}\left(\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\right)\{\theta_n\|x_n - x_{n-1}\| + 2M_2\} + \frac{2\mu}{\tau}\langle F(\bar{x}), \bar{x} - x_{n+1}\rangle.$$

From (C2) and Remark 1, we have $\sum_{n=1}^{\infty} \omega_n = \infty$ and $\limsup_{n\to\infty} \vartheta_n \le 0$. Thus, using Lemma 5 and (41), we get $\Gamma_n \to 0$ as $n \to \infty$. Hence, $x_n \to \bar{x}$ as $n \to \infty$.

Case 2. Assume that $\{\Gamma_n\}$ does not decrease at infinity. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\varphi(n) = \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

By Lemma 6, $\{\varphi(n)\}_{n=n_0}^{\infty}$ is a nondecreasing sequence, $\varphi(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\varphi(n)} \leq \Gamma_{\varphi(n)+1} \text{ and } \Gamma_n \leq \Gamma_{\varphi(n)+1}, \quad \forall n \geq n_0. \tag{42}$$

In view of $\|x_{\varphi(n)} - \bar{x}\|^2 - \|x_{\varphi(n)+1} - \bar{x}\|^2 = \Gamma_{\varphi(n)} - \Gamma_{\varphi(n)+1} \leq 0$ for all $n \geq n_0$ and (22), for all $n \geq n_0$ we have

$$\begin{aligned}
&\delta_{\varphi(n)}(1-\rho_{\varphi(n)}-\alpha_{\varphi(n)}\tau)(1-\delta_{\varphi(n)})\|T_{\varphi(n)}(z_{\varphi(n)})-z_{\varphi(n)}\|^2 \\
&\quad\leq (\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)+1})+\alpha_{\varphi(n)}M_1+(1-\alpha_{\varphi(n)}\tau)\theta_{\varphi(n)}(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1}) \\
&\qquad\qquad +2(1-\alpha_{\varphi(n)}\tau)\theta_{\varphi(n)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2 \\
&\quad\leq \alpha_{\varphi(n)}M_1+(1-\alpha_{\varphi(n)}\tau)\theta_{\varphi(n)}(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)-1}) \\
&\qquad\qquad +2(1-\alpha_{\varphi(n)}\tau)\theta_{\varphi(n)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2 \\
&\quad\leq \alpha_{\varphi(n)}M_1+(1-\alpha_{\varphi(n)}\tau)\theta_{\varphi(n)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|\left(\sqrt{\Gamma_{\varphi(n)}}+\sqrt{\Gamma_{\varphi(n)-1}}\right) \\
&\qquad\qquad +2(1-\alpha_{\varphi(n)}\tau)\theta_{\varphi(n)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2.
\end{aligned} \tag{43}$$

Thus, from (43), conditions (C3) and (C4), and Remark 1, we have

$$\|T_{\varphi(n)}(z_{\varphi(n)})-z_n\| \to 0, \quad n \to \infty. \tag{44}$$

Similarly,

$$\|U_{j_{\varphi(n)}}(y_{\varphi(n)})-y_{\varphi(n)}\| \to 0, \quad n \to \infty. \tag{45}$$

Using similar procedure as above in Case 1, we have $\lim_{n\to\infty}\|x_{\varphi(n)+1}-x_{\varphi(n)}\| = 0$ and for $T := P_C(I-\hat{\lambda}\nabla f)$, we have

$$\lim_{n\to\infty}\|T(x_{\varphi(n)})-x_{\varphi(n)}\| = \lim_{n\to\infty}\|U_j(x_{\varphi(n)})-x_{\varphi(n)}\| = 0$$

for all $j \in \{1,\ldots,M\}$. Since $\{x_{\varphi(n)}\}$ is bounded, there exists a subsequence of $\{x_{\varphi(n)}\}$, still denoted by $\{x_{\varphi(n)}\}$, which converges weakly to p. By similar argument as above in Case 1, we conclude immediately that $p \in \Omega$. In addition, by the similar argument as above in Case 1, we have $\limsup_{n\to\infty}\langle F(\bar{x}), \bar{x}-x_{\varphi(n)}\rangle \leq 0$. Since $\lim_{n\to\infty}\|x_{\varphi(n)+1}-x_{\varphi(n)}\| = 0$, we get $\limsup_{n\to\infty}\langle F(\bar{x}), \bar{x}-x_{\varphi(n)+1}\rangle \leq 0$.

