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Article

Inertial-Like Subgradient Extragradient Methods for Variational Inequalities and Fixed Points of Asymptotically Nonexpansive and Strictly Pseudocontractive Mappings

Lu-Chuan Ceng [1], Adrian Petruşel [2], Ching-Feng Wen [3,4,*] and Jen-Chih Yao [5]

1 Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
2 Department of Mathematics, Babes-Bolyai University, Cluj-Napoca 400084, Romania
3 Center for Fundamental Science and Research Center for Nonliear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 80708, Taiwan
4 Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung 80708, Taiwan
5 Research Center for Interneural Computing, China Medical University Hospital, Taichung 40402, Taiwan; yaojc@mail.cmu.edu.tw
* Correspondence: cfwen@kmu.edu.tw

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Abstract: Let VIP indicate the variational inequality problem with Lipschitzian and pseudomonotone operator and let CFPP denote the common fixed-point problem of an asymptotically nonexpansive mapping and a strictly pseudocontractive mapping in a real Hilbert space. Our object in this article is to establish strong convergence results for solving the VIP and CFPP by utilizing an inertial-like gradient-like extragradient method with line-search process. Via suitable assumptions, it is shown that the sequences generated by such a method converge strongly to a common solution of the VIP and CFPP, which also solves a hierarchical variational inequality (HVI).

Keywords: inertial-like subgradient-like extragradient method with line-search process; pseudomonotone variational inequality problem; asymptotically nonexpansive mapping; strictly pseudocontractive mapping; sequentially weak continuity

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1. Introduction

Throughout this paper we assume that C is a nonempty, convex and closed subset of a real Hilbert space $(H, \|\cdot\|)$, whose inner product is denoted by $\langle\cdot,\cdot\rangle$. Moreover, let P_C denote the metric projection of H onto C.

Suppose $A : H \to H$ is a mapping. In this paper, we shall consider the following variational inequality (VI) of finding $x^* \in C$ such that

$$\langle x - x^*, Ax^*\rangle \geq 0, \quad \forall x \in C. \tag{1}$$

The set of solutions to Equation (1) is denoted by VI(C, A). In 1976, Korpelevich [1] first introduced an extragradient method, which is one of the most popular approximation ones for solving Equation (1) till now. That is, for any initial $u_0 \in C$, the sequence $\{u_n\}$ is generated by

$$\begin{cases} v_n = P_C(u_n - \tau A u_n), \\ u_{n+1} = P_C(u_n - \tau A v_n), \quad \forall n \geq 0, \end{cases} \tag{2}$$

where τ is a constant in $(0, \frac{1}{L})$ for $L > 0$ the Lipschitz constant of mapping A. In the case where VI$(C, A) \neq \emptyset$, the sequence $\{u_n\}$ constructed by Equation (2) is weakly convergent to a point in VI(C, A). Recently, light has been shed on approximation methods for solving problem Equation (1) by many researchers; see, e.g., [2–11] and references therein, to name but a few.

Let $T : C \to C$ be a mapping. We denote by Fix(T) the set of fixed points of T, i.e., Fix$(T) = \{x \in C : x = Tx\}$. T is said to be asymptotically nonexpansive if $\exists \{\theta_n\} \subset [0, +\infty)$ such that $\lim_{n\to\infty} \theta_n = 0$ and $\|T^n u - T^n v\| \leq \|u - v\| + \theta_n \|u - v\|, \forall n \geq 1, u, v \in C$. If $\theta_n \equiv 0$, then T is nonexpansive. Also, T is said to be strictly pseudocontractive if $\exists \zeta \in [0, 1)$ s.t. $\|Tu - Tv\|^2 \leq \|u - v\|^2 + \zeta \|(I - T)u - (I - T)v\|^2$, $\forall u, v \in C$. If $\zeta = 0$, then T reduces to a nonexpansive mapping. One knows that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings. Both strict pseudocontractions and nonexpansive mappings have been studied extensively by a large number of authors via iteration approximation methods; see, e.g., [12–18] and references therein.

Let the mappings $A, B : C \to H$ be both inverse-strongly monotone and let the mapping $T : C \to C$ be asymptotically nonexpansive one with a sequence $\{\theta_n\}$. Let $f : C \to C$ be a δ-contraction with $\delta \in [0, 1)$. By using a modified extragradient method, Cai et al. [19] designed a viscosity implicit rule for finding a point in the common solution set Ω of the VIs for A and B and the FPP of T, i.e., for arbitrarily given $x_1 \in C$, $\{x_n\}$ is the sequence constructed by

$$\begin{cases} u_n = s_n x_n + (1 - s_n) y_n, \\ y_n = P_C(I - \lambda A) P_C(u_n - \mu B u_n), \\ x_{n+1} = P_C[(T^n y_n - \alpha_n \rho F T^n y_n) + \alpha_n f(x_n)], \end{cases}$$

where $\{\alpha_n\}, \{s_n\} \subset (0, 1]$. Under appropriate conditions imposed on $\{\alpha_n\}, \{s_n\}$, they proved that $\{x_n\}$ is convergent strongly to an element $x^* \in \Omega$ provided $\sum_{n=1}^{\infty} \|T^{n+1} y_n - T^n y_n\| < \infty$.

In the context of extragradient techniques, one has to compute metric projections two times for each computational step. Without doubt, if C is a general convex and closed set, the computation of the projection onto C might be quite consuming-time. In 2011, inspired by Korpelevich's extragradient method, Censor et al. [20] first designed the subgradient extragradient method, where a projection onto a half-space is used in place of the second projection onto C. In 2014, Kraikaew and Saejung [21] proposed the Halpern subgradient extragradient method for solving Equation (1), and proved strong convergence of the proposed method to a solution of Equation (1).

In 2018, via the inertial technique, Thong and Hieu [22] studied the inertial subgradient extragradient method, and proved weak convergence of their method to a solution of Equation (1). Very recently, they [23] constructed two inertial subgradient extragradient algorithms with linear-search process for finding a common solution of problem Equation (1) with operator A and the FPP of operator T with demiclosedness property in a real Hilbert space, where A is Lipschitzian and monotone, and T is quasi-nonexpansive. The constructed inertial subgradient extragradient algorithms (Algorithms 1 and 2) are as below:

Algorithm 1: Inertial subgradient extragradient algorithm (I) (see [[23], Algorithm 1]).

Initialization: Given $u_0, u_1 \in H$ arbitrarily. Let $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$.

Iterative Steps: Compute u_{n+1} in what follows:

Step 1. Put $v_n = \alpha_n(u_n - u_{n-1}) + u_n$ and calculate $y_n = P_C(v_n - \tau_n A v_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying $\tau\|Av_n - Ay_n\| \le \mu\|v_n - y_n\|$.

Step 2. Calculate $z_n = P_{T_n}(v_n - \tau_n A y_n)$ with $T_n := \{x \in H : \langle x - y_n, v_n - \tau_n A v_n - y_n\rangle \le 0\}$.

Step 3. Calculate $u_{n+1} = \beta_n T z_n + (1-\beta_n)v_n$. If $v_n = z_n = u_{n+1}$ then $v_n \in \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. Set $n := n+1$ and go to Step 1.

Algorithm 2: Inertial subgradient extragradient algorithm (II) (see [[23], Algorithm 2]).

Initialization: Given $u_0, u_1 \in H$ arbitrarily. Let $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$.

Iterative Steps: Calculate u_{n+1} as follows:

Step 1. Put $v_n = \alpha_n(u_n - u_{n-1}) + u_n$ and calculate $y_n = P_C(v_n - \tau_n A v_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying $\tau\|Av_n - Ay_n\| \le \mu\|v_n - y_n\|$.

Step 2. Calculate $z_n = P_{T_n}(v_n - \tau_n A y_n)$ with $T_n := \{x \in H : \langle x - y_n, v_n - \tau_n A v_n - y_n\rangle \le 0\}$.

Step 3. Calculate $u_{n+1} = \beta_n T z_n + (1-\beta_n)u_n$. If $v_n = z_n = u_n = u_{n+1}$ then $u_n \in \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. Set $n := n+1$ and go to Step 1.

Under mild assumptions, they proved that the sequences generated by the proposed algorithms are weakly convergent to a point in $\mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. Recently, gradient-like methods have been studied extensively by many authors; see, e.g., [24–38].

Inspired by the research work of [23], we introduce two inertial-like subgradient algorithms with line-search process for solving Equation (1) with a Lipschitzian and pseudomonotone operator and the common fixed point problem (CFPP) of an asymptotically nonexpansive operator and a strictly pseudocontractive operator in H. The proposed algorithms comprehensively adopt inertial subgradient extragradient method with line-search process, viscosity approximation method, Mann iteration method and asymptotically nonexpansive mapping. Via suitable assumptions, it is shown that the sequences generated by the suggested algorithms converge strongly to a common solution of the VIP and CFPP, which also solves a hierarchical variational inequality (HVI).

