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Upper Bound of the Third Hankel Determinant for a Subclass of Close-to-Convex Functions Associated with the Lemniscate of Bernoulli

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Abstract: In this paper, our aim is to define a new subclass of close-to-convex functions in the open unit disk $\mathbb{U}$ that are related with the right half of the lemniscate of Bernoulli. For this function class, we obtain the upper bound of the third Hankel determinant. Various other related results are also considered.

Keywords: analytic functions; close-to-convex functions; subordination; lemniscate of Bernoulli Hankel determinant

MSC: primary 05A30, 30C45; secondary 11B65, 47B38

1. Introduction

By $\mathcal{H}(\mathbb{U})$ we denote the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

where $\mathbb{C}$ is the set of complex numbers. We also let $\mathcal{A}$ be the class of analytic functions having the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall\, z \in \mathbb{U}), \tag{1}$$

and which are normalized by the following conditions:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

We denote by $\mathcal{S}$ the class of functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is called starlike in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{U}).$$

The class of all such functions is denoted by $\mathcal{S}^*$. For $f \in \mathcal{S}^*$, one can find that (see [1]):

$$|a_n| \leqq n \quad \text{for} \quad n = 2, 3, \ldots \tag{2}$$

Next, by $\mathcal{K}$, we denote the class of close-to-convex functions in $\mathbb{U}$ that satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{U}),$$

for some $g \in \mathcal{S}^*$.

An example of a function, which is close-to-convex in $\mathbb{U}$, is given by:

$$F(z) = \frac{z - e^{2i\alpha}\cos\alpha z^2}{\left(1 - e^{i\alpha}z\right)^2} \qquad (0 < \alpha < \pi)$$

which maps $\mathbb{U}$ onto the complex z-plane excluding a vertical slit (see [2] where some interesting properties of this function are obtained).

Moreover, by $\mathcal{SL}^*$, we denote the class of functions $f \in \mathcal{A}$ that satisfy the following inequality:

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1 \qquad (\forall\, z \in \mathbb{U}).$$

Thus a function $f \in \mathcal{SL}^*$ is such that $\frac{zf'(z)}{f(z)}$ lies in the region bounded by the right half of the lemniscate of Bernoulli given by the following relation:

$$\left|w^2 - 1\right| < 1,$$

where

$$w = \frac{zf'(z)}{f(z)}.$$

The above defined class was introduced by Sokół et al. (see [3]) and studied by the many authors (see, for example, [4–6]).

Next, if two functions f and g are analytic in $\mathbb{U}$, we say that the function f is subordinate to the function g and write:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$ that is analytic in $\mathbb{U}$ with:

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1,$$

such that:

$$f(z) = g\big(w(z)\big).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see, for example, [7]; see also [8]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \;\rightleftarrows\; f(0) = g(0) \;\text{ and }\; f(\mathbb{U}) \subset g(\mathbb{U}).$$

We next denote by $\mathcal{P}$ the class of analytic functions p which are normalized by $p(0) = 1$ and have the following form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{3}$$

such that:

$$\Re(p(z)) > 0 \qquad (\forall\, z \in \mathbb{U}).$$

In recent years, several interesting subclasses of analytic and multivalent functions have been introduced and investigated (see, for example, [9–16]). Motivated and inspired by recent and ongoing research, we introduce and investigate here a new subclass of close-to-convex functions in $\mathbb{U}$ which are associated with the lemniscate of Bernoulli by using some techniques similar to those that were used earlier by Sokół and Stankiewicz (see [3]).

Definition 1. *A function f of the form of Equation (1) is said to be in the class $\mathcal{KL}^*$ if and only if:*

$$\left| \left(\frac{zf'(z)}{g(z)} \right)^2 - 1 \right| < 1 \tag{4}$$

for some $g \in \mathcal{S}^*$. Equivalently, we have:

$$\frac{zf'(z)}{g(z)} \prec \sqrt{1+z} \qquad (\forall\, z \in \mathbb{U})$$

for some $g \in \mathcal{S}^*$.