From (41), we have

$$\Gamma_{\varphi(n)+1} \leq (1-\omega_{\varphi(n)})\Gamma_{\varphi(n)}+\omega_{\varphi(n)}\vartheta_{\varphi(n)}, \tag{46}$$

where $\omega_{\varphi(n)} = \alpha_{\varphi(n)}\tau$ and

$$\begin{aligned}
\vartheta_{\varphi(n)} = \tfrac{1}{\tau}\left(\tfrac{\theta_{\varphi(n)}}{\alpha_{\varphi(n)}}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|\right)&\{\theta_{\varphi(n)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|+2M_2\} \\
&+\tfrac{2\mu}{\tau}\langle F(\bar{x}), \bar{x}-x_{\varphi(n)+1}\rangle.
\end{aligned}$$

Using $\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)+1} \leq 0$ for all $n \geq n_0$ and $\vartheta_{\varphi(n)} > 0$, the last inequality gives

$$0 \leq -\omega_{\varphi(n)}\Gamma_{\varphi(n)}+\omega_{\varphi(n)}\vartheta_{\varphi(n)}.$$

Since $\omega_{\varphi(n)} > 0$, we obtain $\|x_{\varphi(n)}-\bar{x}\|^2 = \Gamma_{\varphi(n)} \leq \vartheta_{\varphi(n)}$. Moreover, since $\limsup_{n\to\infty}\vartheta_{\varphi(n)} \leq 0$, we have $\lim_{n\to\infty}\|x_{\varphi(n)}-\bar{x}\| = 0$. Thus, $\lim_{n\to\infty}\|x_{\varphi(n)}-\bar{x}\| = 0$ together with $\lim_{n\to\infty}\|x_{\varphi(n)+1}-x_{\varphi(n)}\| = 0$, gives $\lim_{n\to\infty} \Gamma_{\varphi(n)+1} = 0$. Therefore, from (42), we obtain $\lim_{n\to\infty} \Gamma_n = 0$, that is, $x_n \to \bar{x}$ as $n \to \infty$. This completes the proof. □

4. Applications

The mapping $F : H \to H$, given by $F(x) = x - p$ for a fixed point $p \in H$, is one simple example of β-strongly monotone and κ-Lipschitz continuous mapping, where $\beta = 1$ and $\kappa = 1$. If $F(x) = x - p$ for a fixed point $p \in H$, then, BVIPO-FM becomes the problem of finding the projection of p onto $(\bigcap_{j=1}^{M} FixU_j) \bigcap \Gamma$. When $p = 0$, this projection is the minimum-norm solution in $(\bigcap_{j=1}^{M} FixU_j) \bigcap \Gamma$.

Let BVIPO-M denote the bilevel variational inequality problem over the intersection of the set of common solution points of finite number of constrained minimization problems, stated as follows: For a closed convex subset C of a real Hilbert space H, a nonlinear mapping $F : H \to H$ and a real-valued convex functions $f_j : C \to \mathbb{R}$ for $j \in \{0, 1, \ldots, M\}$, BVIPO-M is the problem given by

$$\textit{find } \bar{x} \in \Omega \textit{ such that } \langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in \Omega,$$

where Ω is the solution-set of

$$\textit{find } x^* \in C \textit{ such that } f_j(x^*) = \min_{x \in C} f_j(x), \quad \forall j \in \{0, 1, \ldots, M\}.$$

If the gradient of f_j (∇f_j) is L_j-Lipschitz continuous on C, then, for $0 < \varsigma < \frac{2}{L_j}$ the mapping $P_C(I - \varsigma \nabla f_j)$ is nonexpansive mapping and $\{x^* \in C : f_j(x^*) = \min_{x \in C} f_j(x)\} = Fix(P_C(I - \varsigma \nabla f_j))$. This leads to the following corollary as an immediate consequence of our main theorem for approximation of solution of BVIPO-M, assuming that SVIP(F, Ω) is nonempty.