2. Preliminaries

Let $x \in H$ and $\{x_n\} \subset H$. We use the notation $x_n \to x$ (resp., $x_n \rightharpoonup x$) to indicate the strong (resp., weak) convergence of $\{x_n\}$ to x. Recall that a mapping $T : C \to H$ is said to be:

(i) L-Lipschitzian (or L-Lipschitz continuous) if $\|Tx - Ty\| \le L\|x - y\|,\ \forall x, y \in C$ for some $L > 0$;
(ii) monotone if $\langle Tu - Tv, u - v\rangle \ge 0,\ \forall u, v \in C$;
(iii) pseudomonotone if $\langle Tu, v - u\rangle \ge 0 \Rightarrow \langle Tv, v - u\rangle \ge 0,\ \forall u, v \in C$;
(iv) β-strongly monotone if $\langle Tu - Tv, u - v\rangle \ge \beta\|u - v\|^2,\ \forall u, v \in C$ for some $\beta > 0$;
(v) sequentially weakly continuous if $\forall \{u_n\} \subset C$, the relation holds: $u_n \rightharpoonup u \Rightarrow Tu_n \rightharpoonup Tu$.

For metric projections, it is well known that the following assertions hold:

(i) $\langle P_C u - P_C v, u - v\rangle \ge \|P_C u - P_C v\|^2,\ \forall u, v \in H$;
(ii) $\langle u - P_C u, v - P_C u\rangle \le 0,\ \forall u \in H, v \in C$;
(iii) $\|u - v\|^2 \ge \|u - P_C u\|^2 + \|v - P_C u\|^2,\ \forall u \in H, v \in C$;
(iv) $\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle,\ \forall u, v \in H$;
(v) $\|\tau x + (1-\tau)y\|^2 = \tau\|x\|^2 + (1-\tau)\|y\|^2 - \tau(1-\tau)\|x - y\|^2,\ \forall x, y \in H, \tau \in [0,1]$.

Lemma 1. *[39] Assume that $A : C \to H$ is a continuous pseudomonotone mapping. Then $u^* \in C$ is a solution to the VI $\langle Au^*, v - u^*\rangle \geq 0$, $\forall v \in C$, iff $\langle Av, v - u^*\rangle \geq 0$, $\forall v \in C$.*

Lemma 2. *[40] Let the real sequence $\{t_n\} \subset [0, \infty)$ satisfy the conditions: $t_{n+1} \leq (1 - s_n)t_n + s_n b_n$, $\forall n \geq 1$, where $\{s_n\}$ and $\{b_n\}$ are sequences in $(-\infty, \infty)$ such that (i) $\{s_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} s_n = \infty$, and (ii) $\limsup_{n\to\infty} b_n \leq 0$ or $\sum_{n=1}^{\infty} |s_n b_n| < \infty$. Then $\lim_{n\to\infty} t_n = 0$.*

Lemma 3. *[33] Let $T : C \to C$ be a ζ-strict pseudocontraction. If the sequence $\{u_n\} \subset C$ satisfies $u_n \rightharpoonup u \in C$ and $(I - T)u_n \to 0$, then $u \in \mathrm{Fix}(T)$, where I is the identity operator of H.*

Lemma 4. *[33] Let $T : C \to C$ be a ζ-strictly pseudocontractive mapping. Let the real numbers $\gamma, \delta \geq 0$ satisfy $(\gamma + \delta)\zeta \leq \gamma$. Then $\|\gamma(x - y) + \delta(Tx - Ty)\| \leq (\gamma + \delta)\|x - y\|$, $\forall x, y \in C$.*

Lemma 5. *[41] Let the Banach space X admit a weakly continuous duality mapping, the subset $C \subset X$ be nonempty, convex and closed, and the asymptotically nonexpansive mapping $T : C \to C$ have a fixed point, i.e., $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero, i.e., if the sequence $\{u_n\} \subset C$ satisfies $u_n \rightharpoonup u \in C$ and $(I - T)u_n \to 0$, then $(I - T)u = 0$, where I is the identity mapping of X.*

3. Main Results

Unless otherwise stated, we suppose the following.

- $T : H \to H$ is an asymptotically nonexpansive operator with $\{\theta_n\}$ and $S : H \to H$ is a ζ-strictly pseudocontractive mapping.
- $A : H \to H$ is sequentially weakly continuous on C, L-Lipschitzian pseudomonotone on H, and $A(C)$ is bounded.
- $f : H \to C$ is a δ-contraction with $\delta \in [0, \frac{1}{2})$.
- $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$.
- $\{\sigma_n\} \subset [0, 1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ such that

  (i) $\sup_{n\geq 1} \frac{\sigma_n}{\alpha_n} < \infty$ and $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \geq 1$;
  (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
  (iii) $(\gamma_n + \delta_n)\zeta \leq \gamma_n < (1 - 2\delta)\delta_n$, $\forall n \geq 1$ and $\liminf_{n\to\infty}((1 - 2\delta)\delta_n - \gamma_n) > 0$;
  (iv) $\limsup_{n\to\infty} \beta_n < 1$, $\liminf_{n\to\infty} \beta_n > 0$ and $\liminf_{n\to\infty} \delta_n > 0$.

We first introduce an inertial-like subgradient extragradient algorithm (Algorithm 3) with line-search process as follows:

Algorithm 3: Inertial-like subgradient extragradient algorithm (I).

Initialization: Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$.
Iterative Steps: Compute x_{n+1} in what follows:
Step 1. Put $w_n = \sigma_n(x_n - x_{n-1}) + T^n x_n$ and calculate $y_n = P_C(I - \tau_n A)w_n$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ such that

$$\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\|.$$

Step 2. Calculate $z_n = (1 - \alpha_n)P_{C_n}(w_n - \tau_n Ay_n) + \alpha_n f(x_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n Aw_n - y_n, x - y_n\rangle \leq 0\}$.
Step 3. Calculate

$$x_{n+1} = \gamma_n P_{C_n}(w_n - \tau_n Ay_n) + \delta_n Sz_n + \beta_n T^n x_n.$$

Again set $n := n + 1$ and return to Step 1.

Lemma 6. *In Step 1 of Algorithm 3, the Armijo-like search rule*

$$\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\| \tag{3}$$

is well defined, and the inequality holds: $\min\{\gamma, \frac{\mu l}{L}\} \leq \tau_n \leq \gamma$.

Proof. Since A is L-Lipschitzian, we know that Equation (3) holds for all $\gamma l^m \leq \frac{\mu}{L}$ and so τ_n is well defined. It is clear that $\tau_n \leq \gamma$. Next we discuss two cases. In the case where $\tau_n = \gamma$, the inequality is valid. In the case where $\tau_n < \gamma$, from Equation (3) we derive $\|Aw_n - AP_C(w_n - \frac{\tau_n}{l}Aw_n)\| > \frac{\mu}{\frac{\tau_n}{l}}\|w_n - P_C(w_n - \frac{\tau_n}{l}Aw_n)\|$. Also, since A is L-Lipschitzian, we get $\tau_n > \frac{\mu l}{L}$. Therefore the inequality is true. □

Lemma 7. *Assume that* $\{w_n\}, \{y_n\}, \{z_n\}$ *are the sequences constructed by Algorithm 3. Then*

$$\begin{aligned} \|z_n - p\|^2 \quad &\leq [1 - \alpha_n(1-\delta)]\|x_n - p\|^2 + (1-\alpha_n)\Lambda_n - (1-\alpha_n)(1-\mu)\times \\ &\times [\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + 2\alpha_n\langle (f-I)p, z_n - p\rangle \quad \forall p \in \Omega, \end{aligned} \tag{4}$$

where $u_n := P_{C_n}(w_n - \tau_n Ay_n)$ *and* $\Lambda_n := \sigma_n\|x_n - x_{n-1}\|[2(1+\theta_n)\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|] + \theta_n(2+\theta_n)\|x_n - p\|^2$ *for all* $n \geq 1$.