Thus, clearly, a function $f \in \mathcal{KL}^*$ is such that $\frac{zf'(z)}{g(z)}$ lies in the region bounded by the right half of the lemniscate of Bernoulli given by the following relation:

$$\left| w^2 - 1 \right| < 1.$$

A closer look at the above series development of f suggests that many properties of the function f may be affected (or implied) by the size of its coefficients. The coefficient problem has been reformulated in the more special manner of estimating $|a_n|$, that is, the modulus of the nth coefficient. In 1916, Bieberbach conjectured that the nth coefficient of a univalent function is less or equal to that of the Koebe function.

Closely related to the Bieberbach conjecture is the problem of finding sharp estimates for the coefficients of odd univalent functions, which has the most general form of the square-root transformation of a function $f \in \mathcal{S}$:

$$l(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 \ldots$$

For odd univalent functions, Littlewood and Parley in 1932 proved that, for each postive integer n, the modulus $|c_{2n+1}|$ is less than an absolute constant M. For $M = 1$, the bound becomes the Littlewood–Parley conjecture.

Let $n \geqq 0$ and $q \geqq 1$. Then the qth Hankel determinant is defined as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+q-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+q-1} & \cdot & \cdot & \cdot & \cdot & a_{n+2(q-1)} \end{vmatrix}$$

The Hankel determinant plays a vital role in the theory of singularities [17] and is useful in the study of power series with integer coefficients (see [18–20]). Noteworthy, several authors obtained the sharp upper bounds on $H_2(2)$ (see, for example, [5,21–29]) for various classes of functions. It is a well-known fact for the Fekete-Szegö functional that:

$$\left|a_3 - a_2^2\right| = H_2(1).$$

This functional is further generalized as follows:

$$\left|a_3 - \mu a_2^2\right|$$

for some real or complex number μ. Fekete and Szegö gave sharp estimates of $\left|a_3 - \mu a_2^2\right|$ for μ real and $f \in \mathcal{S}$, the class of normalized univalent functions in $\mathbb{N}$. It is also known that the functional $\left|a_2 a_4 - a_3^2\right|$ is equivalent to $H_2(2)$. Babalola [30] studied the Hankel determinant $H_3(1)$ for some subclasses of analytic functions. In the present investigation, our focus is on the Hankel determinant $H_3(1)$ for the above-defined function class $\mathcal{KL}^*$.

2. A Set of Lemmas

Lemma 1. *(see [31]) Let:*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be in the class $\mathcal{P}$ of functions with positive real part in $\mathbb{U}$. Then, for any number v:

$$\left|p_2 - v p_1^2\right| \begin{cases} -4v + 2 & (v0) \\ 2 & (0v1) \\ 4v - 2 & (v1). \end{cases} \tag{5}$$

When $v < 0$ or $v > 1$, the equality holds true in Equation (5) if and only if:

$$p(z) = \frac{1+z}{1-z}$$

or one of its rotations. If $0 < v < 1$, then the equality holds true in Equation (5) if and only if:

$$p(z) = \frac{1+z^2}{1-z^2}$$

or one of its rotations. If $v = 0$, the equality holds true in Equation (5) if and only if:

$$p(z) = \left(\frac{1+\rho}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\rho}{2}\right)\frac{1-z}{1+z} \qquad (0\rho 1)$$

or one of its rotations. If $v = 1$, then the equality in Equation (5) holds true if $p(z)$ is a reciprocal of one of the functions such that the equality holds true in the case when $v = 0$.

Lemma 2. *[32,33] Let:*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be in the class $\mathcal{P}$ of functions with positive real part in $\mathbb{U}$. Then:

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right)$$

for some $x \quad (|x| \leqq 1)$ *and:*

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z$$

for some $z \quad (|z| \leqq 1)$.

Lemma 3. *[1] Let:*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be in the class $\mathcal{P}$ *of functions with positive real part in* $\mathbb{U}$. *Then:*

$$|p_k| \leqq 2 \qquad (k \in \mathbb{N}).$$

The inequality is sharp.

3. Main Results and Their Demonstrations

In this section, we will prove our main results.