Corollary 1. *If F satisfies condition **(A1)**, $f = f_0$ satisfy condition **(A2)**, and the gradient of each f_j (each ∇f_j) is L_j-Lipschitz continuous on C for all $j \in \{1, \ldots, M\}$, then, for $0 < \varsigma < \frac{2}{\max\{L_1, \ldots, L_M\}}$, replacing each U_j by $P_C(I - \varsigma \nabla f_j)$ for all $j \in \{1, \ldots, M\}$ in Algorithm 1 (in Step 4), the sequence $\{x_n\}$ generated by the algorithm strongly converges to the unique solution of BVIPO-M.*

Let C be closed convex subset C of a real Hilbert space H, $F : H \to H$ is a mapping , $f : C \to \mathbb{R}$ is a real-valued convex function, and each $g_j : C \times C \to \mathbb{R}$ is a bifunction for $j \in \{1, \ldots, M\}$. BVIPO-EM denotes the bilevel variational inequality problem over the intersection of the set of common solution points of a finite number of equilibrium problems and the set of solution points of the constrained minimization problem given by

$$\text{find } \bar{x} \in \Omega \text{ such that } \langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in \Omega,$$

where Ω is the solution-set of

$$\text{find } x^* \in C \text{ such that } f(x^*) = \min_{x \in C} f(x) \text{ and } x^* \in \bigcap_{j=1}^{M} \text{SEP}(g_j, C).$$

If each g_j satisfies Condition CO on C for all $j \in \{1, \ldots, M\}$, then, by Lemma 7 (1) and (3), for each $j \in \{1, \ldots, M\}$, $T_r^{g_j}$ is nonexpansive and $FixT_r^{g_j} = \text{SEP}(g_j, C)$. Applying Theorem 1, we obtain the following result for approximation of solution of BVIPO-EM, assuming that SVIP(F, Ω) is nonempty.

Corollary 2. *If F satisfy condition **(A1)**, f satisfy condition **(A2)**, and each g_j satisfies Condition CO on C for all $j \in \{1, \ldots, M\}$, then, for $r > 0$, replacing each U_j by $T_r^{g_j}$ for all $j \in \{1, \ldots, M\}$ in Algorithm 1 (in Step 4), the sequence $\{x_n\}$ generated by the algorithm strongly converges to the unique solution of BVIPO-EM.*

Let C be closed convex subset C of a real Hilbert space H, $F : H \to H$ is a mapping , $f : C \to \mathbb{R}$ is a real-valued convex function and each $F_j : C \to H$ for $j \in \{1, \ldots, M\}$ is a mapping for $j \in \{1, \ldots, M\}$. Now, suppose that BVIPO-VM denotes the bilevel variational inequality problem over the intersection of the set of common solution points of finite number of variational inequality problems and the set of solution points of the constrained minimization problem given by

$$\textit{find } \bar{x} \in \Omega \textit{ such that } \langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in \Omega,$$

where Ω is the solution-set of

$$\textit{find } x^* \in C \textit{ such that } f(x^*) = \min_{x \in C} f(x) \textit{ and } x^* \in \bigcap_{j=1}^{M} SVIP(F_j, C).$$

Note that if each F_j is η_j-inverse strongly monotone on C for all $j \in \{1, \ldots, M\}$ and $0 < \varrho \leq 2\eta_j$, then,

(a) $P_C(I - \varrho F_j)$ is nonexpansive;
(b) x^* is fixed point of $P_C(I - \varrho F_j)$ iff x^* is the solution of the variational inequality problem VIP(F_j, C), i.e., $Fix(P_C(I - \varrho F_j)) =$ SVIP(F_j, C).

By Theorem 1, we have the following corollary for approximation of solution of BVIPO-VM, assuming that SVIP(F, Ω) is nonempty.