Proof. We observe that

$$\begin{aligned} 2\|u_n - p\|^2 \quad &= 2\|P_{C_n}(w_n - \tau_n Ay_n) - P_{C_n}p\|^2 \leq 2\langle u_n - p, w_n - \tau_n Ay_n - p\rangle \\ &= \|u_n - p\|^2 + \|w_n - p\|^2 - \|u_n - w_n\|^2 - 2\langle u_n - p, \tau_n Ay_n\rangle. \end{aligned}$$

So, it follows that $\|w_n - p\|^2 - \|u_n - w_n\|^2 - 2\langle u_n - p, \tau_n Ay_n\rangle \geq \|u_n - p\|^2$. Since A is pseudomonotone, we deduce from Equation (3) that $\langle Ay_n, y_n - p\rangle \geq 0$ and

$$\begin{aligned} \|u_n - p\|^2 \quad &\leq \|w_n - p\|^2 + 2\tau_n(\langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - u_n\rangle) - \|u_n - w_n\|^2 \\ &\leq \|w_n - p\|^2 + 2\tau_n\langle Ay_n, y_n - u_n\rangle - \|u_n - w_n\|^2 \\ &= \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \tau_n Ay_n - y_n, u_n - y_n\rangle - \|u_n - y_n\|^2. \end{aligned} \tag{5}$$

Since $u_n = P_{C_n}(w_n - \tau_n Ay_n)$ with $C_n := \{x \in H : 0 \geq \langle \tau_n Aw_n - w_n + y_n, y_n - x\rangle\}$, we have $\langle u_n - y_n, w_n - \tau_n Aw_n - y_n\rangle \leq 0$, which together with Equation (3), implies that

$$\begin{aligned} 2\langle w_n - \tau_n Ay_n - y_n, u_n - y_n\rangle \quad &= 2\langle w_n - \tau_n Aw_n - y_n, u_n - y_n\rangle + 2\tau_n\langle Aw_n - Ay_n, u_n - y_n\rangle \\ &\leq 2\mu\|w_n - y_n\|\|u_n - y_n\| \leq \mu(\|w_n - y_n\|^2 + \|u_n - y_n\|^2). \end{aligned}$$

Also, from $w_n = \sigma_n(x_n - x_{n-1}) + T^n x_n$ we get

$$\begin{aligned} &\|w_n - p\|^2 = \|\sigma_n(x_n - x_{n-1}) + T^n x_n - p\|^2 \\ &\leq [(1+\theta_n)\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|]^2 \\ &= (1+\theta_n)^2\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|[2(1+\theta_n)\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|] \\ &= \|x_n - p\|^2 + \theta_n(2+\theta_n)\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|[2(1+\theta_n)\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|] \\ &= \|x_n - p\|^2 + \Lambda_n, \end{aligned}$$

where $\Lambda_n := \theta_n(2+\theta_n)\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|[2(1+\theta_n)\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|]$. Therefore, substituting the last two inequalities for Equation (5), we infer that

$$\begin{aligned}\|u_n - p\|^2 &\le \|w_n - p\|^2 - (1-\mu)\|w_n - y_n\|^2 - (1-\mu)\|u_n - y_n\|^2 \\ &\le \Lambda_n - (1-\mu)\|w_n - y_n\|^2 - (1-\mu)\|u_n - y_n\|^2 + \|x_n - p\|^2, \quad \forall p \in \Omega.\end{aligned} \tag{6}$$

In addition, from Algorithm 3 we have

$$z_n - p = (1-\alpha_n)(u_n - p) + \alpha_n(f - I)p + \alpha_n(f(x_n) - f(p)).$$

Since the function $h(t) = t^2$, $\forall t \in \mathbf{R}$ is convex, from Equation (6) we have

$$\begin{aligned}&\|z_n - p\|^2 \\ &\le [\alpha_n\delta\|x_n - p\| + (1-\alpha_n)\|u_n - p\|]^2 + 2\alpha_n\langle (f-I)p, z_n - p\rangle \\ &\le \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 + \Lambda_n - (1-\mu)\|w_n - y_n\|^2 - (1-\mu)\|u_n - y_n\|^2] \\ &\quad + 2\alpha_n\langle (f-I)p, z_n - p\rangle \\ &= [1 - \alpha_n(1-\delta)]\|x_n - p\|^2 + (1-\alpha_n)\Lambda_n - (1-\alpha_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ &\quad + 2\alpha_n\langle (f-I)p, z_n - p\rangle.\end{aligned}$$

This completes the proof. □

Lemma 8. *Assume that* $\{x_n\}, \{y_n\}, \{z_n\}$ *are bounded vector sequences constructed by Algorithm 3. If* $T^n x_n - T^{n+1}x_n \to 0$, $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$, $w_n - z_n \to 0$ *and* $\exists\{w_{n_k}\} \subset \{w_n\}$ *such that* $w_{n_k} \rightharpoonup z \in H$, *then* $z \in \Omega$.

Proof. In terms of Algorithm 3, we deduce $w_n - x_n = T^n x_n - x_n + \sigma_n(x_n - x_{n-1})$, $\forall n \ge 1$, and hence $\|T^n x_n - x_n\| \le \|w_n - x_n\| + \sigma_n\|x_n - x_{n-1}\| \le \|w_n - x_n\| + \|x_n - x_{n-1}\|$. Using the conditions $x_n - x_{n+1} \to 0$ and $w_n - x_n \to 0$, we get

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{7}$$

Combining the assumptions $w_n - x_n \to 0$ and $w_n - z_n \to 0$ yields

$$\|z_n - x_n\| \le \|w_n - z_n\| + \|w_n - x_n\| \to 0, \quad (n \to \infty).$$

Then, from Equation (4) it follows that

$$\begin{aligned}&(1-\alpha_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ &\le [1-\alpha_n(1-\delta)]\|x_n - p\|^2 + (1-\alpha_n)\Lambda_n - \|z_n - p\|^2 + 2\alpha_n\langle (f-I)p, z_n - p\rangle \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 + \Lambda_n + 2\alpha_n\|(f-I)p\|\|z_n - p\| \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + \Lambda_n + 2\alpha_n\|(f-I)p\|\|z_n - p\|,\end{aligned}$$

where $\Lambda_n := \theta_n(2+\theta_n)\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|[2(1+\theta_n)\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|]$. Since $\alpha_n \to 0$, $\Lambda_n \to 0$ and $x_n - z_n \to 0$, from the boundedness of $\{x_n\}, \{z_n\}$ we get

$$\lim_{n\to\infty} \|w_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - y_n\| = 0.$$

Thus as $n \to \infty$,

$$\|w_n - u_n\| \le \|w_n - y_n\| + \|y_n - u_n\| \to 0 \quad \text{and} \quad \|x_n - u_n\| \le \|x_n - w_n\| + \|w_n - u_n\| \to 0.$$

Furthermore, using Algorithm 3 we have $x_{n+1} - z_n = \gamma_n(u_n - z_n) + \delta_n(Sz_n - z_n) + \beta_n(T^n x_n - z_n)$, which hence implies

$$\begin{aligned}\delta_n\|Sz_n - z_n\| &= \|x_{n+1} - z_n - \beta_n(T^n x_n - z_n) - \gamma_n(u_n - z_n)\| \\ &= \|x_{n+1} - x_n + \delta_n(x_n - z_n) - \gamma_n(u_n - x_n) - \beta_n(T^n x_n - x_n)\| \\ &\leq \|x_{n+1} - x_n\| + \|x_n - z_n\| + \|u_n - x_n\| + \|T^n x_n - x_n\|.\end{aligned}$$

Note that $x_n - x_{n+1} \to 0$, $z_n - x_n \to 0$, $x_n - u_n \to 0$, $x_n - T^n x_n \to 0$ and $\liminf_{n\to\infty} \delta_n > 0$. So we obtain

$$\lim_{n\to\infty} \|z_n - Sz_n\| = 0. \tag{8}$$

Noticing $y_n = P_C(I - \tau_n A)w_n$, we have $\langle x - y_n, w_n - \tau_n Aw_n - y_n\rangle \leq 0$, $\forall x \in C$, and hence

$$\langle w_n - y_n, x - y_n\rangle + \tau_n\langle Aw_n, y_n - w_n\rangle \leq \tau_n\langle Aw_n, x - w_n\rangle, \quad \forall x \in C. \tag{9}$$

Since A is Lipschitzian, we infer from the boundedness of $\{w_{n_k}\}$ that $\{Aw_{n_k}\}$ is bounded. From $w_n - y_n \to 0$, we get the boundedness of $\{y_{n_k}\}$. Taking into account $\tau_n \geq \min\{\gamma, \frac{\mu l}{L}\}$, from Equation (9) we have $\liminf_{k\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \geq 0$, $\forall x \in C$. Moreover, note that $\langle Ay_n, x - y_n\rangle = \langle Ay_n - Aw_n, x - w_n\rangle + \langle Aw_n, x - w_n\rangle + \langle Ay_n, w_n - y_n\rangle$. Since A is L-Lipschitzian, from $w_n - y_n \to 0$ we get $Aw_n - Ay_n \to 0$. According to Equation (9) we have $\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0$, $\forall x \in C$.