Theorem 1. *Let* $f \in \mathcal{KL}^*$ *and be of the form of Equation (1). Then:*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \frac{1}{48}(62 - 75\mu) & \left(\mu < \frac{38}{75}\right) \\ \frac{1}{2} & \left(\frac{38}{75} \leqq \mu \leqq \frac{86}{75}\right) \\ \frac{1}{48}(75\mu - 62) & \left(\mu > \frac{86}{75}\right). \end{cases}$$

It is asserted also that:

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{3}\left(3\mu - \frac{38}{25}\right)|a_2|^2 \leqq \frac{1}{2} \qquad \left(\frac{38}{75} < \mu \leqq \frac{62}{75}\right)$$

and:

$$\left|a_3 - \mu a_2^2\right| + \frac{1}{3}\left(\frac{86}{25} - 3\mu\right)|a_2|^2 \leqq \frac{1}{2} \qquad \left(\frac{62}{75} < \mu \leqq \frac{86}{75}\right).$$

Proof. If $f \in \mathcal{KL}^*$, then it follows from definition that:

$$\frac{zf'(z)}{g(z)} \prec \phi(z) \qquad (\text{for some } g \in \mathcal{S}^*), \tag{6}$$

where:

$$\phi(z) = (1+z)^{\frac{1}{2}}.$$

Define a function $p(z)$ by:

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots.$$

It is clear that $p(z) \in \mathcal{P}$. This implies that:

$$w(z) = \frac{p(z) - 1}{p(z) + 1}.$$

In addition, from Equation (6), we have:

$$\frac{zf'(z)}{g(z)} \prec \phi(z)$$

with:

$$\phi(w(z)) = \left(\frac{2p(z)}{p(z)+1}\right)^{\frac{1}{2}}.$$

We now have:

$$\begin{aligned}\left(\frac{2p(z)}{p(z)+1}\right)^{\frac{1}{2}} &= 1 + \frac{1}{4}p_1 z + \left[\frac{1}{4}p_2 - \frac{5}{32}p_1^2\right]z^2 + \left[\frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3\right]z^3 \\ &+ \left[\frac{1}{4}p_4 - \frac{5}{16}p_1p_3 + \frac{39}{128}p_2p_1^2 - \frac{5}{32}p_2^2 - \frac{141}{2048}p_1^4\right]z^4 + \cdots.\end{aligned}$$

Similarly, we get:

$$\begin{aligned}\frac{zf'(z)}{g(z)} &= 1 + [2a_2 - b_2]z + \left[3a_3 - 2a_2b_2 - b_3 + b_2^2\right]z^2 \\ &+ \left[4a_4 - 2a_2b_3 - 3a_3b_2 + 2b_2b_3 + 2a_2b_2^2 - b_4 - b_2^3\right]z^3 + \cdots.\end{aligned}$$

Therefore, upon comparing the corresponding coefficients and by using Equation (2) , we find that:

$$a_2 = \frac{5}{8}p_1, \tag{7}$$

$$a_3 = \frac{1}{4}p_2 + \frac{19}{96}p_1^2 \tag{8}$$

$$a_4 = \frac{7}{48}p_3 + \frac{9}{64}p_1p_2 + \frac{91}{1536}p_1^3. \tag{9}$$

We thus obtain:

$$\left|a_3 - \mu a_2^2\right| = \frac{1}{4}\left|p_2 - \frac{1}{48}(75\mu - 38)p_1^2\right|. \tag{10}$$

Finally, by applying Lemma 1 in conjunction with Equation (10), we obtain the result asserted by Theorem 1. □

Theorem 2. *Let $f \in \mathcal{KL}^*$ and be of the form of Equation (1). Then:*

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{9105}{36416}. \tag{11}$$

Proof. Making use of Equations (7)–(9), we have:

$$\begin{aligned}a_2a_4 - a_3^2 &= \left(\frac{35}{384}p_1p_3 + \frac{45}{512}p_1^2p_2 + \frac{455}{12288}p_1^4\right) - \left(\frac{1}{4}p_2 + \frac{19}{96}p_1^2\right)^2 \\ &= \frac{35}{384}p_1p_3 - \frac{1}{16}p_2^2 - \frac{17}{1536}p_1^2p_2 - \frac{79}{36864}p_1^4 \\ &= \frac{1}{36864}\left(3360p_1p_3 - 2304p_2^2 - 408p_1^2p_2 - 79p_1^4\right).\end{aligned}$$