Corollary 3. *If F satisfy condition **(A1)**, f satisfy condition **(A2)** and each F_j is η_j-inverse strongly monotone on C for all $j \in \{1, \ldots, M\}$, then for $0 < \varrho \leq 2\min\{\eta_1, \ldots, \eta_M\}$, replacing each U_j by $P_C(I - \varrho F_j)$ for all $j \in \{1, \ldots, M\}$ in Algorithm 1 (in Step 4), the sequence $\{x_n\}$ generated by the algorithm strongly converges to the unique solution of BVIPO-VM.*

5. Numerical Results

Example 1. *Consider the bilevel variational inequality problem*

$$\textit{find } \bar{x} \in \Omega \textit{ such that } \langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in \Omega,$$

where Ω is the solution-set of

$$\textit{find } x^* \in C \textit{ such that } f_j(x^*) = \min_{x \in C} f_j(x), \quad \forall j \in \{0, 1, \ldots, M\}$$

for $H = \mathbb{R}^N$, $C = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : -2 \leq x_i \leq 2, \ \forall i \in \{1, \ldots, N\}\}$, and F and f_j are given by

$$F(x) = F(x_1, \ldots, x_N) = (\gamma_1 x_1, \ldots, \gamma_N x_N),$$

$$f_j(x) = \frac{1}{2}\|(I - P_{D_j})A_j x\|^2, \quad \forall j \in \{0, 1, \ldots, M\},$$

where $\gamma_i > 0$ for all $i \in \{1, \ldots, N\}$,

$$D_j = \left\{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N : \frac{-1}{j+1} \leq x_i \leq \frac{1}{j+2}, \ \forall i \in \{1, \ldots, N\} \right\},$$

$A_j : \mathbb{R}^N \to \mathbb{R}^N$ is given by $A_j = \sigma_j I_{N\times N}$ for $\sigma_j > 0$ ($I_{N\times N}$ is $N \times N$ identity matrix) for $j \in \{0,1,\dots,M\}$. Note the following:

- **(i)** *F is β-strongly monotone and κ-Lipschitz continuous on $H = \mathbb{R}^N$, where $\beta = \min\{\gamma_i : i = 1,\dots,N\}$ and $\kappa = \max\{\gamma_i : i = 1,\dots,N\}$.*
- **(ii)** *A_j is bounded linear operator, $\|A_j\| = \sigma_j$; and A_j is self-adjoint operator.*
- **(iii)** *The gradient of each f_j (each ∇f_j) is L_j-Lipschitz continuous on C for all $j \in \{0,1,\dots,M\}$, where $L_j = \sigma_j^2$ and ∇f_j is given by (see [38])*

$$\nabla f_j(x) = A_j(I - P_{D_j})A_j x = \sigma_j^2 x - \sigma_j P_{D_j}(\sigma_j x).$$

- **(iv)** *For each $j \in \{0,1,\dots,M\}$,*

$$\{x^* \in C : f_j(x^*) = \min_{x\in C} f_j(x)\} = \Gamma_j,$$

where $\Gamma_j = \left\{x \in \mathbb{R}^N : \frac{-1}{\sigma_j(j+1)} \le x_i \le \frac{1}{\sigma_j(j+2)},\ \ \forall i = 1,\dots,N\right\}$. Hence,

$$\Omega = \bigcap_{j=1}^{M} \Gamma_j = \left\{x \in \mathbb{R}^N : LB \le x_i \le UB,\ \ \forall i \in \{1,\dots,N\}\right\},$$

where $LB = \max\left\{\frac{-1}{\sigma_j(j+1)} : j \in \{0,1,\dots,M\}\right\}$ and $UB = \min\left\{\frac{1}{\sigma_j(j+2)} : j \in \{0,1,\dots,M\}\right\}$.
- **(v)** *0 is the solution of the given bilevel variational inequality problem, i.e., SVIP(Ω, F)$= \{0\}$.*

We set $\sigma_j = 2^j$ for each $j \in \{0,1,\dots,M\}$ and $M = 4$. Therefore,

$$\Omega = \left\{x \in \mathbb{R}^N : \frac{-1}{80} \le x_i \le \frac{1}{96},\ \ \forall i \in \{1,\dots,N\}\right\}$$

and the gradient of $f = f_0$ is L-Lipschitz continuous on C where $L = L_0 = \sigma_0^2 = 1$. We will test our experiment for different dimension N and different parameters.