We claim $x_n - Tx_n \to 0$ below. Indeed, note that

$$\begin{aligned}\|Tx_n - x_n\| &\leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - x_n\| \\ &\leq (2 + \theta_1)\|x_n - T^n x_n\| + \|T^{n+1}x_n - T^n x_n\|.\end{aligned}$$

Hence from Equation (7) and the assumption $T^n x_n - T^{n+1}x_n \to 0$ we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{10}$$

We now choose a sequence $\{\varepsilon_k\} \subset (0,1)$ such that $\varepsilon_k \downarrow 0$ as $k \to \infty$. For each $k \geq 1$, we denote by m_k the smallest natural number satisfying

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \varepsilon_k \geq 0, \quad \forall j \geq m_k.$$

From the decreasing property of $\{\varepsilon_k\}$, it is easy to see that $\{m_k\}$ is increasing. Considering that $\{y_{m_k}\} \subset C$ implies $Ay_{m_k} \neq 0$, $\forall k \geq 1$, we put

$$\mu_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}.$$

So we have $\langle Ay_{m_k}, \mu_{m_k}\rangle = 1$, $\forall k \geq 1$. Thus, from Equation (9), we have $\langle x + \varepsilon_k\mu_{m_k} - y_{m_k}, Ay_{m_k}\rangle \geq 0$, $\forall k \geq 1$. Also, since A is pseudomonotone, we get

$$\langle A(x + \varepsilon_k\mu_{m_k}), x + \varepsilon_k\mu_{m_k} - y_{m_k}\rangle \geq 0, \quad \forall k \geq 1.$$

Consequently,

$$\langle x - y_{m_k}, Ax\rangle \geq \langle x + \varepsilon_k\mu_{m_k} - y_{m_k}, Ax - A(x + \varepsilon_k\mu_{m_k})\rangle - \varepsilon_k\langle \mu_{m_k}, Ax\rangle, \quad \forall k \geq 1. \tag{11}$$

We show $\lim_{k\to\infty} \varepsilon_k \mu_{m_k} = 0$. In fact, since $w_{n_k} \rightharpoonup z$ and $w_n - y_n \to 0$, we get $y_{n_k} \rightharpoonup z$. So, $\{y_n\} \subset C$ guarantees $z \in C$. Also, since A is sequentially weakly continuous on C, we deduce that $Ay_{n_k} \rightharpoonup Az$. So, we get $Az \neq 0$. It follows that $0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|$. Since $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $\varepsilon_k \downarrow 0$ as $k \to \infty$, we obtain that

$$0 \leq \limsup_{k\to\infty} \|\varepsilon_k \mu_{m_k}\| = \limsup_{k\to\infty} \frac{\varepsilon_k}{\|Ay_{m_k}\|} \leq \frac{\limsup_{k\to\infty} \varepsilon_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0.$$

Thus $\varepsilon_k \mu_{m_k} \to 0$.

The last step is to show $z \in \Omega$. Indeed, we have $x_{n_k} \rightharpoonup z$. From Equation (10) we also have $x_{n_k} - Tx_{n_k} \to 0$. Note that Lemma 5 yields the demiclosedness of $I - T$ at zero. Thus $z \in \mathrm{Fix}(T)$. Moreover, since $w_n - z_n \to 0$ and $w_{n_k} \rightharpoonup z$, we have $z_{n_k} \rightharpoonup z$. From Equation (8) we get $z_{n_k} - Sz_{n_k} \to 0$. By Lemma 5 we know that $I - S$ is demiclosed at zero, and hence we have $(I - S)z = 0$, i.e., $z \in \mathrm{Fix}(S)$. In addition, taking $k \to \infty$, we infer that the right hand side of Equation (11) converges to zero by the Lipschitzian property of A, the boundedness of $\{y_{m_k}\}$, $\{\mu_{m_k}\}$, and the limit $\lim_{k\to\infty} \varepsilon_k \mu_{m_k} = 0$. Therefore, $\langle Ax, x - z\rangle = \liminf_{k\to\infty} \langle Ax, x - y_{m_k}\rangle \geq 0$, $\forall x \in C$. From Lemma 3 we get $z \in \mathrm{VI}(C, A)$, and hence $z \in \Omega$. This completes the proof. □

Theorem 1. *Let $\{x_n\}$ be the sequence constructed by Algorithm 3. Suppose that $T^n x_n - T^{n+1} x_n \to 0$. Then*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} x_n - x_{n+1} \to 0, \\ x_n - T^n x_n \to 0, \\ \sup_{n\geq 1} \|(T^n - f)x_n\| < \infty, \end{cases}$$

where $x^* \in \Omega$ is only a solution of the HVI: $\langle (f - I)x^*, p - x^*\rangle \leq 0$, $\forall p \in \Omega$.

Proof. Without loss of generality, we may assume that $\{\beta_n\} \subset [a, b] \subset (0, 1)$. We can claim that $P_\Omega \circ f$ is a contractive map. Banach's Contraction Principle ensures that it has a unique fixed point, i.e., $P_\Omega f(x^*) = x^*$. So, there exists a unique solution $x^* \in \Omega$ to the HVI

$$\langle (I - f)x^*, p - x^*\rangle \geq 0, \quad \forall p \in \Omega. \tag{12}$$

It is clear that the necessity of the theorem is valid. In fact, if $x_n \to x^* \in \Omega$, then as $n \to \infty$, we obtain that $\|x_n - x_{n+1}\| \to 0$, $\|x_n - T^n x_n\| \leq \|x_n - x^*\| + \|x^* - T^n x_n\| \leq (2 + \theta_n)\|x_n - x^*\| \to 0$, and

$$\begin{aligned} \sup_{n\geq 1} \|T^n x_n - f(x_n)\| &\leq \sup_{n\geq 1} (\|T^n x_n - x^*\| + \|x^* - f(x^*)\| + \|f(x^*) - f(x_n)\|) \\ &\leq \sup_{n\geq 1} [(1 + \theta_n)\|x_n - x^*\| + \|x^* - f(x^*)\| + \delta\|x^* - x_n\|] \\ &\leq \sup_{n\geq 1} [(2 + \theta_n)\|x_n - x^*\| + \|x^* - f(x^*)\|] < \infty. \end{aligned}$$

We now assume that $\lim_{n\to\infty} (\|x_n - x_{n+1}\| + \|x_n - T^n x_n\|) = 0$ and $\sup_{n\geq 1} \|(T^n - f)x_n\| < \infty$, and prove the sufficiency by the following steps.

Step 1. We claim the boundedness of $\{x_n\}$. In fact, take a fixed $p \in \Omega$ arbitrarily. From Equation (6) we get

$$\|w_n - p\|^2 - (1 - \mu)\|w_n - y_n\|^2 - (1 - \mu)\|u_n - y_n\|^2 \geq \|u_n - p\|^2, \tag{13}$$

which hence yields

$$\|w_n - p\| \geq \|u_n - p\|, \quad \forall n \geq 1. \tag{14}$$

By the definition of w_n, we have

$$\begin{aligned} \|w_n - p\| &\le (1+\theta_n)\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\| \\ &= (1+\theta_n)\|x_n - p\| + \alpha_n \cdot \tfrac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|. \end{aligned} \tag{15}$$

From $\sup_{n\ge1} \frac{\sigma_n}{\alpha_n} < \infty$ and $\sup_{n\ge1} \|x_n - x_{n-1}\| < \infty$, we deduce that $\sup_{n\ge1} \frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\| < \infty$, which immediately implies that $\exists M_1 > 0$ s.t.

$$M_1 \ge \frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|, \quad \forall n \ge 1. \tag{16}$$

From Equations (14)–(16), we obtain

$$\|u_n - p\| \le \|w_n - p\| \le (1+\theta_n)\|x_n - p\| + \alpha_n M_1, \quad \forall n \ge 1. \tag{17}$$

Note that $A(C)$ is bounded, $y_n = P_C(I - \tau_n)Aw_n$, $f(H) \subset C \subset C_n$ and $u_n = P_{C_n}(w_n - \tau_n Ay_n)$. Hence, we know that $\{Ay_n\}$ is a bounded sequence. So, from $\sup_{n\ge1} \|(T^n - f)x_n\| < \infty$, it follows that

$$\begin{aligned} \|u_n - f(x_n)\| &= \|P_{C_n}(w_n - \tau_n Ay_n) - P_{C_n} f(x_n)\| \le \|w_n - \tau_n Ay_n - f(x_n)\| \\ &\le \|w_n - T^n x_n\| + \|T^n x_n - f(x_n)\| + \tau_n\|Ay_n\| \\ &\le \|x_n - x_{n-1}\| + \|(T^n - f)x_n\| + \gamma\|Ay_n\| \le M_0, \end{aligned}$$

where $\sup_{n\ge1}(\|x_n - x_{n-1}\| + \|(T^n - f)x_n\| + \gamma\|Ay_n\|) \le M_0$ for some $M_0 > 0$. Taking into account $\lim_{n\to\infty} \frac{\theta_n(2+\theta_n)}{\alpha_n(1-\beta_n)} = 0$, we know that $\exists n_0 \ge 1$ such that

$$\theta_n(2+\theta_n) \le \frac{\alpha_n(1-\beta_n)(1-\delta)}{2} \ (\le \frac{\alpha_n(1-\delta)}{2}), \quad \forall n \ge n_0.$$