With the value of p_2 and p_3 from Lemma 2, using triangular inequality and replacing $|x| < 1$ by ρ and p_1 by p, we have:

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &= \frac{1}{36864}\left[19p^4 + 1680p\left(4-p^2\right) + 324\left(4-p^2\right)p^2\rho\right.\\ &\quad \left.+\rho^2\left(4-p^2\right)\left(264p^2 - 1680p + 2304\right)\right]\\ &= F\left(p,\rho\right). \end{aligned} \tag{12}$$

Differentiating Equation (12) with respect to ρ, we have:

$$\frac{\partial F}{\partial \rho} = \frac{1}{36864}\left[324\left(4-p^2\right)p^2 + 2\rho\left(4-p^2\right)\left(264p^2 - 1680p + 2304\right)\right].$$

It is clear that:

$$\frac{\partial F(p,\rho)}{\partial \rho} > 0,$$

which shows that $F(p,\rho)$ is an increasing function on the closed interval $[0,1]$. This implies that the maximum value occurs at $\rho = 1$, that is:

$$\max\{F(p,\rho)\} = F(p,1) = G(p).$$

We now have:

$$G\left(p\right) = \frac{1}{36864}\left[-569p^4 + 48p^2 + 9216\right]. \tag{13}$$

Differentiating Equation (13) with respect to p, we have:

$$G'\left(p\right) = \frac{1}{36864}\left[-2276p^3 + 96p\right]$$

Differentiating the above equation again with respect to p, we have:

$$G''\left(p\right) = \frac{1}{36864}\left[-6828p^2 + 96\right] < 0.$$

For $p = 0$, this shows that the maximum value of $G(p)$ occurs at $p = 0$. Hence we obtain:

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{9105}{36416},$$

which completes the proof of Theorem 2. □

Theorem 3. *Let $f \in \mathcal{KL}^*$ and of the form of Equation (1). Then:*

$$|a_2a_3 - a_4| \leqq \frac{7}{24}.$$

Proof. We make use of Equations (7)–(9), along with Lemma 2. Since $p_1 \leqq 2$, by Lemma 3, let $p_1 = p$ and assume without restriction that $p \in [0,2]$. Then, taking the absolute value and applying the triangle inequality with $\rho = |x|$, we obtain:

$$\begin{aligned}|a_2a_3 - a_4| &\leqq \frac{1}{1536}\left\{55p^3 + 100p\rho\left(4-p^2\right) + 112\left(4-p^2\right)\right.\\ &\quad \left.+56\rho^2\left(p-2\right)\left(4-p^2\right)\right\}\\ &=: F(\rho).\end{aligned}$$

Differentiating $F(\rho)$ with respect to ρ, we have:

$$F'(\rho) = \frac{1}{1536}\left\{100p\left(4-p^2\right)+112\rho\,(p-2)\left(4-p^2\right)\right\}.$$

For $0<\rho<1$ and fixed $p\in(0,\,2)$, it can easily be seen that:

$$\frac{\partial F}{\partial \rho}<0.$$

This shows that $F_1\,(p,\,\rho)$ is a decreasing function of ρ, which contradicts our assumption. Therefore, we have:

$$\max F\,(p,\,\rho) = F\,(p,\,0) = G\,(p)\,.$$

This implies that:

$$G'\,(p) = \frac{1}{1536}\left\{165p^2-224p\right\}$$

and:

$$G''\,(p) = \frac{1}{1536}\left\{330p-224\right\}<0$$

for $p=0$. Thus, clearly, $p=0$ is the point of maximum. Hence we get the required result asserted by Theorem 3. □

To prove Theorem 4, we need Lemma 4.

Lemma 4. *If a function f of the form of Equation (1) is in the class $\mathcal{KL}^*$, then:*

$$|a_2|\leqq\frac{5}{4},\quad |a_3|\leqq\frac{31}{24},\quad |a_4|\leqq\frac{85}{64}\quad \text{and}\quad |a_5|\leqq\frac{859}{640}.$$

These estimates are sharp.