Take $\theta = \frac{1}{2}$ and $\gamma_i = i$ for each $i \in \{1,\dots,N\}$. Thus, F is 1-strongly monotone and N-Lipschitz continuous on $\mathbb{R}^N$. Hence, notice that the positive real constants μ, ς, and λ_n are chosen to be $0 < \varsigma < \frac{2}{\max\{L_1,L_2,L_3,L_4\}} = \frac{1}{128}$, $0 < \mu < \min\{\frac{2}{N^2}, \frac{1}{2}\}$, and $0 < a \le \lambda_n \le b < \frac{2}{N}$. We describe the numerical results of Algorithm 1 (applying Corollary 1) for the positive real constants μ and ς given by $\varsigma = \frac{1}{200}$ and

$$\mu = \begin{cases} \frac{1}{3}, & \text{if } N = 1,2 \\ \frac{2}{N^2-1}, & \text{if } N = 3,4,5,\dots \end{cases}.$$

In Figures 1 and 2 and Table 1, the real sequences $\{\alpha_n\}$, $\{\varepsilon_n\}$, $\{\rho_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$, $\{\theta_n\}$ are chosen as follows:

Data 1. $\alpha_n = \frac{1}{2n+4}$, $\varepsilon_n = \frac{1}{(n+2)^2}$, $\rho_n = \frac{n+3}{2n+4}$, $\lambda_n = \frac{2}{N+1}$, $\beta_n = \frac{n+3}{2n+2}$, $\theta_n = \bar{\theta}_n$.
Data 2. $\alpha_n = \frac{1}{3n^{0.5}+1}$, $\varepsilon_n = \frac{1}{3n^{1.5}+n}$, $\rho_n = \frac{2n^{0.5}-1}{3n^{0.5}+1}$, $\lambda_n = \frac{1}{N}$, $\beta_n = \frac{1}{2}$, $\theta_n = \bar{\theta}_n$.
Data 3. $\alpha_n = \frac{1}{5n}$, $\varepsilon_n = \frac{1}{n^3}$, $\rho_n = \frac{4n-1}{5n+1}$, $\lambda_n = \frac{1}{N+1}$, $\beta_n = \frac{10n+91}{11n+110}$, $\theta_n = \bar{\theta}_n$.

The stopping criteria in Table 1 is defined as $\|x_n - x_{n-1}\| \le 10^{-3}$.

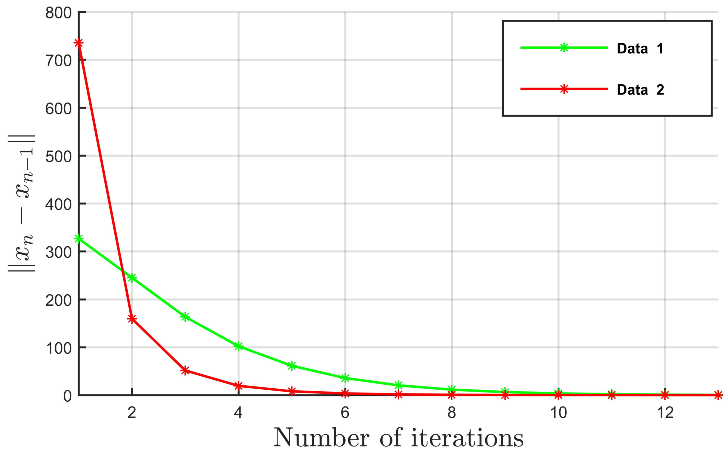

Figure 1. For $N = 100$ and x_0, x_1 are for randomly generated starting points x_0 and x_1 (the same starting points for Data 1 and Data 2).