So, from Algorithm 3 and Equation (17) it follows that for all $n \ge n_0$,

$$\begin{aligned} \|z_n - p\| &\le \alpha_n\delta\|x_n - p\| + (1-\alpha_n)\|u_n - p\| + \alpha_n\|(f - I)p\| \\ &\le [1 - \alpha_n(1-\delta) + \theta_n]\|x_n - p\| + \alpha_n(M_1 + \|(f - I)p\|) \\ &\le [1 - \tfrac{\alpha_n(1-\delta)}{2}]\|x_n - p\| + \alpha_n(M_1 + \|(f - I)p\|), \end{aligned}$$

which together with Lemma 4 and $(\gamma_n + \delta_n)\zeta \le \gamma_n$, implies that for all $n \ge n_0$,

$$\begin{aligned} &\|x_{n+1} - p\| = \|\beta_n(T^n x_n - p) + \gamma_n(z_n - p) + \delta_n(Sz_n - p) + \gamma_n(u_n - z_n)\| \\ &\le \beta_n(1+\theta_n)\|x_n - p\| + (1-\beta_n)\|z_n - p\| + \gamma_n\alpha_n\|u_n - f(x_n)\| \\ &\le \beta_n(1+\theta_n)\|x_n - p\| + (1-\beta_n)[(1 - \tfrac{\alpha_n(1-\delta)}{2})\|x_n - p\| + \alpha_n(M_0 + M_1 + \|(f - I)p\|)] \\ &\le [1 - \tfrac{\alpha_n(1-\beta_n)(1-\delta)}{2} + \beta_n\tfrac{\alpha_n(1-\beta_n)(1-\delta)}{2}]\|x_n - p\| + \alpha_n(1-\beta_n)(M_0 + M_1 + \|(f - I)p\|) \\ &= [1 - \tfrac{\alpha_n(1-\beta_n)^2(1-\delta)}{2}]\|x_n - p\| + \tfrac{\alpha_n(1-\beta_n)^2(1-\delta)}{2} \cdot \tfrac{2(M_0 + M_1 + \|(f - I)p\|)}{(1-\delta)(1-\beta_n)}. \end{aligned}$$

By induction, we obtain $\|x_n - p\| \le \max\{\|x_{n_0} - p\|, \frac{2(M_0 + M_1 + \|(f - I)p\|)}{(1-\delta)(1-b)}\}, \forall n \ge n_0$. Therefore, we derive the boundedness of $\{x_n\}$ and hence the one of sequences $\{u_n\}, \{w_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}, \{Sz_n\}, \{T^n x_n\}$.

Step 2. We claim that $\exists M_4 > 0$ s.t.

$$(1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4, \quad \forall n \ge n_0.$$

In fact, using Lemmas 4 and 7 and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned}
&\|x_{n+1}-p\|^2 = \|\beta_n(T^n x_n - p) + \gamma_n(z_n - p) + \delta_n(Sz_n - p) + \gamma_n(u_n - z_n)\|^2 \\
&\le \beta_n\|T^n x_n - p\|^2 + (1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n-p) + \delta_n(Sz_n - p)]\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n\|u_n - f(x_n)\|\|x_{n+1}-p\| \\
&\le \beta_n\|T^n x_n - p\|^2 + (1-\beta_n)\{[1-\alpha_n(1-\delta)]\|x_n-p\|^2 + (1-\alpha_n)\Lambda_n \\
&\quad - (1-\alpha_n)(1-\mu)[\|w_n-y_n\|^2 + \|u_n-y_n\|^2] + 2\alpha_n\langle (f-I)p, z_n-p\rangle\} \\
&\quad + 2(1-\beta_n)\alpha_n\|u_n - f(x_n)\|\|x_{n+1}-p\| \\
&\le \beta_n\|T^n x_n - p\|^2 + (1-\beta_n)\{[1-\alpha_n(1-\delta)]\|x_n-p\|^2 + (1-\alpha_n)\Lambda_n \\
&\quad - (1-\alpha_n)(1-\mu)[\|w_n-y_n\|^2 + \|u_n-y_n\|^2] + \alpha_n M_2\},
\end{aligned} \tag{18}$$

where

$$\Lambda_n := \theta_n(2+\theta_n)\|x_n-p\|^2 + \sigma_n\|x_n-x_{n-1}\|[2(1+\theta_n)\|x_n-p\| + \sigma_n\|x_n-x_{n-1}\|],$$

and

$$\sup_{n\ge 1} 2(\|(f-I)p\|\|z_n-p\| + \|u_n - f(x_n)\|\|x_{n+1}-p\|) \le M_2$$

for some $M_2 > 0$. Also, from Equation (16) we have

$$\begin{aligned}
\Lambda_n &= \theta_n(2+\theta_n)\|x_n-p\|^2 + \sigma_n\|x_n-x_{n-1}\|[2(1+\theta_n)\|x_n-p\| + \sigma_n\|x_n-x_{n-1}\|] \\
&\le \theta_n(2+\theta_n)\|x_n-p\|^2 + \alpha_n M_1\|[2(1+\theta_n)\|x_n-p\| + \alpha_n M_1] \\
&= \alpha_n\{\tfrac{\theta_n}{\alpha_n}(2+\theta_n)\|x_n-p\|^2 + M_1\|[2(1+\theta_n)\|x_n-p\| + \alpha_n M_1]\} \le \alpha_n M_3,
\end{aligned} \tag{19}$$

where

$$\sup_{n\ge 1}\{\frac{\theta_n}{\alpha_n}(2+\theta_n)\|x_n-p\|^2 + M_1\|[2(1+\theta_n)\|x_n-p\| + \alpha_n M_1]\} \le M_3$$

for some $M_3 > 0$. Note that

$$\theta_n(2+\theta_n) \le \frac{\alpha_n(1-\beta_n)(1-\delta)}{2}, \quad \forall n \ge n_0.$$

Substituting Equation (19) for Equation (18), we obtain that for all $n \ge n_0$,

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n(1+\theta_n)^2\|x_n-p\|^2 + (1-\beta_n)\{[1-\alpha_n(1-\delta)]\|x_n-p\|^2 + (1-\alpha_n)\alpha_n M_3 \\
&\quad - (1-\alpha_n)(1-\mu)[\|w_n-y_n\|^2 + \|u_n-y_n\|^2] + \alpha_n M_2\} \\
&\le [1 - \tfrac{\alpha_n(1-\beta_n)(1-\delta)}{2}]\|x_n-p\|^2 + \alpha_n M_3 \\
&\quad - (1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n-y_n\|^2 + \|u_n-y_n\|^2] + \alpha_n M_2 \\
&\le \|x_n-p\|^2 - (1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n-y_n\|^2 + \|u_n-y_n\|^2] + \alpha_n M_4,
\end{aligned}$$

where $M_4 := M_2 + M_3$. This immediately implies that for all $n \ge n_0$,

$$(1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n-y_n\|^2 + \|u_n-y_n\|^2] \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n M_4. \tag{20}$$

Step 3. We claim that $\exists M > 0$ s.t.

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\le [1 - \tfrac{(1-2\delta)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n]\|x_n-p\|^2 + \tfrac{[(1-2\delta)\delta_n - \gamma_n]\alpha_n}{1-\alpha_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_{n+1}\| \\
&\quad + \tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_n\| + \tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\langle f(p)-p, x_n-p\rangle \\
&\quad + \tfrac{\gamma_n+\delta_n}{(1-2\delta)\delta_n-\gamma_n}(\tfrac{\theta_n}{\alpha_n}\cdot\tfrac{2M^2}{1-b} + \tfrac{\sigma_n}{\alpha_n}\|x_n-x_{n-1}\|3M)\}.
\end{aligned}$$

In fact, we get

$$\begin{aligned} \|w_n - p\|^2 &\le [(1+\theta_n)\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|]^2 \\ &= \|x_n - p\|^2 + \theta_n(2+\theta_n)\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|[2(1+\theta_n)\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|] \\ &\le \|x_n - p\|^2 + \theta_n 2M^2 + \sigma_n\|x_n - x_{n-1}\|3M, \end{aligned} \tag{21}$$

where $M \ge \sup_{n\ge1}\{(1+\theta_n)\|x_n - p\|, \sigma_n\|x_n - x_{n-1}\|\}$ for some $M > 0$. From Algorithm 3 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\beta_n(T^n x_n - p) + \gamma_n(z_n - p) + \delta_n(Sz_n - p) + \gamma_n(u_n - z_n)\|^2 \\ &\le \|\beta_n(T^n x_n - p) + \gamma_n(z_n - p) + \delta_n(Sz_n - p)\|^2 + 2\gamma_n\alpha_n\langle u_n - f(x_n), x_{n+1} - p\rangle \\ &\le \beta_n\|T^n x_n - p\|^2 + (1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(Sz_n - p)]\|^2 \\ &\quad + 2\gamma_n\alpha_n\langle u_n - p, x_{n+1} - p\rangle + 2\gamma_n\alpha_n\langle p - f(x_n), x_{n+1} - p\rangle, \end{aligned}$$

which together with Lemma 4, leads to

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \beta_n(1+\theta_n)^2\|x_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 + 2\gamma_n\alpha_n\|u_n - p\|\|x_{n+1} - p\| \\ &\quad + 2\gamma_n\alpha_n\langle p - f(x_n), x_{n+1} - p\rangle \\ &\le \beta_n(1+\theta_n)^2\|x_n - p\|^2 + (1-\beta_n)[(1-\alpha_n)\|u_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, z_n - p\rangle] \\ &\quad + \gamma_n\alpha_n(\|u_n - p\|^2 + \|x_{n+1} - p\|^2) + 2\gamma_n\alpha_n\langle p - f(x_n), x_{n+1} - p\rangle. \end{aligned}$$

From Equations (17) and (21) we know that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \theta_n 2M^2 + \sigma_n\|x_n - x_{n-1}\|3M.$$