Proof. The proof of Lemma 4 is similar to that of a known result which was proved by Sokół (see [6]). Therefore, we here choose to omit the details involved in the proof of Lemma 4. □

Theorem 4. *Let $f\in\mathcal{KL}^*$ and be of the form of Equation (1). Then:*

$$|H_3(1)|\leqq\frac{1509169}{1092480}.$$

Proof. Since:

$$|H_3(1)|\leqq|a_3|\left|\left(a_2a_4-a_3^2\right)\right|+|a_4|\,|(a_2a_3-a_4)|+|a_5|\left|\left(a_1a_3-a_2^2\right)\right|. \tag{14}$$

By Theorem 2, we have:

$$\left|a_2a_4-a_3^2\right|\leqq\frac{9105}{36416}. \tag{15}$$

In addition, by Theorem 3, we get:

$$|a_2a_3-a_4|\leqq\frac{7}{24}. \tag{16}$$

Now, using the fact that $a_1=1$, as well as Theorem 1 with $\mu=1$, Lemma 4, Equations (15) and (16) in conjunction with Equation (14), we have the required result asserted by Theorem 4. □

4. Conclusions

Using the concept of the principle of subordination, we have introduced a new subclass of close-to-convex functions in $\mathbb{U}$, associated with the limniscate of Bernoulli. We have then derived the upper bound on $H_3(1)$ for this subclass of close-to-convex functions in $\mathbb{U}$, which is associated with the limniscate of Bernoulli. Our main results are stated and proved as Theorems 1–4. These general results are motivated essentially by the earlier works which are pointed out in this presentation.

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References

1. Duren, P.L. *Univalent Functions*; Grundlehren der Mathematischen Wissenschaften, Band 259; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
2. Goodman, A.W.; Saff, E.B. Functions that are convex in one direction. *Proc. Am. Math. Soc.* **1979**, *73*, 183–187. [CrossRef]
3. Sokół, J.; Stankiewicz, J. Radius of convexity of some subclasses of strongly starlike functions. *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* **1996**, *19*, 101–105.
4. Ali, R.M.; Cho, N.E.; Ravichandran, V.; Kumar, S.S. Differential subordination for functions associated with the lemniscate of Bernoulli. *Taiwan J. Math.* **2012**, *16*, 1017–1026. [CrossRef]
5. Raza, M.; Malik, S.N. Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. *J. Inequal. Appl.* **2013**, *2013*, 412. [CrossRef]
6. Sokół, J. Coefficient estimates in a class of strongly starlike functions. *Kyungpook Math. J.* **2009**, *49*, 349–353. [CrossRef]
7. Miller, S.S.; Mocanu, P.T. Differential subordination and univalent functions. *Mich. Math. J.* **1981**, *28*, 157–171. [CrossRef]
8. Miller, S.S.; Mocanu, P.T. *Differential Subordination*: *Theory and Applications*; Series on Monographs and Textbooks in Pure and Applied Mathematics, No. 225; Marcel Dekker Incorporated: New York, NY, USA; Basel, Switzerland, 2000.
9. Srivastava, H.M.; Tahir, M.; Khan, B.; Ahmad, Q.Z.; Khan, N. Some general classes of q-starlike functions associated with the Janowski functions. *Symmetry* **2019**, *11*, 292. [CrossRef]
10. Aldweby, H.; Darus, M. On Fekete-Szegö problems for certain subclasses defined by q-derivative. *J. Funct. Spaces* **2017**, *2017*, 7156738. [CrossRef]
11. Noor, K.I.; Khan, N.; Darus, M.; Ahmad, Q.Z.; Khan, B. Some properties of analytic functions associated with conic type regions. *Intern. J. Anal. Appl.* **2018**, *16*, 689–701.
12. Hussain, S.; Khan, S.; Zaighum, M.A.; Darus, M. Applications of a q-Sălăgean type operator on multivalent functions. *J. Inequal. Appl.* **2018**, *2018*, 301. [CrossRef]
13. Kanas, S.; Srivastava, H.M. Linear operators associated with k-uniformly convex functions. *Integral Transforms Spec. Funct.* **2000**, *9*, 121–132. [CrossRef]
14. Srivastava, H.M.; Eker, S.S. Some applications of a subordination theorem for a class of analytic functions. *Appl. Math. Lett.* **2008**, *21*, 394–399. [CrossRef]
15. Rasheed, A.; Hussain, S.; Zaighum, M.A.; Darus, M. Class of analytic function related with uniformly convex and Janowski's functions. *J. Funct. Spaces* **2018**, *2018*, 4679857. [CrossRef]
16. Srivastava, H.M.; Owa, S. *Current Topics in Analytic Function Theory*; World Scientific Publishing: Hackensack, NJ, USA, 1992.
17. Dienes, P. *The Taylor Series: An Introduction to the Theory of Functions of a Complex Variable*; New York-Dover Publishing Company: Mineola, NY, USA, 1957.
18. Cantor, D.G. Power series with integral coefficients. *Bull. Am. Math. Soc.* **1963**, *69*, 362–366. [CrossRef]
19. Edrei, A. Sur les determinants recurrents et less singularities d'une fonction donee por son developpement de Taylor. *Comput. Math.* **1940**, *7*, 20–88.