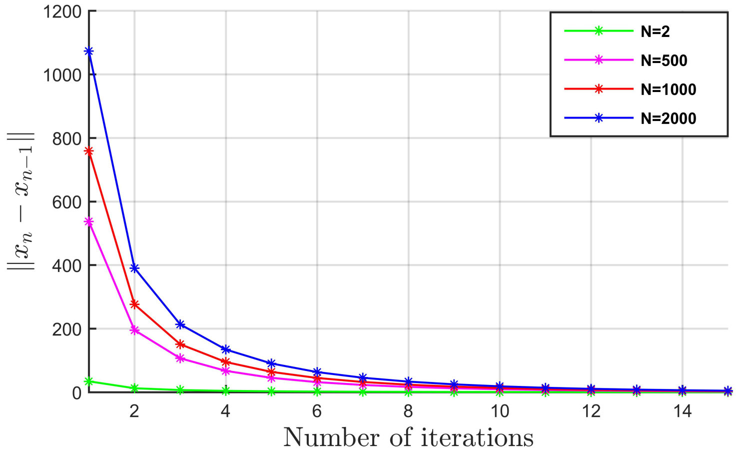

Figure 2. For Data 3 and for randomly generated starting points x_0 and x_1.

Table 1. For starting points $x_0 = 10(1, \ldots, 1) \in \mathbb{R}^N$ and $x_1 = 10x_0$.

	$N = 10$			$N = 1200$		
	Iter(n)	**CPU(s)**	$\|x_n\|$	**Iter(n)**	**CPU(s)**	$\|x_n\|$
Data 1	10	0.0165	0.4231	9	0.0182	0.5739
Data 2	9	0.0186	0.4461	8	0.0193	0.3755
Data 3	10	0.0178	0.1356	8	0.0191	0.4524

Figure 3 demonstrates the behavior of Algorithm 1 for different parameters ρ_n (Case 1: $\rho_n = \frac{1}{5n+3}$; Case 2: $\rho_n = \frac{2n+2}{5n+3}$; Case 2: $\rho_n = \frac{3n+3}{5n+3}$; Case 4: $\rho_n = \frac{4n+2}{5n+3}$), where $\alpha_n = \frac{1}{5n+3}$, $\varepsilon_n = \frac{1}{(5n+3)^3}$, $\lambda_n = \frac{1}{N+1}$, $\beta_n = \frac{n+10}{10n+90}$, $\theta_n = \bar{\theta}_n$.

From Figures 1–3 and Table 1, it is clear to see that your algorithm depends of the dimension, starting points, and parameters. From Figure 3, we can see that the sequence generated by the algorithm converges faster to the solution of the problem for the choice of ρ_n, where ρ ($\lim_{n\to\infty} \rho_n = \rho$) is very close to 0.

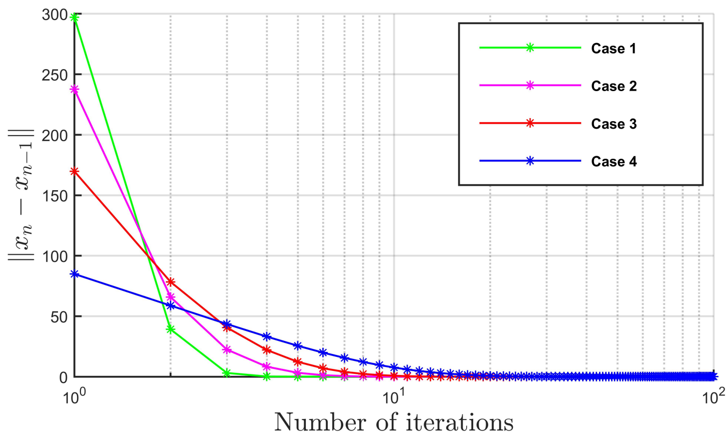

Figure 3. For $N = 50$ and starting points $x_0 = 100(1, \ldots, 1) \in \mathbb{R}^N$ and $x_1 = -50(1, \ldots, 1) \in \mathbb{R}^N$.