Hence, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 \quad &\le [1 - \alpha_n(1-\beta_n)]\|x_n - p\|^2 + \beta_n\theta_n 2M^2 + (1-\beta_n)(1-\alpha_n)(\theta_n 2M^2 \\ &\quad + \sigma_n\|x_n - x_{n-1}\|3M) + 2\alpha_n\delta_n\langle f(x_n) - p, z_n - p\rangle + \gamma_n\alpha_n(\|x_n - p\|^2 \\ &\quad + \|x_{n+1} - p\|^2) + (1-\beta_n)\alpha_n(\theta_n 2M^2 + \sigma_n\|x_n - x_{n-1}\|3M) \\ &\quad + 2\gamma_n\alpha_n\langle f(x_n) - p, z_n - x_{n+1}\rangle \\ &\le [1 - \alpha_n(1-\beta_n)]\|x_n - p\|^2 + 2\gamma_n\alpha_n\|f(x_n) - p\|\|z_n - x_{n+1}\| \\ &\quad + 2\alpha_n\delta_n\delta\|x_n - p\|^2 + 2\alpha_n\delta_n\langle f(p) - p, x_n - p\rangle + 2\alpha_n\delta_n\|f(x_n) - p\|\|z_n - x_n\| \\ &\quad + \gamma_n\alpha_n(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + (1-\beta_n)(\tfrac{\theta_n 2M^2}{1-\beta_n} + \sigma_n\|x_n - x_{n-1}\|3M), \end{aligned}$$

which immediately yields

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\le [1 - \tfrac{(1-2\delta)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n]\|x_n - p\|^2 + \tfrac{[(1-2\delta)\delta_n - \gamma_n]\alpha_n}{1-\alpha_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - p\|\|z_n - x_{n+1}\| \\ &\quad + \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - p\|\|z_n - x_n\| + \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\langle f(p) - p, x_n - p\rangle \\ &\quad + \tfrac{\gamma_n + \delta_n}{(1-2\delta)\delta_n - \gamma_n}(\tfrac{\theta_n}{\alpha_n}\cdot\tfrac{2M^2}{1-b} + \tfrac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|3M)\}. \end{aligned} \tag{22}$$

Step 4. We claim the strong convergence of $\{x_n\}$ to a unique solution $x^* \in \Omega$ to the HVI Equation (12). In fact, setting $p = x^*$, from Equation (22) we know that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le [1 - \tfrac{(1-2\delta)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n]\|x_n - x^*\|^2 + \tfrac{[(1-2\delta)\delta_n-\gamma_n]\alpha_n}{1-\alpha_n\gamma_n} \cdot \{\tfrac{2\gamma_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n) - x^*\|\|z_n - x_{n+1}\| \\
&\quad + \tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n) - x^*\|\|z_n - x_n\| + \tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\langle f(x^*) - x^*, x_n - x^*\rangle \\
&\quad + \tfrac{\gamma_n+\delta_n}{(1-2\delta)\delta_n-\gamma_n}(\tfrac{\theta_n}{\alpha_n} \cdot \tfrac{2M^2}{1-b} + \tfrac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|3M)\}.
\end{aligned}$$

According to Lemma 4, it is sufficient to prove that $\limsup_{n\to\infty}\langle (f-I)x^*, x_n - x^*\rangle \le 0$. Since $x_n - x_{n+1} \to 0$, $\alpha_n \to 0$ and $\{\beta_n\} \subset [a,b] \subset (0,1)$, from Equation (20) we get

$$\begin{aligned}
&\limsup_{n\to\infty}(1-\alpha_n)(1-b)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\
&\le \limsup_{n\to\infty}[\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4] \\
&\le \limsup_{n\to\infty}(\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| = 0,
\end{aligned}$$

which hence leads to

$$\lim_{n\to\infty}\|w_n - y_n\| = \lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{23}$$

Obviously, the assumptions $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ guarantee that $\|w_n - x_n\| \le \|T^n x_n - x_n\| + \|x_n - x_{n-1}\| \to 0$ $(n \to \infty)$. Thus,

$$\|x_n - y_n\| \le \|x_n - w_n\| + \|w_n - y_n\| \to 0, \quad (n \to \infty).$$

Since $z_n = (1-\alpha_n)u_n + \alpha_n f(x_n)$ with $u_n := P_{C_n}(w_n - \tau_n A y_n)$, from Equation (23) and the boundedness of $\{x_n\}$, $\{u_n\}$, we get

$$\|z_n - y_n\| \le \alpha_n(\|f(x_n)\| + \|u_n\|) + \|u_n - y_n\| \to 0, \quad (n \to \infty), \tag{24}$$

and hence

$$\|z_n - x_n\| \le \|z_n - y_n\| + \|y_n - x_n\| \to 0, \quad (n \to \infty).$$

Obviously, combining Equations (23) and (24) guarantees that

$$\|w_n - z_n\| \le \|w_n - y_n\| + \|y_n - z_n\| \to 0, \quad (n \to \infty).$$

Since $\{x_n\}$ is bounded, we know that $\exists \{x_{n_k}\} \subset \{x_n\}$ s.t.

$$\limsup_{n\to\infty}\langle (f-I)x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle (f-I)x^*, x_{n_k} - x^*\rangle. \tag{25}$$

Next, we may suppose that $x_{n_k} \rightharpoonup \tilde{x}$. Hence from Equation (25) we get

$$\limsup_{n\to\infty}\langle (f-I)x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle (f-I)x^*, x_{n_k} - x^*\rangle = \langle (f-I)x^*, \tilde{x} - x^*\rangle. \tag{26}$$

From $w_n - x_n \to 0$ and $x_{n_k} \rightharpoonup \tilde{x}$ it follows that $w_{n_k} \rightharpoonup \tilde{x}$.

Since $T^n x_n - T^{n+1}x_n \to 0$, $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$, $w_n - z_n \to 0$ and $w_{n_k} \rightharpoonup \tilde{x}$, from Lemma 8 we conclude that $\tilde{x} \in \Omega$. Therefore, from Equations (12) and (26) we infer that

$$\limsup_{n\to\infty} \langle (f-I)x^*, x_n - x^* \rangle = \langle (f-I)x^*, \tilde{x} - x^* \rangle \leq 0.$$

Note that

$$\sum_{n=0}^{\infty} \frac{(1-2\delta)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n = \infty.$$

It is clear that

$$\limsup_{n\to\infty}\{ \tfrac{2\gamma_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-x^*\|\|z_n - x_{n+1}\| + \tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-x^*\|\|z_n - x_n\| \\ + \tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\langle f(x^*)-x^*, x_n - x^*\rangle + \tfrac{\gamma_n+\delta_n}{(1-2\delta)\delta_n-\gamma_n}(\tfrac{\theta_n}{\alpha_n}\cdot\tfrac{2M^2}{1-b} + \tfrac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|3M)\} \leq 0.$$

Consequently, all conditions of Lemma 4 are satisfied, and hence we immediately deduce that $x_n \to x^*$. This completes the proof. □

Next, we introduce another inertial-like subgradient extragradient algorithm (Algorithm 4) with line-search process as the following.

It is remarkable that Lemmas 6–8 are still valid for Algorithm 4.

Algorithm 4: Inertial-like subgradient extragradient algorithm (II).

Initialization: Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$.

Iterative Steps: Compute x_{n+1} in what follows:

Step 1. Put $w_n = \sigma_n(x_n - x_{n-1}) + T^n x_n$ and calculate $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ such that

$$\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\|.$$

Step 2. Calculate $z_n = (1-\alpha_n)P_{C_n}(w_n - \tau_n A y_n) + \alpha_n f(x_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \leq 0\}$.

Step 3. Calculate

$$x_{n+1} = \gamma_n P_{C_n}(w_n - \tau_n A y_n) + \delta_n S z_n + \beta_n T^n w_n.$$

Again set $n := n+1$ and return to Step 1.

Theorem 2. *Let $\{x_n\}$ be the sequence constructed by Algorithm 4. Suppose that $T^n x_n - T^{n+1}x_n \to 0$. Then*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} x_n - x_{n+1} \to 0, \\ x_n - T^n x_n \to 0, \\ \sup_{n\geq 1}\|(T^n - f)x_n\| < \infty, \end{cases}$$

where $x^* \in \Omega$ is only a solution of the HVI: $\langle (I-f)x^*, p - x^*\rangle \geq 0$, $\forall p \in \Omega$.

Proof. Using the same reasoning as in the proof of Theorem 1, we know that there is only a solution $x^* \in \Omega$ of Equation (12), and that the necessity of the theorem is true.

We claim the sufficiency of the theorem below. For the purpose, we suppose that $\lim_{n\to\infty}(\|x_n - x_{n+1}\| + \|x_n - T^n x_n\|) = 0$ and $\sup_{n\geq 1}\|(T^n - f)x_n\| < \infty$. Then we prove the sufficiency by the following steps.