20. Pólya, G.; Schoenberg, I.J. Remarks on de la Vallée Poussin means and convex conformal maps of the circle. *Pac. J. Math.* **1958**, *8*, 259–334. [CrossRef]
21. Mahmood, S.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Khan, B.; Ali, I. Upper bound of the third Hankel determinant for a subclass of q-starlike functions. *Symmetry* **2019**, *11*, 347. [CrossRef]
22. Janteng, A.; Abdulhalirn, S.; Darus, M. Coefficient inequality for a function whose derivative has positive real part. *J. Inequal. Pure Appl. Math.* **2006**, *50*, 1–5.
23. Mishra, A.K.; Gochhayat, P. Second Hankel determinant for a class of analytic functions defined by fractional derivative. *Int. J. Math. Math. Sci.* **2008**, *2008*, 153280. [CrossRef]
24. Singh, G.; Singh, G. On the second Hankel determinant for a new subclass of analytic functions. *J. Math. Sci. Appl.* **2014**, *2*, 1–3.
25. Srivastava, H.M.; Ahmad, Q.Z.; Khan, N.; Khan, N.; Khan, B. Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain. *Mathematics* **2019**, *7*, 181. [CrossRef]
26. Srivastava, H.M.; Khan, B.; Khan, N.; Ahmad, Q.Z. Coefficient inequalities for q-starlike functions associated with the Janowski functions. *Hokkaido Math. J.* **2019**, *48*, 407–425. [CrossRef]
27. Shi, L.; Srivastava, H.M.; Arif, M.; Hussain, S.; Khan, H. An investigation of the third hankel determinant problem for certain subfamilies of univalent functions involving the exponential function. *Symmetry* **2019**, *11*, 598. [CrossRef]
28. Güney, H.Ö; Murugusundaramoorthy, G.; Srivastava, H.M. The second hankel determinant for a certain class of bi-close-to-convex function. *Results Math.* **2019**, *74*, 93. [CrossRef]
29. Sun, Y.; Wang, Z.-G.; Rasila, A. On Hankel determinants for subclasses of analytic functions and close-to-convex harmonic mapping. *arXiv* **2017**, arXiv:1703.09485.
30. Babalola, K.O. On $H_3(1)$ Hankel determinant for some classes of univalent functions. *Inequal. Theory Appl.* **2007**, *6*, 1–7.
31. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis*; Li, Z., Ren, F., Yang, L., Zhang, S., Eds.; International Press: Cambridge, UK, 1994; pp. 157–169.
32. Libera, R.J.; Zlotkiewicz, E.J. Early coefficient of the inverse of a regular convex function. *Proc. Am. Math. Soc.* **1982**, *85*, 225–230. [CrossRef]
33. Libera, R.J.; Zlotkiewicz, E.J. Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$. *Proc. Am. Math. Soc.* **1983**, *87*, 251–257. [CrossRef]