Example 2. *Consider BVIPO-FM is given by*

$$\text{find } \bar{x} \in \Omega \text{ such that } \langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in \Omega,$$

where Ω *is the solution set of*

$$\text{find } x^* \in C \text{ such that } f(x^*) = \min_{x \in C} f(x) \text{ and } x^* \in FixU$$

for $H = \mathbb{R}^N = C$ *and* F, f, *and* U *are given by*

$$F(x) = F(x_1, \ldots, x_N) = (a_1 x_1 + b_1, \ldots, a_N x_N + b_N),$$

$$f(x) = \frac{1}{2}\|(I - P_D)2x\|^2,$$

$$Ux = x,$$

where $a_i > 0$, $b_i > 0$ *for all* $i \in \{1, \ldots, N\}$ *and*

$$D = \left\{ x \in \mathbb{R}^N : \frac{-2\max\{b_i : i = 1, \ldots, N\}}{\min\{a_i : i = 1, \ldots, N\}} \leq x_i \leq 0, \quad \forall i \in \{1, \ldots, N\} \right\}.$$

We took $a_i = i$ *and* $b_i = N + 1 - i$. *Thus,* F *is* β*-strongly monotone and* κ*-Lipschitz continuous on* $H = \mathbb{R}^N$, *where* $\beta = 1$ *and* $\kappa = N$. *The gradient of* f *is* L*-Lipschitz continuous on* C, *where* $L = 1$ *and* ∇f *is given by* $\nabla f(x) = 4x - 2P_Q(2x)$. *Moreover,* $\Omega = \{x \in \mathbb{R}^N : -N \leq x_i \leq 0, \ \ \forall i = 1, \ldots, N\}$ *and*

$$SVIP(\Omega, F) = \left\{ \left(-N, \frac{-(N-1)}{2}, \frac{-(N-2)}{3}, \ldots, \frac{-1}{N} \right) \right\}.$$

Table 2 illustrates the numerical result of our algorithm, solving BVIPO-FM given in this example for different dimensions and different stopping criteria $\frac{\|x_n - x_{n-1}\|}{\|x_1 - x_0\|} \leq TOL$, *where the parameters are given in the following:* $\alpha_n = \frac{1}{5n-1}$, $\varepsilon_n = \frac{1}{(5n-1)^2}$, $\rho_n = \frac{1}{5}$, $\lambda_n = \frac{1}{N}$, $\beta_n = \frac{1}{2}$, $\theta_n = \bar{\theta}_n$.

Table 2. For starting points $x_0 = 100(1, \ldots, 1) \in \mathbb{R}^N$ and $x_1 = 100x_0$.

	TOL = 10^{-2}		TOL = 10^{-3}		TOL = 10^{-4}	
	Iter(n)	**CPU(s)**	**Iter(n)**	**CPU(s)**	**Iter(n)**	**CPU(s)**
$N = 2$	4	0.00345	23	0.3163	107	0.9705
$N = 10$	10	0.02217	36	0.4802	116	1.2201
$N = 100$	21	0.20681	47	0.9207	149	1.8491

For TOL= 10^{-5}, $N = 4$, $x_0 = (1, 2, 3, 4)$, *and* $x_1 = (5, 6, 7, 8)$, *the approximate solution obtained after 319 iterations is*

$$x_{319} = (-3.978599508, -1.487950389, -0.641608433, -0.24194702778).$$

6. Conclusions

We have proposed a strongly convergent inertial algorithm for a class of bilevel variational inequality problem over the intersection of the set of common fixed points of finite number of nonexpansive mappings and the set of solution points of the constrained minimization problem of real-valued convex function (BVIPO-FM). The contribution of our result in this paper is twofold. First, it provides effective way of solving BVIPO-FM, where iterative scheme combines inertial term to speed up the convergence of the algorithm. Second, our result can be applied to find a solution to the bilevel variational inequality problem over the solution set of the problem P, where the problem P (the lower level problem) can be converted as a common fixed point of a finite number of nonexpansive mappings.

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