Step 1. We claim the boundedness of $\{x_n\}$. In fact, using the same reasoning as in Step 1 of the proof of Theorem 1, we obtain that inequalities Equations (13)–(17) hold. Noticing $\lim_{n\to\infty}\frac{\theta_n(2+\theta_n)}{\alpha_n(1-\beta_n)} = 0$, we infer that $\exists n_0 \geq 1$ s.t.

$$\theta_n(2+\theta_n) \leq \frac{\alpha_n(1-\beta_n)(1-\delta)}{2} \; (\leq \frac{\alpha_n(1-\delta)}{2}), \quad \forall n \geq n_0.$$

So, from Algorithm 4 and Equation (17) it follows that for all $n \geq n_0$,

$$\begin{aligned}\|z_n - p\| &\leq \alpha_n\delta\|x_n - p\| + (1-\alpha_n)[(1+\theta_n)\|x_n - p\| + \alpha_n M_1] + \alpha_n\|(f-I)p\| \\ &\leq [1 - \tfrac{\alpha_n(1-\delta)}{2}]\|x_n - p\| + \alpha_n(M_1 + \|(f-I)p\|),\end{aligned}$$

which together with Lemma 4 and $(\gamma_n + \delta_n)\zeta \leq \gamma_n$, implies that for all $n \geq n_0$,

$$\begin{aligned}&\|x_{n+1} - p\| = \|\beta_n(T^n w_n - p) + \gamma_n(z_n - p) + \delta_n(Sz_n - p) + \gamma_n(u_n - z_n)\| \\ &\leq \beta_n(1+\theta_n)\|w_n - p\| + (1-\beta_n)\|z_n - p\| + \gamma_n\alpha_n\|u_n - f(x_n)\| \\ &\leq [1 - \tfrac{\alpha_n(1-\beta_n)(1-\delta)}{2} + \beta_n\theta_n(2+\theta_n)]\|x_n - p\| + \beta_n(1+\theta_n)\alpha_n M_1 \\ &\quad + \alpha_n(1-\beta_n)(M_0 + M_1 + \|(f-I)p\|) \\ &\leq [1 - \tfrac{\alpha_n(1-\beta_n)(1-\delta)}{2} + \beta_n\tfrac{\alpha_n(1-\beta_n)(1-\delta)}{2}]\|x_n - p\| + \alpha_n(1-\beta_n)(M_0 + M_1\tfrac{1+\theta_n}{1-\beta_n} + \|(f-I)p\|) \\ &= [1 - \tfrac{\alpha_n(1-\beta_n)^2(1-\delta)}{2}]\|x_n - p\| + \tfrac{\alpha_n(1-\beta_n)^2(1-\delta)}{2} \cdot \tfrac{2(M_0 + M_1\frac{1+\theta_n}{1-\beta_n} + \|(f-I)p\|)}{(1-\delta)(1-\beta_n)}.\end{aligned}$$

Hence,

$$\|x_n - p\| \leq \max\{\|x_{n_0} - p\|, \frac{2(M_0 + M_1\frac{2}{1-b} + \|(f-I)p\|)}{(1-\delta)(1-b)}\}, \quad \forall n \geq n_0.$$

Thus, sequence $\{x_n\}$ is bounded.

Step 2. We claim that for all $n \geq n_0$,

$$\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4 \geq (1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2],$$

with constant $M_4 > 0$. Indeed, utilizing Lemmas 4 and 7 and the convexity of $\|\cdot\|^2$, one reaches

$$\begin{aligned}&\|x_{n+1} - p\|^2 = \|\beta_n(T^n w_n - p) + \gamma_n(z_n - p) + \delta_n(Sz_n - p) + \gamma_n(u_n - z_n)\|^2 \\ &\leq \beta_n\|T^n w_n - p\|^2 + (1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(Sz_n - p)]\|^2 \\ &\quad + 2(1-\beta_n)\alpha_n\|u_n - f(x_n)\|\|x_{n+1} - p\| \\ &\leq \beta_n(1+\theta_n)^2\|w_n - p\|^2 + (1-\beta_n)\{[1 - \alpha_n(1-\delta)]\|x_n - p\|^2 + (1-\alpha_n)\Lambda_n \\ &\quad - (1-\alpha_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + 2\alpha_n\langle (f-I)p, z_n - p\rangle\} \\ &\quad + 2(1-\beta_n)\alpha_n\|u_n - f(x_n)\|\|x_{n+1} - p\| \\ &\leq \beta_n(1+\theta_n)^2(\|x_n - p\|^2 + \Lambda_n) + (1-\beta_n)\{[1 - \alpha_n(1-\delta)]\|x_n - p\|^2 + (1-\alpha_n)\Lambda_n \\ &\quad - (1-\alpha_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + \alpha_n M_2\},\end{aligned} \tag{27}$$

where $\Lambda_n := \theta_n(2+\theta_n)\|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|[2(1+\theta_n)\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|]$, and $\sup_{n\geq 1} 2(\|(f-I)p\|\|z_n-p\|+\|u_n-f(x_n)\|\|x_{n+1}-p\|) \leq M_2$ for some $M_2>0$. Also, from Equation (16) we have

$$\begin{aligned}\Lambda_n &= \theta_n(2+\theta_n)\|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|[2(1+\theta_n)\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|]\\ &\leq \alpha_n\{\tfrac{\theta_n}{\alpha_n}(2+\theta_n)\|x_n-p\|^2+M_1\|[2(1+\theta_n)\|x_n-p\|+\alpha_n M_1]\}\leq \alpha_n M_3,\end{aligned}\tag{28}$$

where $\sup_{n\geq 1}\{\frac{\theta_n}{\alpha_n}(2+\theta_n)\|x_n-p\|^2+M_1\|[2(1+\theta_n)\|x_n-p\|+\alpha_n M_1]\}\leq M_3$ for some $M_3>0$. Note that $\theta_n(2+\theta_n)\leq\frac{\alpha_n(1-\beta_n)(1-\delta)}{2}$, $\forall n\geq n_0$. Substituting Equation (28) for Equation (27), we obtain that for all $n\geq n_0$,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq [1-\alpha_n(1-\beta_n)(1-\delta)+\beta_n\theta_n(2+\theta_n)]\|x_n-p\|^2+\beta_n(1+\theta_n)^2\alpha_n M_3\\ &\quad+(1-\beta_n)(1-\alpha_n)\alpha_n M_3-(1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n-y_n\|^2\\ &\quad+\|u_n-y_n\|^2]+(1-\beta_n)\alpha_n M_2\\ &\leq \|x_n-p\|^2-(1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+\alpha_n M_4,\end{aligned}$$

where $M_4:=M_2+4M_3$. This immediately implies that for all $n\geq n_0$,

$$(1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4.$$

Step 3. We claim that $\exists M>0$ s.t.

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\leq [1-\tfrac{(1-2\delta)\delta_n-\gamma_n}{1-\alpha_n\gamma_n}\alpha_n]\|x_n-p\|^2+\tfrac{[(1-2\delta)\delta_n-\gamma_n]\alpha_n}{1-\alpha_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_{n+1}\|\\ &\quad+\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_n\|+\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\langle f(p)-p,x_n-p\rangle\\ &\quad+\tfrac{\gamma_n+\delta_n}{(1-2\delta)\delta_n-\gamma_n}(\tfrac{\theta_n}{\alpha_n}\cdot\tfrac{2M^2(1+b(1+\theta_n)^2)}{1-b}+\tfrac{\sigma_n}{\alpha_n}\|x_n-x_{n-1}\|\tfrac{3M(1+b\theta_n(2+\theta_n))}{1-b})\}.\end{aligned}\tag{29}$$

In fact, we get

$$\|w_n-p\|^2\leq[(1+\theta_n)\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|]^2\leq\|x_n-p\|^2+\theta_n 2M^2+\sigma_n\|x_n-x_{n-1}\|3M,\tag{30}$$

where $\exists M>0$ s.t. $\sup_{n\geq 1}\{(1+\theta_n)\|x_n-p\|,\sigma_n\|x_n-x_{n-1}\|\}\leq M$. From Algorithm 4 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\beta_n(T^n w_n-p)+\gamma_n(z_n-p)+\delta_n(Sz_n-p)+\gamma_n(u_n-z_n)\|^2\\ &\leq \beta_n\|T^n w_n-p\|^2+(1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n-p)+\delta_n(Sz_n-p)]\|^2\\ &\quad+2\gamma_n\alpha_n\langle u_n-p,x_{n+1}-p\rangle+2\gamma_n\alpha_n\langle p-f(x_n),x_{n+1}-p\rangle,\end{aligned}$$

which together with Lemma 4, leads to

$$\begin{aligned}&\|x_{n+1}-p\|^2\leq\beta_n(1+\theta_n)^2\|w_n-p\|^2+(1-\beta_n)\|z_n-p\|^2+2\gamma_n\alpha_n\|u_n-p\|\|x_{n+1}-p\|\\ &\quad+2\gamma_n\alpha_n\langle p-f(x_n),x_{n+1}-p\rangle\\ &\leq\beta_n(1+\theta_n)^2(\|x_n-p\|^2+\theta_n 2M^2+\sigma_n\|x_n-x_{n-1}\|3M)+(1-\beta_n)[(1-\alpha_n)\|u_n-p\|^2\\ &\quad+2\alpha_n\langle f(x_n)-p,z_n-p\rangle]+\gamma_n\alpha_n(\|u_n-p\|^2+\|x_{n+1}-p\|^2)\\ &\quad+2\gamma_n\alpha_n\langle p-f(x_n),x_{n+1}-p\rangle.\end{aligned}$$

By Step 3 of Algorithm 4, and from Equation (30) we know that $\|u_n - p\|^2 \le \|x_n - p\|^2 + \theta_n 2M^2 + \sigma_n\|x_n - x_{n-1}\|3M$. Hence, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le [1-\alpha_n(1-\beta_n)]\|x_n - p\|^2 + \beta_n\theta_n 2M^2 + (1-\beta_n)(1-\alpha_n)(\theta_n 2M^2 \\
&\quad + \sigma_n\|x_n - x_{n-1}\|3M) + 2\alpha_n\delta_n\langle f(x_n) - p, z_n - p\rangle + \gamma_n\alpha_n(\|x_n - p\|^2 \\
&\quad + \|x_{n+1} - p\|^2) + (1-\beta_n)\alpha_n(\theta_n 2M^2 + \sigma_n\|x_n - x_{n-1}\|3M) \\
&\quad + 2\gamma_n\alpha_n\langle f(x_n) - p, z_n - x_{n+1}\rangle + \beta_n(1+\theta_n)^2(\theta_n 2M^2 + \sigma_n\|x_n - x_{n-1}\|3M) \\
&\le [1-\alpha_n(1-\beta_n)]\|x_n - p\|^2 + 2\gamma_n\alpha_n\|f(x_n) - p\|\|z_n - x_{n+1}\| \\
&\quad + 2\alpha_n\delta_n\langle f(x_n) - p, x_n - p\rangle + 2\alpha_n\delta_n\langle f(x_n) - p, z_n - x_n\rangle \\
&\quad + \gamma_n\alpha_n(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + (1-\beta_n)[\theta_n\tfrac{2M^2(1+\beta_n(1+\theta_n)^2)}{1-\beta_n} \\
&\quad + \sigma_n\|x_n - x_{n-1}\|\tfrac{3M(1+\beta_n\theta_n(2+\theta_n))}{1-\beta_n}] \\
&\le [1-\alpha_n(1-\beta_n)]\|x_n - p\|^2 + 2\gamma_n\alpha_n\|f(x_n) - p\|\|z_n - x_{n+1}\| \\
&\quad + 2\alpha_n\delta_n\delta\|x_n - p\|^2 + 2\alpha_n\delta_n\langle f(p) - p, x_n - p\rangle + 2\alpha_n\delta_n\|f(x_n) - p\|\|z_n - x_n\| \\
&\quad + \gamma_n\alpha_n(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + (1-\beta_n)[\theta_n\tfrac{2M^2(1+b(1+\theta_n)^2)}{1-b} \\
&\quad + \sigma_n\|x_n - x_{n-1}\|\tfrac{3M(1+b\theta_n(2+\theta_n))}{1-b}],
\end{aligned}$$

which immediately yields Equation (29).

Step 4. We claim the strong convergence of $\{x_n\}$ to a unique solution $x^* \in \Omega$ of HVI Equation (12). In fact, using the same reasoning as in Step 4 of the proof of Theorem 1, we derive the desired conclusion. This completes the proof. □

Next, we shall show how to solve the VIP and CFPP in the following illustrating example.

The initial point $x_0 = x_1$ is randomly chosen in $\mathbf{R} = (-\infty, \infty)$. Take $f(x) = \frac{1}{4}\sin x$, $\gamma = l = \mu = \frac{1}{2}$, $\sigma_n = \alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{3}$, $\gamma_n = \frac{1}{6}$, and $\delta_n = \frac{1}{2}$. Then we know that $\delta = \frac{1}{4}$ and $f(\mathbf{R}) \subset [-\frac{1}{4}, \frac{1}{4}]$.

We first provide an example of Lipschitz continuous and pseudomonotone mapping A, asymptotically nonexpansive mapping T and strictly pseudocontractive mapping S with $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $C = [-1.5, 1]$ and $H = \mathbf{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. Let $A, T, S : H \to H$ be defined as $Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$, $Tx := \frac{4}{5}\sin x$ and $Sx := \frac{1}{3}x + \frac{1}{2}\sin x$ for all $x \in H$. Now, we first show that A is pseudomonotone and Lipschitz continuous with $L = 2$ such that $A(C)$ is bounded. Indeed, it is clear that $A(C)$ is bounded. Moreover, for all $x, y \in H$ we have

$$\begin{aligned}
\|Ax - Ay\| &= |\tfrac{1}{1+\|\sin x\|} - \tfrac{1}{1+\|x\|} - \tfrac{1}{1+\|\sin y\|} + \tfrac{1}{1+\|y\|}| \\
&\le |\tfrac{\|\sin y\| - \|\sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)}| + |\tfrac{\|y\| - \|x\|}{(1+\|x\|)(1+\|y\|)}| \\
&\le \|\sin x - \sin y\| + \|x - y\| \le 2\|x - y\|.
\end{aligned}$$

This implies that A is Lipschitz continuous with $L = 2$. Next, we show that A is pseudomonotone. For any given $x, y \in H$, it is clear that the relation holds:

$$\langle Ax, y - x\rangle = (\frac{1}{1+|\sin x|} - \frac{1}{1+|x|})(y - x) \ge 0 \Rightarrow \langle Ay, y - x\rangle = (\frac{1}{1+|\sin y|} - \frac{1}{1+|y|})(y - x) \ge 0.$$

Furthermore, it is easy to see that T is asymptotically nonexpansive with $\theta_n = (\frac{4}{5})^n$, $\forall n \ge 1$, such that $\|T^{n+1}x_n - T^n x_n\| \to 0$ as $n \to \infty$. Indeed, we observe that

$$\|T^n x - T^n y\| \le \frac{4}{5}\|T^{n-1}x - T^{n-1}y\| \le \cdots \le (\frac{4}{5})^n\|x - y\| \le (1 + \theta_n)\|x - y\|,$$

and

$$\|T^{n+1}x_n - T^n x_n\| \le (\frac{4}{5})^{n-1}\|T^2 x_n - Tx_n\| = (\frac{4}{5})^{n-1}\|\frac{4}{5}\sin(Tx_n) - \frac{4}{5}\sin x_n\| \le 2(\frac{4}{5})^n \to 0,\ (n\to\infty).$$

It is clear that $\mathrm{Fix}(T) = \{0\}$ and

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = \lim_{n\to\infty}\frac{(4/5)^n}{1/(n+1)} = 0.$$

Moreover, it is readily seen that $\sup_{n\ge1}|(T^n - f)x_n| = \sup_{n\ge1}|\frac{4}{5}\sin(T^{n-1}x_n) - \frac{1}{4}\sin x_n| \le \frac{21}{20} < \infty$. In addition, it is clear that S is strictly pseudocontractive with constant $\zeta = \frac{1}{4}$. Indeed, we observe that for all $x, y \in H$,

$$\|Sx - Sy\|^2 \le [\frac{1}{3}\|x-y\| + \frac{1}{2}\|\sin x - \sin y\|]^2 \le \|x-y\|^2 + \frac{1}{4}\|(I-S)x - (I-S)y\|^2.$$

It is clear that $(\gamma_n + \delta_n)\zeta = (\frac{1}{6}+\frac{1}{2})\cdot\frac{1}{4} \le \frac{1}{6} = \gamma_n < (1-2\delta)\delta_n = (1-2\cdot\frac{1}{4})\cdot\frac{1}{2} = \frac{1}{4}$ for all $n \ge 1$. Therefore, $\Omega = \mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A) = \{0\} \ne \emptyset$. In this case, Algorithm 3 can be rewritten as follows:

$$\begin{cases} w_n = T^n x_n + \frac{1}{n+1}(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = \frac{1}{n+1}f(x_n) + \frac{n}{n+1}P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \frac{1}{3}T^n x_n + \frac{1}{6}P_{C_n}(w_n - \tau_n A y_n) + \frac{1}{2}Sz_n, \quad \forall n \ge 1, \end{cases}$$

where C_n and τ_n are picked up as in Algorithm 3. Thus, by Theorem 1, we know that $\{x_n\}$ converges to $0 \in \Omega$ if and only if $|x_n - x_{n+1}| + |x_n - T^n x_n| \to 0,\ (n\to\infty)$.

On the other hand, Algorithm 4 can be rewritten as follows:

$$\begin{cases} w_n = T^n x_n + \frac{1}{n+1}(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = \frac{1}{n+1}f(x_n) + \frac{n}{n+1}P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \frac{1}{3}T^n w_n + \frac{1}{6}P_{C_n}(w_n - \tau_n A y_n) + \frac{1}{2}Sz_n, \quad \forall n \ge 1, \end{cases}$$

where C_n and τ_n are picked up as in Algorithm 4. Thus, by Theorem 2, we know that $\{x_n\}$ converges to $0 \in \Omega$ if and only if $|x_n - x_{n+1}| + |x_n - T^n x_n| \to 0,\ (n\to\infty)$.

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